ACTEX Learning

Learn Today. Lead Tomorrow.

ACTEX Study Manual for

SOA Exam FM

Spring 2017 Edition

Volumes I & II

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ACTEX Learning
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Preface

ACTEX first published a study manual for the Society of Actuaries' Exam FM ("Financial Mathematics") in 2004. That manual was prepared by lead author Matthew Hassett, assisted by Michael Ratliff, Toni Coombs Garcia, and Amy Steeby. The manual has been regularly updated and expanded to keep pace with changes in the syllabus for Exam FM and to increase the number of sample problems and practice exams. Now this new edition of the ACTEX Study Manual for Exam FM, edited by lead author John Dinius, has been extensively revised and edited to reflect changes in the SOA's Exam FM syllabus effective with the June 2017 administration of the exam.

The revised syllabus for Exam FM has eliminated most of the material on Financial Derivatives, and has added sections on "The Determination of Interest Rates" and "Interest Rate Swaps." The SOA has issued new study notes on these two topics, as well as a new study note on "Using Duration and Convexity to Approximate Present Value." This material is covered in two new modules in this manual (Modules 8 and 9) plus an expansion of the Asset-Liability Management module (Module 7). In addition, many sections of the other modules in the manual have been revised to provide additional explanations, examples, and solutions.

This manual has 9 modules that are arranged in 3 groups of 3 modules each. The first 3 modules present basic concepts (the time value of money, annuities, and loan repayment). The next 3 modules apply these concepts to a range of topics (bonds, yield rates, and the term structure of interest rates). The final 3 modules examine more complex real-world concepts (asset-liability management, the factors that influence market interest rates, and interest rate swaps). After each group of three modules there is a "midterm exam," providing the student an opportunity to check his/her progress. Also included at the end of the manual are 11 practice exams of 35 problems each. These are intended to provide realistic exam-taking experience to complete the student's preparation for Exam FM.

This manual is written so that it can be read without reference to any other text. However, we strongly recommend that the student obtain and read one of the official textbooks for SOA Exam FM, and use that text *in combination with* this manual. The following pages provide recommendations on how to prepare for actuarial exams and suggestions on how to use this manual most effectively.

A note about Errors:
If you find a possible error in this manual, please let us know. Use the "Feedback" link on the ACTEX homepage (www.actexmadriver.com) and describe the issue. We will review all comments and respond to you with an answer. Any confirmed errata will be posted on the ACTEX website under the "Errata" link.

On passing exams

How to Learn Actuarial Mathematics and Pass Exams

On the next page you will find a list of study tips for learning the material in the Exam FM syllabus and passing Exam FM. But first it is important to state the basic learning philosophy that we are using in this guide:

You must master the basics before you proceed to the more difficult problems

Think about your basic calculus course. There were some very challenging applications in which you used derivatives to solve hard max-min problems.

It is important to learn how to solve these hard problems, but if you did not have the basic skills of taking derivatives and doing algebraic simplification, you could not do the more advanced problems. Thus every calculus book has you practice derivative skills before presenting the tougher sections on applied problems.

You should approach interest theory the same way. The first 2 or 3 modules give you the basic tools you will need to solve the problems in the later modules. Learn these concepts and methods (and the related formulas) very well, as you will need them in each of the remaining modules.

This guide is designed to progress from simpler problems to harder ones.

In each module we start with the basic concepts and simple examples, and then progress to more difficult material so that you will be prepared to attack actual exam problems by the end of the module.

The same philosophy is used in our practice exams at the end of this manual. The first few practice exams have simpler problems, and the problems become more difficult as you progress through the practice exams.

A good strategy when taking an exam is to answer all of the easier problems before you tackle the harder ones.

An exam is scored in percentage terms, and a multiple choice exam like Exam FM will have a mix of problems at different difficulty levels.

If an exam has ten problems and three are very hard, getting the right answers to only the three hard problems and missing the others gets you a score of 30%. This is actually a possibility if the very hard problems are the first ones on the exam and you try to solve them first.

A useful exam strategy is to go through the exam and quickly solve all the more basic problems before spending extra time on the hard ones. Strive to answer all of the easy problems correctly.

Study Tips

1) Develop a schedule so that you will complete your studying in time for the exam. Divide your schedule into time for each module, plus time at the end to review and to solve practice problems. Your schedule will depend on how much time you have before the exam, but a reasonable approach might be to complete one module per week.

2) If possible, join a study group of your peers who are studying for Exam FM.

3) For each module:
   a) Read the module in the FM manual.
   b) As you read through the examples in the text, make sure that you can correctly compute the answers.
   c) Summarize each concept you learn in the manual's margins or in a notebook.
   d) Understand the main idea of each concept and be able to summarize it in your own words. Imagine that you are trying to teach someone else this concept.
   e) While reading, create flash cards for the formulas, to facilitate memorization.
   f) Learn the calculator skills thoroughly and know *all* of your calculator functions.
   g) Do a review of the corresponding chapter in the recommended text.
   h) Do the Basic Review Problems and review your solutions.
   i) Do the Sample Exam Problems and review your solutions.
      i) If you have been stuck on a problem for more than 20 minutes, it is OK to refer to the solutions. Just make sure that when you are finished with the problem, you can recite the concept that you missed and summarize it in your own words. If you get stuck on a problem, think about what principles were used in this question and see if you could write a different problem with similar structure (as if you were the exam writer).
      ii) Mark each sample exam problem as an Easy, Medium, or Hard problem.

4) After learning the material in each module, it is a good idea to go back to previous modules and do a quick half-hour or 1-hour review, so that information isn't forgotten.

5) Go back and redo the sample exam problems that you have marked as Medium or Hard when you looked through them the first time.

6) At the end of modules 3, 6, and 9, we have included practice exams that are like midterms. Taking these tests will help you consolidate your knowledge.

7) After learning the material in all of the modules and taking the midterms, go to the practice exams.
   a) The first 6 practice exams are relatively straightforward to enable you to review the basics of each topic. You may want to attempt them in a *non-timed* environment to evaluate your skills and understanding.
   b) The final 5 practice exams introduce more difficult questions in order to replicate the exam experience. You should take each of these in a *timed* environment to give yourself experience with exam conditions.

   Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like the ones in this manual.

Contents

Introduction

As you begin your preparation for the Society of Actuaries' Exam FM, you should be aware that studying Financial Mathematics (or "interest theory," as I like to call it) is not a matter of learning mathematics. Instead, financial mathematics involves *applying* mathematics to situations that involve financial transactions. This will require you to learn a new language, the language of the financial world, and then to apply your *existing* math skills to solve problems that are presented in this new language. It is important that you spend adequate time to fully understand the meanings of all the terms that will be introduced in this manual. Nearly all of the problems on Exam FM will be word problems (rather than just formulas), and it is very difficult to solve these problems unless you understand the language that is being used.

In this manual, we assume that you have a solid working knowledge of differential and integral calculus and some familiarity with probability. We also assume that you have an excellent knowledge of algebraic methods. Depending on what mathematics courses you have taken (and how recently), you may need to review these topics in order to understand some of the material and work the problems in this manual.

Throughout the manual, a large number of the examples and practice problems are solved using the Texas Instruments BA II Plus calculator, which is the financial calculator approved for use on Exam FM. It is essential for you to have a BA II Plus calculator in order to understand the solutions presented here, and also to solve the problems on the actual exam. This calculator is available in a standard model, and also as the "BA II Plus *Professional*." The Professional model, which is somewhat more expensive, is a bit easier to work with, which could be important when taking a time-limited exam.

Over the years, most actuarial students have found that the best way to prepare for Exam FM is to work a very large number of problems (hundreds and hundreds of problems). There are many examples, exercises, problems, and practice exams included in this manual. Many more problems can be found on the Society of Actuaries website (www.soa.org) or by searching the Web. You should plan to spend a significant proportion of your study time working problems and reviewing the solutions that are provided in the manual and on the websites.

Financial mathematics is an integral part of an actuary's skill set, and you can expect to apply interest theory regularly throughout your career. A strong understanding of the topics covered in this manual will provide you a valuable tool for understanding financial and economic matters both on and off the job.

Best of luck to you in learning Financial Mathematics and passing Exam FM!

John Dinius
January 2017

Module **1**

Interest Rates and the Time Value of Money

Section 1.1

Time Value of Money

Interest theory deals with the **time value of money**. For example, a dollar invested today at 6% interest per year is worth $1.06 one year from today. Because a dollar invested today can provide *more* than one dollar a year from now, it follows that receiving a dollar today has a *greater value* than receiving one dollar a year from now. In other words, money has a "time value," and in order to assess the value of a payment, we need to know not only the *amount* of the payment, but also *when the payment occurs*. That is the underlying principle of interest theory.

In the example of the investment at 6% interest, the dollar that is invested today is called the **principal**, and the $0.06 increase in value is called **interest**. What happens to the investment *after* the first year depends on whether it is earning **compound interest** or **simple interest**. We illustrate this with an example based on an investment of 100 that earns 6% interest for two years.

a) Compound interest: Interest is earned during each year on the total amount in the account at the beginning of that year. The amounts in the account at the end of Year 1 and Year 2 are:

   Year 1: $100+0.06(100)=100(1.06)=106$

   Year 2: $106+0.06(106)=106(1.06)=100(1.06)^2=112.36$

   Interest is "compounded" at the end of Year 1. That is, the interest earned during Year 1 is "converted" to principal at the end of Year 1 and it becomes part of the principal that earns interest during Year 2.

b) Simple interest: In each year interest is earned on only the original principal of 100. The amounts in the account at the end of Year 1 and Year 2 are:

Year 1: $100+0.06(100)=100(1.06)=106$

Year 2: $106+0.06(100)=100*(1+2(0.06))=112$

Because the interest earned during the first year is not converted to principal ("compounded") at the end of the first year, it does not earn interest during the second year. The principal is 100 in both years, and the amount of interest earned in each year is 6.

Compound interest is the most widely used method of computing interest, especially for multi-period investments. Simple interest is generally used only for shorter-term investments (usually less than one year). Because it is so widely used, we will begin our study of interest theory with compound interest.

Note: In this manual, amounts of money will generally be given without an indication of what currency is being used. You may want to think of these amounts as U.S. or Canadian dollars ($100, in the case of the above example), or you may just treat them as amounts of money with no specific denomination.

Section 1.2

Present Value and Future Value

The value of an investment today (time 0) is its **present value** [PV], and its value n periods from today is called its **future value** [FV] as of time n. More broadly, if we know the value of an investment as of a particular date and we want to find its value as of an *earlier* date, we are calculating a *present value* as of the earlier date. And if we want to find the value as of a *later* date, then we are calculating a *future value* (or an **accumulated value**) as of that later date. If funds are invested at a compound interest rate of i per period for n periods, the basic relationships are:

(1.1)

$$FV = PV(1+i)^n \qquad PV = \frac{FV}{(1+i)^n}$$

Example (1.2)

Let $n = 10$ and $i = 0.06$.

a) If $PV = 1{,}000$, then $FV = 1{,}000(1.06)^{10} = 1{,}790.85$

b) If $FV = 1{,}000$, then $PV = \dfrac{1{,}000}{(1.06)^{10}} = 558.39$

Calculation a) demonstrates that if we invest 1,000 today at 6% interest, in 10 years it will have accumulated to a future value of 1,790.85.

Calculation b) shows that if we need 1,000 ten years from now, we can accumulate that amount by investing 558.39 now at 6% interest.

Exercise (1.3)

Using an interest rate of 5% compounded annually, find a) the present value (today) of 20,000 payable in 15 years, and b) the future value 6 years from today of 5,000 deposited today.

Answer: a) 9,620.34 b) 6,700.48

Calculator Note

The BA II Plus calculator has 5 "Time Value of Money" keys:

N	Number of periods
I/Y	Interest rate per period (usually per year)
PV	Present value
PMT	Periodic payment
FV	Future value

In this module we will not look at any problems that involve periodic payments. The PMT key will be used starting in Module 2. Using the other four keys, we can solve compound interest problems like Example **(1.2)**, as we illustrate next.

To begin any new problem, it is wise to clear the Time Value of Money [TVM] registers to erase any entries from prior problems. Note that the legend "CLR TVM" appears above the FV key on the BA II Plus calculator. To clear the TVM registers use the keystrokes 2ND CLR TVM. This sets all 5 of the TVM values to 0.

Before we do the actual calculation, we must choose a sign convention for the values we enter into the calculator as well as the answers we calculate. In this manual, we will use the following convention: **Money that you receive is positive; money that you pay out is negative.** Thus, if you put 1,000 into an account now, you should enter it into the calculator as $-1{,}000$ to indicate that it is "out of pocket." (You can make an entry negative by using the +|- key.)

In Example **(1.4)**, we will rework Example **(1.2)** using the calculator.

Example (1.4)

We will redo the calculations of Example **(1.2)** using the BA II Plus's TVM functions.

To find the future value of 1,000 in 10 years at 6% compound interest per year, use the following keystrokes:

10 [N] 6 [I/Y] 1,000 [+|-] [PV] [CPT] [FV]

You will see in the display: FV = 1,790.85.
The answer is <u>positive</u>, since this is money that you will <u>receive</u>.
Note that the interest rate is 6% = 0.06, but it is entered into the calculator as 6, not 0.06. The calculator treats your entry as a percentage; that is, the amount entered is divided by 100 when the calculation is done.

To find the present value at 6% compound interest of 1,000 to be received 10 years in the future, use the following keystrokes:

10 [N] 6 [I/Y] 1000 [FV] [CPT] [PV] .

You will see in the display: PV = −558.39.
The answer is <u>negative</u>; this is money that you must put <u>into</u> the account.

Note: You may have noticed that there is an asterisk (*) above the value -558.39 in your calculator's display. The asterisk indicates that -558.39 is a <u>computed</u> value, not a number that was entered. As long as the inputs (the values for N, I/Y, PMT, and FV) are not changed, the asterisk will continue to appear over the -558.39 (even if you perform other calculations and then press RCL PV). But if one of the inputs (such as N or FV) is changed, then when you press RCL PV, the -558.39 will appear <u>without</u> the asterisk, indicating that it is <u>not</u> a computed value based on the <u>current</u> entries in the TVM variables.

Exercise (1.5)

Rework Exercise **(1.3)** using the calculator's TVM functions.

Section 1.3

Functions of Investment Growth

An investor might wish to plot the growth of an investment over time. Two functions are commonly used:

The **accumulation function**, $a(t)$, is the value at time t of an initial investment of 1 made at time 0.

The **amount function**, $A(t)$, is the value at time t of an initial investment of $A(0)$ made at time 0.

For compound interest at a constant annual rate i, these functions are:

(1.6)

Compound interest:

$$a(t) = (1+i)^t \qquad A(t) = A(0)(1+i)^t$$

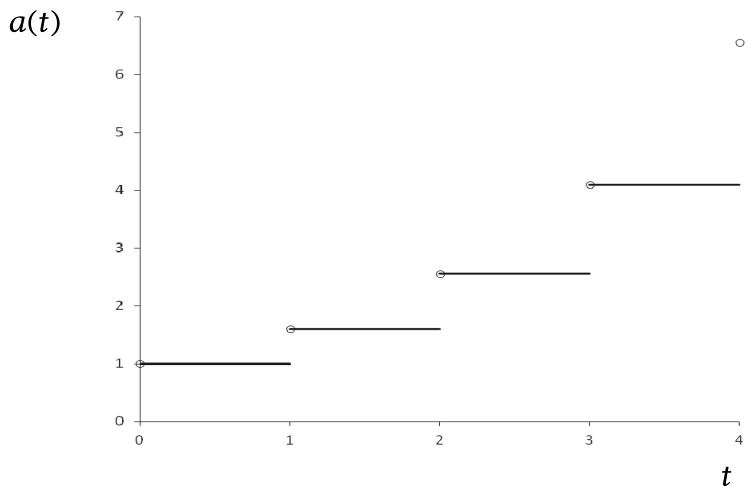
For compound interest at a rate of i = 60% per year, the values of $a(t)$ at the end of each of the first 4 years are as shown in the following table:

t	0	1	2	3	4
$a(t)$	1	1.60	2.56	4.096	6.5536

Note: An extremely high interest rate (60%) is used here so that the following graphs will clearly show the exponential (and non-linear) form of the accumulation function.

What is the value of $a(t)$ when t is *not* an integer?
For example, what is the value of $a(t)$ for t = 1.5?
There can be instances where interest is considered to be earned only at the *end* of each year. In that case, the value of the accumulation function at t = 1.5 would be the same as at t = 1, that is, it would be 1.60. At the end of the second year, all of the interest for the period from t = 1 to t = 2 would be credited, and the accumulated value would increase instantaneously from 1.60 to 2.56.

Thus, if interest is considered to be earned only *at the end of each year*, the graph of $a(t)$ is a *step function*:

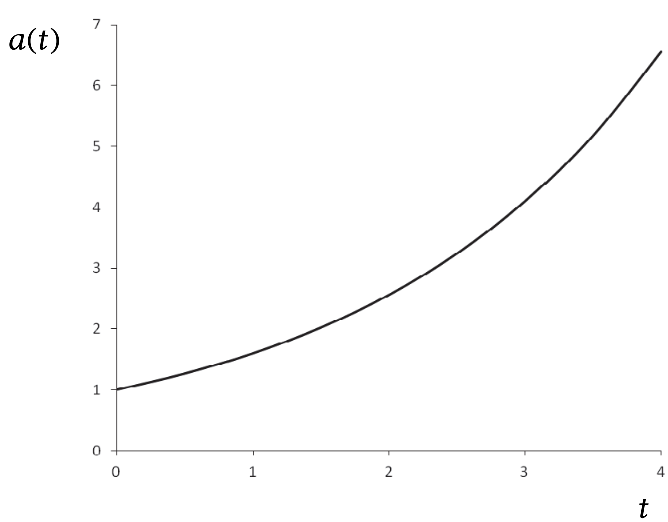


Real-world contracts that involve interest should specify how interest for partial periods will be calculated. For our purposes in studying interest theory, we will generally assume that interest is earned *continuously*. When interest is earned continuously, the formulas $a(t)=(1+i)^t$ and $A(t)=A(0)\left(1+i\right)^t$ are valid for *all* values of t, not just *integer* values, and the accumulation and amount functions are continuous functions (not step functions). Unless an exam question specifies that interest is credited only at the end of each year (or the end of each month or each quarter, etc.), you should assume that interest is earned continuously.

In the current example, the value of the accumulation function at time 1.5 is:

$$a(1.5)=(1.60)^{1.5}=2.0239$$

If interest is *earned continuously*, the graph of $a(t)$ is a smooth, continuous function:



For *simple* interest at a constant annual rate i, the accumulation and amount functions are:

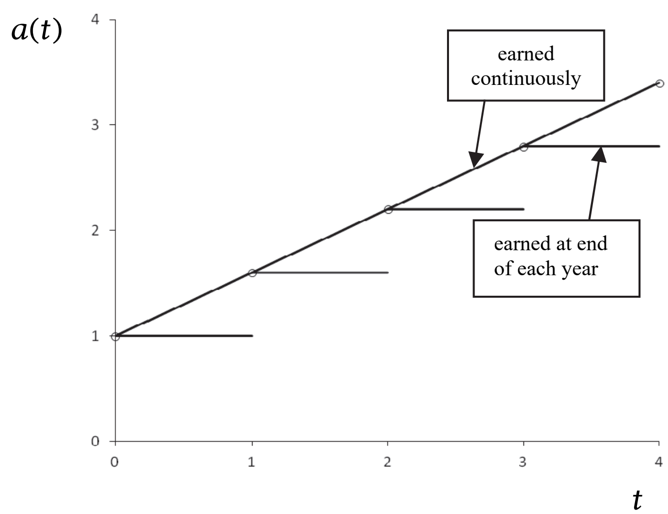
(1.7) Simple interest:

$$a(t) = (1 + i \cdot t) \qquad A(t) = A(0)\left(1 + i \cdot t\right)$$

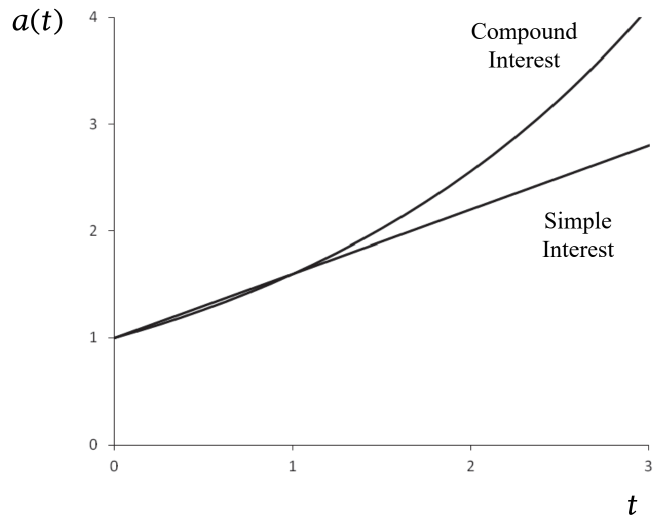
For simple interest at a rate i = 60% per year, the values of $a(t)$ at the end of each of the first 4 years are:

t	0	1	2	3	4
$a(t)$	1	1.60	2.20	2.80	3.40

Again, $a(t)$ can be a step function if interest is considered to be earned only at the end of each year. However, as with compound interest, we will generally treat simple interest as being earned continuously, so that $a(t)$ is a continuous function. The following graph includes plots for both the step function and the continuous function.



The following graph compares the growth of 2 investments. In each case, an amount of 1 is invested at time 0. One investment earns *simple* interest at a 60% annual rate; the other earns *compound* interest at a 60% annual rate.



Both investments have a value of 1.60 at the end of 1 year. After the first year, the one earning compound interest grows much faster, as it earns "interest on interest." The simple-interest investment earns interest on only the original principal of 1, so its growth (its slope) is constant at 0.60 per year. Note, however, that the investment at simple interest has a *larger* value between time 0 and time 1 than the investment earning compound interest, since $1+t\cdot i>(1+i)^t$ for values of t between 0 and 1.

The simple-interest investment is growing faster (has a steeper slope) than the compound-interest investment at the beginning of the first year. But the compound-interest investment grows faster and faster as the year progresses (because it earns interest on a larger and larger principal). By the end of the first year, the compound-interest investment has caught up with the simple-interest investment, and thereafter exceeds it.

Section 1.4

Effective Rate of Interest

We will now use the amount function to define the **effective rate of interest** for any specified time period. For the one-year time period $[t, t+1]$, the beginning amount is $A(t)$, the ending amount is $A(t+1)$, and the amount of interest earned over the interval is $A(t+1) - A(t)$.

The effective rate of interest for this period is defined as:

$$i_{[t,t+1]} = \frac{\text{interest earned between } t \text{ and } t+1}{\text{value of investment at time } t} = \frac{A(t+1) - A(t)}{A(t)} = \frac{a(t+1) - a(t)}{a(t)}$$

(1.8)

Note: The notation $i_{[t_1,t_2]}$ will be used in this module to represent an effective interest rate for the period $[t_1, t_2]$. This is not standard actuarial notation, and it will not be used in the other modules of this manual.

Example (1.9)

Let the interest rate be 6% and the time interval be [1,2].

For compound interest:

$$i_{[1,2]} = \frac{a(2) - a(1)}{a(1)} = \frac{1.1236 - 1.06}{1.06} = 0.06$$

For simple interest:

$$i_{[1,2]} = \frac{a(2) - a(1)}{a(1)} = \frac{1.12 - 1.06}{1.06} = 0.0566$$

Exercise (1.10)

Let the interest rate be 6% and the time interval be [2,3].
Find $i_{[2,3]}$ for a) compound interest at 6%, and b) simple interest at 6%.

Answers: a) 0.06 b) 0.0536

Note that over multi-year periods a compound interest rate of 6% per year gives a constant effective rate of 6% for each one-year period, while a simple interest rate of 6% leads to declining effective rates over time. This is because the investment at compound interest always earns 6% on the entire beginning-of-year balance, but the investment at simple interest earns 6% on only the original principal.

The above discussion involves effective rates over a 1-year period of time. These are called **annual effective rates**. We will usually express rates of compound interest as annual effective rates, but we can also calculate an effective rate for a shorter or longer time period (e.g., a quarterly effective rate, or a 2-year effective rate). In each case, the effective rate equals the amount of interest earned during the period (e.g., a 3-month period or a 2-year period), divided by the value of the investment at the beginning of the period.

Example (1.11)

An investment earns a 6% annual interest rate.
We will calculate the quarterly (3-month) effective rates for the periods [0.25,0.50] and [1.25,1.50]. We will do each of these calculations based on a 6% rate of compound interest, and also for a 6% rate of simple interest.

At 6% compound interest

For the period [0.25,0.50]:

$$i_{[0.25,0.50]} = \frac{a(0.50)-a(0.25)}{a(0.25)} = \frac{1.06^{0.5}-1.06^{0.25}}{1.06^{0.25}} = 0.01467$$

For the period [1.25,1.50]:

$$i_{[1.25,1.50]} = \frac{a(1.50)-a(1.25)}{a(1.25)} = \frac{1.06^{1.50}-1.06^{1.25}}{1.06^{1.25}} = 0.01467$$

At 6% simple interest

For the period [0.25,0.50]:

$$i_{[0.25,0.50]} = \frac{a(0.50)-a(0.25)}{a(0.25)} = \frac{\left[1+(0.5)0.06\right]-\left[1+(0.25)0.06\right]}{1+(0.25)0.06} = 0.01478$$

For the period [1.25,1.50]:

$$i_{[1.25,1.50]} = \frac{a(1.50)-a(1.25)}{a(1.25)} = \frac{\left[1+(1.50)0.06\right]-\left[1+(1.25)0.06\right]}{1+(1.25)0.06} = 0.01395$$

During the period [0.25,0.50], 6% simple interest generated a higher quarterly effective rate than 6% compound interest. As we noted previously, early in the first year simple interest produces faster growth than compound interest at the same numerical interest rate (but compound interest catches up at the end of the first year).

During the period [1.25,1.50], of course, compound interest produces a higher effective rate than simple interest. The quarterly effective rate for compound interest during this period is 0.01467, the same as it was for the period [0.25,0.50]. But the quarterly effective rate for simple interest has decreased from 0.01478 to 0.01395.

Exercise (1.12)

An investment earns a 6% annual interest rate. Calculate the quarterly (3-month) effective rates for the periods [0.50,0.75] and [1.50,1.75]:

a) at 6% compound interest, and
b) at 6% simple interest

Are the rates calculated in a) (at compound interest) higher or lower than the rates calculated in b) (at simple interest)?

Answer: a) 0.01467 for each period b) 0.01456 and 0.01376;
Compound interest produces a higher effective rate in each period.

Section 1.5

Nominal Rates of Interest

In many instances where payments (such as loan payments) are made more frequently than once a year (e.g., monthly, quarterly, or semi-annually), the interest rate is expressed as a **nominal annual rate**. A nominal annual rate of interest is equal to the effective interest rate *per period* multiplied by the number of periods per year.

For example, if an investment is earning interest at a 2% quarterly effective rate, you could multiply 2% by 4 and refer to this as a "nominal annual rate of 8%, convertible quarterly" (or "compounded quarterly"). This gives us a simple way of referring to the interest rate on an *annual scale*, but 8% is *not* the rate you *actually* earn each year. In this example, you actually earn *more* than 8%. One dollar accumulates to $(1.02)^4 = 1.0824$ in one year, so a nominal annual rate of 8% convertible quarterly is *equivalent* to an annual effective rate of 8.24%.

Many students find this confusing, so we will go over it again for reinforcement:

1. **The effective rate per period is your starting point**
   Example: 2% per quarter (a 2% "quarterly effective rate")

2. **Calculate the nominal annual rate.**
   Nominal Rate = (Rate/period) × (Number of periods per year)
   Example $2\% \times 4 = 8\%$
   (an 8% "nominal annual rate convertible quarterly")

3. **The annual effective rate is the annual rate you *actually* earn with compounding of interest**
   Example. End-of-year accumulated value is $(1.02)^4 = 1.0824$
   The annual effective rate is 8.24%.

A nominal rate is an artificial rate that gives you a way of *talking about* a periodic rate (such as a quarterly or monthly effective rate) in familiar *annual* terms. The annual effective rate is *not* artificial. It is the rate you *actually* earn in a year. Similarly, a quarterly effective rate is the rate you *actually* earn in one quarter.

It is important to understand that interest calculations are *always* done using effective rates (whether annual, quarterly, monthly, etc.) Nominal rates are *not* used in calculations; a nominal rate must first be converted to an effective rate, and then calculations are performed using the effective rate.

Exercise (1.13)

Suppose you earn interest at a rate of 1% per month, compounded monthly (i.e., a 1% monthly effective rate).

a) What is your nominal annual rate?
b) What is your annual effective rate?

Answer: a) 12% convertible monthly b) 12.6825%

In the general case of m conversion periods per year, we denote the nominal rate by $i^{(m)}$. The effective interest rate per period is $\frac{i^{(m)}}{m}$, and the annual effective rate is:

(1.14)

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$

This has the important consequence that:

(1.15)

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

You will often see the statement that interest is "**convertible**" or "**compounded**" ***m* times per year.** This means that the interest earned during each period (of length $1/m$ years) is "compounded" (converted to principal) at the end of that period and earns interest during the following period.

Example (1.16)

Suppose interest is convertible monthly and the nominal rate is $i^{(12)} = 0.09$. Then the annual effective rate is:

$$\left(1 + \frac{0.09}{12}\right)^{12} - 1 = 1.0075^{12} - 1 = 0.0938 \quad \text{(or 9.38\%)}$$

This process can easily be reversed to find the nominal rate if we are given the effective rate.

Example (1.17)

Interest is convertible semi-annually and results in an annual effective rate of 10.25%. Find the nominal annual rate convertible semi-annually.

Solution.

$m = 2$, so we need to find $i^{(2)}$.

By **(1.15)**,

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = 1.1025$$

$$\left(1+\frac{i^{(2)}}{2}\right) = \sqrt{1.1025} = 1.05$$

$$i^{(2)} = 2(1.05-1) = 10\%$$

Thus the nominal annual rate is 10% convertible semi-annually, and the semi-annual effective rate is 5%.

Note that you can derive a formula that solves for $i^{(m)}$ given i and m. It is:

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

It is not necessary to memorize this formula. Formula **(1.15)** is intuitive and easy to remember, and we can always substitute the given values for i and m into it to solve for $i^{(m)}$. This is the approach we used in Example **(1.17)**.

Calculator Note

The BA II Plus calculator has an interest conversion worksheet that can be used to solve these problems. The legend above the [2] key is ICONV, which stands for "interest conversion." You can activate the worksheet by using the keystrokes. [2ND] ICONV. The worksheet has three variables:

NOM for nominal annual rate
EFF for annual effective rate
C/Y for number of conversion periods per year

You can scroll among these variables using the ↑ and ↓ keys.

In Example **(1.16)** we found the effective rate corresponding to a nominal annual rate of 9% convertible monthly. To do this on the BA II Plus calculator, enter the ICONV worksheet and scroll to the line for NOM. Key in 9 and hit the [Enter] key. Then scroll ↑ to the line for C/Y and key in 12 and hit the [Enter] key. Then scroll ↑ to the line for EFF and use the [CPT] key to compute the effective rate. The rate displayed is EFF = 9.38 (to two decimal places).This means 9.38%, so the rate is 0.0938.

In **(1.17)** we found the nominal rate corresponding to an effective rate of 10.25% convertible semi-annually. To do this on the BA II Plus calculator, enter the ICONV worksheet and scroll to the line for EFF. Key in 10.25 and hit the [Enter] key. Then scroll ↓ to the line for C/Y and key in 2 and hit the [Enter] key. Then scroll ↓ to the line for NOM and use the [CPT] key to compute the effective rate. The rate displayed is NOM = 10 (that is, 10%, or 0.10).

The ICONV worksheet can be used to calculate EFF or NOM, but not C/Y (the number of conversion periods per year.)

To exit the ICONV worksheet, press the [C/E] key. This key will also allow you to exit any other BAII Plus worksheets. (Sometimes you will have to press [C/E] more than once.)

Exercise (1.18)

a) Given $i^{(12)} = 6\%$, find the annual effective rate i.
b) Given an annual effective rate of $i = 5\%$, find $i^{(12)}$.

Answers: a) 6.168% b) 4.889%

Section 1.6

Rate of Discount

Investments can be structured in many ways. Consider an investor who would like to earn 6% for one year. Two common approaches are:

a) Invest a given sum at the beginning of the year. If you invest $1,000 at the beginning of the year at 6% per year, you will receive a payment of $1,060 at the end of the year.

b) Target a given sum at the end of the year, and "discount" that amount to determine how much to invest. Suppose that you want to have 1,000 at the end of the year. The present value of 1,000 at 6% interest is $1{,}000/1.06 = 943.40$.

   You would invest 943.40 and be repaid 1,000. The difference of 56.60 is referred to as the amount of **discount**, and $56.60/1{,}000 = 0.0566$ is the **rate of discount**. This rate equals the amount of interest that will be earned during the year (56.60), divided by the *end-of-year* value (1,000). (As we already know, dividing the interest earned by the *beginning-of-year* value produces the annual effective interest rate: $56.60/943.40 = 0.06$).

The rate of discount is used extensively in interest theory and actuarial mathematics. We can easily derive an expression for the rate of discount in terms of *i*. If you wish to invest at an annual effective rate *i* and obtain a future value of 1, the present value to invest is:

$$PV = \frac{1}{(1+i)}$$

The annual effective rate of discount, *d*, is defined as:

(1.19)
$$d = 1 - \frac{1}{(1+i)}$$

From this definition of the rate of discount, we can use algebra to develop the key relationship:

(1.20)
$$d = \frac{i}{(1+i)}$$

The annual effective rate of discount, d, can also be defined as the amount of interest earned during a one-year period, divided by the value of the investment at the *end* of that year.

$$d = \frac{\text{interest earned between } t \text{ and } t+1}{\text{value of investment at time } t+1} = \frac{A(t+1)-A(t)}{A(t+1)} = \frac{a(t+1)-a(t)}{a(t+1)}$$

(1.21)

When $t = 0$ and $A(0) = 1$, this equation is identical to Formula **(1.20)**, because the amount of interest earned is i, and the *end-of-year* balance is $(1+i)$, so d equals i divided by $(1+i)$. The interest rate, of course, is the amount of interest earned (i), divided by the *beginning-of-year* balance (which is 1).

Example (1.22)

For $i = 0.06$, $d = \dfrac{0.06}{1.06} = 0.0566$

Exercise (1.23)

Given $i = 0.10$, find d.

Answer: 0.0909

You should be aware that the word "discount" is used in various ways besides those discussed in this section. It is important to make sure you understand how the word is being used each time it appears. For example, the verb form, "to discount," typically means "to calculate a present value." The phrase "discount at a rate of 5%" likely means "calculate a present value using a 5% rate of *interest*" (not a 5% rate of discount). Even the phrase "discount rate" frequently refers to a rate of *interest* at which discounting (present valuing) is to be done. Always read exam questions carefully to make sure you know whether they are referring to interest rates or rates of discount.

Section 1.7

Present Value Factor

Another variable that is used in actuarial interest problems is the present value factor, v:

(1.24)

$$v = \frac{1}{1+i}$$

The variable v equals the value at the beginning of the year of 1 unit payable at the end of the year.

From the definition of d in **(1.19)**, we see that:

(1.25)

$$d = 1 - v \text{ and } v = 1 - d$$

And from formula **(1.20)**, we can derive the following important relation:

(1.26)

$$d = iv$$

The difference $i - d$ simplifies nicely:

$$i - d = i - \frac{i}{(1+i)} = \frac{i^2}{(1+i)} = id$$

(1.27)

$$i - d = id$$

The preceding relationships are often useful in solving exam problems.

Example (1.28)

Given $d = 0.07$, find v and i.

Solution.

$v = 1 - d = 0.93$. Then $1 + i = \frac{1}{v} = 1.0753$. It follows that $i = 0.0753$.

Exercise (1.29)

Given $d = 0.05$, find v and i.

Answers: $v = 0.95$ $i = 0.0526$

Note that we can now write:

$$PV = \frac{FV}{(1+i)^n} = v^n \cdot FV$$

The use of v as the present value factor is common in actuarial texts and is essential for actuarial exams. Many other financial professions do not use v.

Section 1.8

Nominal Rates of Discount

A rate of discount can also be quoted as a *nominal* annual rate. For example, if the quarterly effective rate of discount is 2%, we can say that the nominal annual rate of discount is 8% convertible quarterly. The annual effective rate of discount would *not* be 8%, as we shall see below.

The nominal rate of discount convertible m-thly is denoted by $d^{(m)}$. For example, a nominal rate of discount convertible quarterly would be denoted by $d^{(4)}$. It is related to the annual effective rate of discount d by the equation:

(1.30)

$$1-d=\left(1-\frac{d^{(m)}}{m}\right)^m$$

Example (1.31)

Find the annual effective rate of discount that is equivalent to a nominal annual rate of discount of 8% convertible quarterly.

Solution.

$$1-d=\left(1-\frac{0.08}{4}\right)^4=(0.98)^4=0.9224 \qquad d=1-0.9224=0.0776$$

Example (1.32)

Find the nominal annual rate of discount convertible semi-annually that corresponds to an annual effective rate of discount of 6%.

Solution.

$$\left(1-\frac{d^{(2)}}{2}\right)^2=1-d=1-0.06=0.94$$

$$\left(1-\frac{d^{(2)}}{2}\right)=\sqrt{0.94}=0.969536$$

$$d^{(2)}=2\cdot(1-0.969536)=0.060928$$

Calculator Note

In the ICONV worksheet, if you enter either EFF or NOM as a *negative* number, the BA II Plus will interpret the negative number as a rate of *discount* and solve for the corresponding (NOM or EFF) rate of discount (which will also be displayed as a negative number).

For example, in **(1.31)** we found the annual effective rate of discount corresponding to a nominal rate of discount of 8% convertible quarterly. To do this on the BA II Plus calculator, open the ICONV worksheet and scroll to the line for NOM. Key in 8 [+|-] and hit the [Enter] key. Then scroll ↑ to the line for C/Y, key in 4, and hit the [Enter] key. Then scroll ↑ to the line for EFF and use the [CPT] key to compute the effective rate.

The calculator displays EFF = -7.76 (to two decimal places). That value (7.76%) is the same rate of discount that was calculated in Example **(1.31)**.

You can clear the computed values in the ICONV *worksheet by keying in* 2ND *CLR WORK (2nd function of the CE/C key). The value of C/Y will remain unchanged until you enter a new value, but NOM and EFF will be set to 0.*

Exercise (1.33)

Find:

a) the annual effective rate of discount that is equivalent to a nominal rate of discount of 7.5% convertible every 4 months (m=3), and

b) the nominal annual rate of discount convertible monthly that is equivalent to an annual effective rate of discount of 6%

Answers: a) 7.31% b) 6.17%

Some problems may require conversion of a nominal interest rate convertible m times per year to an equivalent nominal rate of discount convertible p times per year. The equation for this problem is:

$$\left(1+\frac{i^{(m)}}{m}\right)^m=\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

In this equation the left-hand side represents $1+i$ and the right-hand side represents $\frac{1}{v}=1+i$. Notice that the exponent in the right-hand expression is $-p$, not p.

Example (1.34)

Find the rate of discount convertible semi-annually that is equivalent to a nominal rate of interest of 8% convertible monthly.

Solution.

$$\left(1+\frac{0.08}{12}\right)^{12} = 1.0830 = \left(1-\frac{d^{(2)}}{2}\right)^{-2}$$

$$\sqrt{1.083} = 1.0407 = \left(1-\frac{d^{(2)}}{2}\right)^{-1}$$

$$d^{(2)} = 2\cdot\left(1-1.0407^{-1}\right) = 0.0782$$

As was previously mentioned for nominal rates of interest, you should never perform calculations using nominal rates of discount. Always convert a nominal rate to an effective rate and use the effective rate for your calculations.

Section 1.9

Continuous Compounding and the Force of Interest

We have already noted that unless an exam problem states otherwise, interest should be regarded as being *earned continuously* throughout the year, so that the accumulation function, $a(t)$, is a continuous exponential function of t. In this section, we will consider the case where the interest rate is said to be <u>compounded</u> *continuously*. That is, interest is *converted to principal* as soon as it is earned, and immediately begins earning interest. As we will see, this is equivalent to earning a nominal annual rate of interest $i^{(m)}$ where m is infinite.

If an investment earns compound interest at an annual effective rate of 8%, its value increases annually by a factor of 1.08. However, if it earns interest at a nominal annual rate of 8% convertible semi-annually, then it is earning a semi-annual effective rate of 4%, and its value increases annually by a factor of $1.0816\left(= 1.04^2\right)$. This is equivalent to an annual effective interest rate of 8.16%.

The following table shows the equivalent annual effective rates earned by money invested at a nominal annual rate of 8%, but with a range of conversion periods from semi-annual to monthly to hourly. If m is the number of conversion periods per year, then the equivalent annual effective rate is:

$$\left(1+\frac{0.08}{m}\right)^m - 1$$

Nominal Rate: 8%		
Conversion Period	***m***	**Ann. Eff. Rate**
Semi-annual	2	8.16000%
Quarterly	4	8.24322%
Monthly	12	8.29995%
Daily	365	8.32776%
Hourly	8,760	8.32867%

Note that a higher frequency of conversion (a larger value of m) produces a higher equivalent annual effective interest rate.

Now consider the situation mentioned in the first paragraph above, where interest is convertible to principal *continuously* as it is earned. In that case, m is infinite, and the annual effective interest rate is $\lim_{m\to\infty}\left(1+\frac{0.08}{m}\right)^m - 1$. This is equal to $e^{0.08} - 1$, which is approximately 0.0832871, so the equivalent annual effective interest rate is 8.32871%. Note that the result of compounding interest every hour (as shown in the above table) matches the result of continuous compounding to 5 significant digits.

When interest is compounded continuously, we call the interest rate a "**force of interest**." The force of interest is the rate *per year* at which the investment is earning interest, expressed *as a percentage of the current value of the investment*. In the preceding example, the force of interest was 8%, so if the investment has a value of 100 at time 0, then it is earning interest at a rate of 8.00 per year at time 0. But as soon as interest is earned, that interest is added to the principal, and the increased principal earns interest at a rate of 8% per year. (For example, after 1 day, the principal equals 100.02 and is earning interest at a rate of 8.0016 per year. On the last day of the year, the principal has grown to 108.305 and is earning interest at a rate of 8.6644 per year.)

Based on a force of interest of 0.08, at the end of the year, 8.3287 of interest will have been earned on a beginning-of-year investment of 100. By definition, the annual effective interest rate for the year is 8.3287%. You might want to think of 8.3287-per-year as the *average* rate at which the value of the investment increased during the year. At the beginning of the year, interest was accumulating at a slower rate; at the end of the year it accumulated at a faster rate. But *on average* the investment earned interest at a rate equal to 8.3287% of the beginning-of-year principal. Since 8.3287% is the annual effective interest rate for this investment, we see that the annual effective interest rate is really an *average* rate at which interest is accumulating during the year, expressed as a percentage of the *beginning-of-year balance*. By contrast, the force of interest is the *instantaneous* rate at which interest is being earned, expressed as a percentage of the *current balance* at any given moment.

The Greek letter delta (δ) is used to represent the force of interest. For a constant force of interest δ, the equivalent annual effective rate, i, can be calculated using the following formula:

(1.35)

$$i = \lim_{m\to\infty}\left(1+\frac{\delta}{m}\right)^m - 1 = e^{\delta} - 1$$

Note: This relationship between i and δ is derived using calculus on page M1-26.

Remember, continuous compounding is not a different type of compound interest. The accumulation and amount functions are still exponential. The force of interest, δ, is simply a different way to *describe* the rate at which an investment is increasing with compound interest. We are able to use the above formula to translate rates expressed as a force of interest to the equivalent annual effective rate (or vice-versa), just as we are able to translate a nominal rate into the equivalent annual effective rate.

For a constant force of interest δ and the equivalent annual effective interest rate i, we have the following relationships:

(1.36)

$$1+i = e^{\delta} \qquad \delta = \ln(1+i)$$

(1.37)

$$(1+i)^n = e^{n\delta} \qquad v^n = (1+i)^{-n} = e^{-n\delta}$$

The preceding discussion assumes a constant force of interest. That is, we have assumed that δ is a constant. However, there are also situations where the force of interest varies over time. In that case, δ is a function of t and is expressed as $\delta(t)$ or δ_t, the *instantaneous* force of interest at time t, which is defined as:

(1.38)

$$\boxed{\delta(t) = \frac{a'(t)}{a(t)}}$$

We can analyze this formula as follows: The accumulation function, $a(t)$, is the value of an investment at time t (based on an investment of 1 at time 0). Its derivative, $a'(t)$, is the rate at which that investment is earning interest, expressed as a rate per year. The ratio of these two values, $\frac{a'(t)}{a(t)}$, is the rate (per year) at which the investment is earning interest at time t, *expressed as a percentage of* $a(t)$. This is the definition of a force of interest. And because it is measured at time t and can change as a function of time, $\delta(t)$ or δ_t is called the *instantaneous* force of interest at time t.

For the accumulation function with a constant force of interest δ (that is, when $a(t) = e^{\delta t}$), this definition yields:

$$\delta(t) = \frac{de^{\delta t} / dt}{e^{\delta t}} = \frac{\delta e^{\delta t}}{e^{\delta t}} = \delta$$

Thus $\delta(t)$ is equal to δ for all values of t, so definition **(1.38)** is valid for a constant force of interest, as well as for a varying force of interest.

The following example involves a varying force of interest:

Example (1.39)

Let $a(t) = (t+1)^2$.

(*Note: This is an unrealistic accumulation function, but it is easy to analyze.*)

Then $\delta(t) = \frac{d\left[(t+1)^2\right]/dt}{(t+1)^2} = \frac{2(t+1)}{(t+1)^2} = \frac{2}{t+1}$.

There is another useful relationship which enables us to create an expression for $a(t)$ if only $\delta(t)$ is given. Note that $\frac{d}{dt}\ln[a(t)] = \frac{a'(t)}{a(t)} = \delta(t)$.

Thus: $\int_0^k \delta(t)dt = \ln\left[a(t)\right]\Big|_0^k = \ln\left[a(k)\right] - \ln\left[a(0)\right] = \ln\left[a(k)\right] - \ln(1) = \ln[a(k)]$

This implies that:

(1.40)

$$e^{\int_0^t \delta(u)du} = e^{\ln\left[a(t)\right]} = a(t)$$

Example (1.41)

Given $\delta(t) = \frac{2}{(t+1)}$, find an expression for $a(t)$.

Solution.

To find $a(t)$, we first need to integrate $\delta(t)$:

$$\int_0^t \delta(u)du = \int_0^t \frac{2}{(u+1)}du = 2\ln(u+1)\Big|_0^t = 2\ln(t+1)$$

Then we can write:

$$a(t) = e^{2\ln(t+1)} = (e^{\ln(t+1)})^2 = (t+1)^2$$

Note: If your calculus is rusty, you may need to review calculus before doing these problems.

Exercise (1.42)

Given $\delta(t) = \frac{6}{(2t+1)}$, find $a(t)$.

Answer: $a(t) = e^{3\ln(2t+1)} = (2t+1)^3$

Deriving the Relationship Between δ and *i*:

In this section, we have used the relationship $\delta = \ln(1+i)$. The following analysis derives that relationship from the definition of δ.

Consider the accumulation function, $a(t)$, for an investment of 1 made at time 0 that earns interest at a rate of δ per year *compounded continuously*, i.e., δ is the force of interest. At any time t, the value of this investment is $a(t)$ and interest is accumulating at a rate of $\delta \cdot a(t)$. That is, $a(t)$ is growing continuously at a rate of $\delta \cdot a(t)$ per year, leading to the following equation:

$$da(t) / dt = \delta \cdot a(t)$$

We can solve for $a(1)$ as follows:

$$\frac{da(t)}{a(t)} = \delta \cdot dt$$

$$\int_{t=0}^{1} \frac{da(t)}{a(t)} = \int_{t=0}^{1} \delta \cdot dt$$

$$\left[\ln a(t)\right]\Big|_0^1 = \delta \cdot t\Big|_0^1$$

$$\ln\left[a(1)\right] - \ln\left[a(0)\right] = \delta \cdot (1-0)$$

$$\ln\left[a(1)\right] - \ln\left[1\right] = \delta$$

$$\ln\left[a(1)\right] - 0 = \delta$$

$$a(1) = e^{\delta}$$

Since $a(1) = 1+i$, we have:

$$1+i = e^{\delta}$$

$$i = e^{\delta} - 1$$

$$\delta = \ln(1+i)$$

These are the key relationships between a constant force of interest δ and the equivalent annual effective interest rate i.

Section 1.10

Rates for Treasury Bills

In this section, we will consider a real-world example: the methods used to quote rates for United States Treasury bills and Government of Canada Treasury bills. The rates quoted for these securities are neither effective rates nor nominal rates. For Exam FM, you need to know how these rates are calculated and be able to convert them to annual effective rates.

United States Treasury bills (T-bills) are short-term securities issued by the United States Treasury. An investor who purchases a Treasury bill is lending money to the United States government for a term of 4 weeks, 13 weeks, 26 weeks, or 52 weeks. The investor will receive the face amount of the T-bill on its maturity date. For example, an investor might pay 960 for a 52-week T-bill with a face amount of 1,000. At the end of 52 weeks, the U.S. Treasury will pay the maturity value of 1,000, and the investor will have earned a return of:

$$\frac{1{,}000}{960} - 1 = 4.167\%$$

This is the *effective* interest rate for the *52-week (364-day) term* of the T-bill. Technically, the *annual* effective rate is a bit higher than this, since 4.167% was earned in slightly less than one year.

Quoted rates for U.S. T-bills are neither effective interest rates nor nominal rates. Instead, they are expressed as "bank discount yield," which is calculated by the following formula:

(1.43) For a U.S. Treasury bill:

$$\text{Quoted Rate} = \frac{360}{\text{Days to Maturity}} \times \frac{\text{Amount of Interest}}{\text{Maturity Value}}$$

This formula produces a rate that is not consistent with any of the standard interest functions that we have studied. The next two examples demonstrate how this formula can be used to calculate the quoted rate for a U.S. T-bill if its price is known, or to calculate its price from the quoted rate.

Example (1.44)

A newly-issued U.S. Treasury bill will mature in 182 days for its face amount of 1,000. It is priced at 985. What is the quoted rate, and what is the annual effective yield?

Solution.
Since the bond is priced at 985 and will mature for 1,000, the amount of interest is 15. We therefore have all the values needed to find the quoted rate using Formula **(1.43)**:

$$\text{Quoted Rate} = \frac{360}{182} \times \frac{15}{1{,}000} = 0.02967$$

The quoted rate is 2.967%.

In 182 days, the value of the T-bill increases by a factor of:

$$\frac{1{,}000}{985} = 1.01523$$

Its annual effective yield is the equivalent annual effective interest rate:

$$1.015228^{365/182} - 1 = 3.077\%$$

We can also compute the equivalent annual effective rate of discount:

$$d = \frac{i}{1+i} = \frac{0.03077}{1.03077} = 2.985\%$$

Rates quoted for U.S. Treasury bills are similar to, but not the same as, nominal rates of discount. The last term in Formula **(1.43)**, $\frac{\text{Amount of Interest}}{\text{Maturity Value}}$, is the definition of an effective rate of discount for the term of the T-bill (interest earned, divided by ending balance). However, the other part of the formula, $\frac{360}{\text{Days to Maturity}}$, has a numerator that assumes 30-day months and a denominator that uses the actual number of days until maturity. Because of this inconsistency, the quoted rate for a Treasury bill is *smaller than the nominal rate of discount* that is actually being earned. The next example and the comments that follow it demonstrate this relationship.

Example (1.45)

A newly-issued U.S. Treasury bill matures in 182 days for its face amount of 1,000. Its rate is quoted as 4%. What is its current price, and what is its annual effective yield?

Solution.
From Formula **(1.43)**, we can develop the following formula for the amount of interest that will be earned:

$$\text{Quoted Rate} \times \frac{\text{Days to Maturity}}{360} \times \text{Maturity Value} = \text{Amount of Interest}$$

Using the values given for this T-bill, we have:

$$0.04 \cdot \frac{182}{360} \cdot 1{,}000 = 20.22$$

The amount of interest is 20.22. The current price of this T-bill is:

$$1{,}000 - 20.22 = 979.78$$

To find the yield, we note that the price increases from 979.78 to 1,000 in 182 days, so the annual effective yield is:

$$\left(\frac{1{,}000}{979.78}\right)^{365/182} - 1 = 4.182\%$$

The 4% quoted rate in Example **(1.45)** is similar to a nominal rate of discount $\left(d^{(2)}\right)$, but it is slightly different because it is calculated using the *actual number of days to maturity* and the assumption of a *360-day year*. To analyze how this affects the calculation, we will compare the calculations for this Treasury bill to the corresponding calculations for a 6-month zero-coupon bond with a 1,000 maturity value and a yield that is described as "a nominal rate of discount of 4%, convertible semi-annually."

The 6-month effective rate of discount for the bond is: $\frac{d^{(2)}}{2} = \frac{0.04}{2} = 0.02$.

This corresponds to $0.04 \cdot \frac{182}{360} = 0.02022$ for the 4% T-bill.

The interest earned during the bond's 6-month term is: $0.02 \cdot 1{,}000 = 20$.

This corresponds to $0.02022 \cdot 1{,}000 = 20.22$ for the T-bill.

The bond's price is $1{,}000 - 20 = 980$.

This corresponds to $1{,}000 - 20.22 = 979.78$ for the T-bill.

The 6-month effective interest rate for the bond is $\frac{1{,}000}{980} - 1 = 2.0408\%$.

This corresponds to $\frac{1{,}000}{979.78} - 1 = 2.0637\%$ for the T-bill.

The annual effective interest rate earned by the bond is $1.020408^2 - 1 = 4.123\%$.
This corresponds to $1.020637^{365/182} - 1 = 4.182\%$ for the T-bill.

Conclusion: The yield for a 6-month T-bill with a quoted rate of 4% is slightly higher than for a 6-month zero-coupon bond earning a nominal rate of discount of 4% convertible semi-annually.

The Canadian government also issues Treasury bills. Rates for Government of Canada T-bills are quoted differently from U.S. T-bills. Basically, they are *interest rates* rather than rates of discount, and they assume a *365-day* year rather than 360, as the following formula indicates.

(1.46)

For a Government of Canada Treasury bill:

$$\text{Quoted Rate} = \frac{365}{\text{Days to Maturity}} \times \frac{\text{Amount of Interest}}{\text{Current Price}}$$

Example (1.47)

A newly-issued Government of Canada Treasury bill matures in 182 days for its face amount of 1,000. It is priced at 985. What is the quoted rate, and what is the annual effective yield?

Solution.
The amount of interest the bond will earn is 15. Applying Formula **(1.46)**, we have:

$$\text{Quoted Rate} = \frac{365}{182} \times \frac{15}{985} = 0.03054$$

The quoted rate is 3.054%.

In 182 days, the value of the T-bill increases by a factor of:

$$\frac{1{,}000}{985} = 1.01523$$

Its annual effective yield is the equivalent annual effective interest rate:

$$1.015228^{365/182} - 1 = 3.077\%$$

The price, term, yield, and maturity value for the Government of Canada T-bill of Example **(1.47)** are the same as for the U.S. T-bill of Example **(1.44)**. But because the quotation convention is different, the Canadian security is quoted at 3.054% and the U.S. security is quoted at 2.967%. Neither of these quotes matches the annual effective yield (3.077% for both securities), because the quoted values are not effective rates. They are similar to (but not identical to) nominal rates: a nominal rate of discount in the case of U.S. T-bills, and a nominal rate of interest in the case of Government of Canada T-bills.

Example (1.48)

A newly-issued Government of Canada Treasury bill matures in 182 days for its face amount of 1,000. Its rate is quoted as 4%. What is its current price, and what is its annual effective yield?

Solution.
From Formula **(1.46)**, we can develop the following formula for the amount of interest that will be earned:

$$\text{Quoted Rate} \times \frac{\text{Days to Maturity}}{365} \times \text{Current Price} = \text{Amount of Interest}$$

$$0.04 \times \frac{182}{365} \times \text{Current Price} = \text{Amount of Interest}$$

We don't know the Current Price for this T-bill, but we do know that:

$$\text{Current Price} + \text{Amount of Interest} = \text{Maturity Value} = 1{,}000$$

This gives us 2 equations in 2 unknowns.
Using P for Price and I for Interest, we have:

$$0.04 \times \frac{182}{365} \times P = I$$

$$P + I = 1{,}000$$

We can then solve for P and I as follows:

$$I = 1{,}000 - P$$

$$0.04 \times \frac{182}{365} \times P = 1{,}000 - P$$

$$P \cdot \left(0.04 \times \frac{182}{365} + 1\right) = 1{,}000$$

$$P = 980.44$$

$$I = 19.56$$

To find the yield, we note that the price increases from 980.44 to 1,000 in 182 days, so the annual effective yield is:

$$\left(\frac{1{,}000}{980.44}\right)^{365/182} - 1 = 4.0401\%$$

Note: The 4% quoted rate is similar to a 4% nominal rate of interest convertible semi-annually, which is equivalent to a 4.04% annual effective rate (almost identical to the 4.0401% rate calculated above). Since Government of Canada T-bill quotes are based on a 365-day year, the yield of a T-bill quoted at 4% is nearly identical to the yield based on a 4% nominal rate of interest.

The term, maturity value, and quoted rate for the Government of Canada T-bill in Example **(1.48)** are the same as for the U.S. T-bill of Example **(1.45)**. But because the quotation conventions are not the same, the Canadian security has a price of 980.44 and a yield of 4.0401%, while the U.S. security has a price of 979.78 and a yield of 4.182%.

For Exam FM, it is important to understand the special formulas used to quote rates for U.S. and Canadian T-bills. When performing calculations for T-bills, it is usually best to convert the rates to equivalent annual effective rates.

Section 1.11

Relating Discount, Force of Interest and Interest Rate

A very important relationship is:

(1.49) $d < d^{(m)} < \delta < i^{(m)} < i \qquad \text{(given that } i > 0 \text{ and } m > 1\text{)}$

This relationship is useful when checking whether a calculated value is reasonable, or determining which answer choices are plausible. It is not hard to see that $d < \delta < i$, since for $i > 0$:

$$\frac{i}{1+i} < \ln(1+i) < i$$

For a concrete example, let $i = 0.05$ and $m = 4$. Then:

$$\delta = \ln(1.05) = 0.0488 \qquad d = \frac{.05}{1.05} = 0.0476$$

$$i^{(4)} = 0.049089 \qquad d^{(4)} = 0.048494$$

It should be noted that, just as δ is the limit of $i^{(m)}$ as m becomes infinite, δ is also the limit of $d^{(m)}$ as m becomes infinite. That is, the force of discount and the force of interest are equal. For a given rate of compound growth, the effective rate of interest i is a larger number than the effective rate of discount d. The corresponding nominal rates of interest, $i^{(m)}$, decrease as m increases, and the nominal rates of discount, $d^{(m)}$, increase as m increases. At the limit, when m becomes infinite, $i^{(m)}$ and $d^{(m)}$ are both equal to δ:

(1.50) $i^{(\infty)} = d^{(\infty)} = \delta$

As an example of equivalent rates, consider the U.S. T-bill of Example **(1.44)** and the Government of Canada T-bill of Example **(1.47)**. In both cases, the T-bill is priced at 985 and matures for 1,000 at the end of 182 days. Therefore, interest on the two securities is accumulating at the same rate. However, the *quoted* rates for the U.S. and Canadian securities are not the same, and neither rate matches the annual effective rate. Those three different rates are shown in the following table, along with the continuously compounded yield (force of interest) for that same T-bill. The four measures are arranged in order of increasing size.

U.S. T-bill Quote	Force of Interest	Gov. of Canada T-bill Quote	Annual Effective Rate
2.967%	< 3.031%	< 3.054%	< 3.077%

Formula **(1.49)** provides an indication of why these different measures are related in this way. As mentioned earlier, the U.S. quotation method for T-bills is similar to a nominal rate of discount ($d^{(m)}$), so we would expect it to have a smaller value than the force of interest. By contrast, the Canadian quotation method is essentially a nominal rate of interest ($i^{(m)}$), so it is larger than the force of interest. The annual effective rate has the largest value.

Section 1.12

Solving for PV, FV, *n*, and *i* with Compound Interest

Time value of money calculations can be done easily with the BAII Plus. This section will show you how to solve for PV, FV, n, and i. In most cases, we will first use algebra to solve the problem, in order to establish the logic, and then show how to use the calculator's Time Value of Money [TVM] keys to save time.

Example (1.51)

You want to have 120,000 in a college fund in 18 years. How much should you deposit now into an account earning a 6% annual effective rate of interest?

Solution.

You need to have FV = 120,000. Thus:

$$PV = 120{,}000\,v^{18} = \frac{120{,}000}{1.06^{18}} = 42{,}041.25$$

For the BA II Plus the keystrokes are:

18 [N] 6 [I/Y] 120000 [FV] [CPT] [PV]

The equation used in the last example, $PV = 120{,}000\,v^{18}$, is called an **equation of value**. It is an equation expressing that the value a lender pays to a borrower equals the value that the borrower pays to the lender. In this case, the lender is you, and you are lending (or investing) the present value of 120,000 by depositing that amount into an account. The borrower is the bank (or other financial institution) that accepts the deposit and agrees to pay 6% interest. The value of what you deposit into the account at time 0 is unknown, so we call it PV. The amount that the bank pays you is 120,000, but since that amount is paid at time 18, the *value at time* 0 of what the bank pays you is $120{,}000\,v^{18}$. In each of the following problems we will develop the appropriate equation of value in order to solve for the unknown value.

An important concept that is always present in an equation of value is that of **valuation date**. In the above example, we chose time 0 as the valuation date, and we determined the value of the payments as of time 0. (PV is an amount paid at time 0, so its value at time 0 is PV. And 120,000 is an amount paid at time 18, so its value at time 0 is $120{,}000\,v^{18}$.) An equation of value is not valid unless the value of each payment is determined as of the same date as every other payment. It is therefore critical to choose a valuation date and make sure that each term in the equation of value represents the value as of that date. As we will see, a wise choice of valuation date can make a problem much easier to solve.

Exercise (1.52)

How much should you deposit into the fund described in **(1.51)** if you want to have 100,000 in 16 years?

Answer: 39,364.63

Example (1.53)

You deposit 1,000 into an account earning interest at an annual effective rate of 5.75%. How much will you have in 5 years?

Solution.
The equation of value is:

$$FV = 1{,}000(1.0575)^5 = 1{,}322.52$$

For the BA II Plus, the keystrokes are:

5 [N] 5.75 [I/Y] 1000 [+|-] [PV] [CPT] [FV]

Note that in this case the valuation date is time 5. Both FV and $1{,}000(1.0575)^5$ represent values determined as of time 5. We could have chosen time 0 as the valuation date instead, in which case the equation of value would be: $\dfrac{FV}{(1.0575)^5} = 1{,}000$. Both the left and right sides of this equation represent values as of time 0.

Exercise (1.54)

In Example **(1.53)**, how much will be in the account at the end of 10 years?

Answer: 1,749.06

Example (1.55)

You deposit 1,000 into an account earning a force of interest of 0.06. How long will it take to double your money?

Solution.
Doubling your money gives FV = 2,000. The equation of value is:

$$2{,}000 = 1{,}000e^{0.06t}$$

It follows that:

$$2 = e^{0.06t} \qquad \ln(2) = 0.06t$$

$$t = \frac{\ln(2)}{0.06} = 11.5525$$

Note: Since the BA II Plus calculator's TVM functions do not use force of interest, there is no TVM method for solving this problem.

In Example **(1.55)**, the valuation date was unknown. It was time t. We know that 2,000 is an amount that will be paid at time t, and $1{,}000e^{.06t}$ is the value at time t of your deposit of 1,000 made at time 0. Once we solve for t, of course, we know that the valuation date was time 11.5525.

Exercise (1.56)

For the account in Example **(1.55)** how long would it take to triple your money?

Answer: 18.3102 years

Now let's look at a variation on the preceding example that requires careful thinking:

Example (1.57)

You deposit 1,000 into an account earning an annual effective rate of 6%, but with interest payable only at the end of each year. If the account value is withdrawn before the end of a year, no interest is payable for that year. After how many years will the account balance be at least 2,000?

Solution.

If interest were being earned continuously, 2,000 would be reached after exactly 11.8957 years:

$$1{,}000 \cdot 1.06^t = 2{,}000$$

$$1.06^t = 2$$

$$t = \frac{\ln 2}{\ln 1.06} = 11.8957$$

However, because interest is not considered to be earned until the end of the year, you will not have 2,000 after 11.8957 years, but you will have *more than* 2,000 at the end of the 12th year. The answer is therefore 12 years.

Exercise (1.58)

You deposit 1,000 into an account earning an annual effective rate of 5%, but with interest payable only at the end of each year. After how many years will the account balance be at least 2,000?

Answer: 15

Example (1.59)

You make an investment where you pay 1,000 now and get 1,500 back in 5 years. What interest rate will you earn?

Solution.
The equation of value is:

$$1{,}000(1+i)^5 = 1{,}500$$

Thus:

$$(1+i)^5 = 1.5$$
$$1+i = 1.5^{1/5} = 1.08447$$
$$i = 8.447\%$$

For the BA II Plus the keystrokes are
5 [N] 1000 [+|-] [PV] 1500 [FV] [CPT] [I/Y]

Exercise (1.60)

You make an investment where you pay 1,000 now and will get 2,000 back in 12 years. What annual effective interest rate will you earn?

Answer: 5.9463%

The problems can be made more complex, as you will see when you move to the Sample Exam problems at the end of this module. One way to make a problem a bit more complex is to state it using a nominal interest rate.

Example (1.61)

You deposit 1,000 into an account earning 5.75% convertible semi-annually. How much will you have in 5 years?

Solution.
Now we have an effective interest rate of $\frac{0.0575}{2} = 0.02875$ per semi-annual period for $2 \times 5 = 10$ periods.
(Note that we cannot use the nominal rate of 5.75% for our calculations. We must convert it to an effective rate.)

The equation of value is:

$$FV = 1{,}000(1.02875)^{10} = 1{,}327.70$$

The BA II Plus keystrokes are:
10 [N] 2.875 [I/Y] 1000 [+|-] [PV] [CPT] [FV]

Calculator Note

The BA II Plus calculator has an option that allows you to choose 2 interest conversion periods per year (or 4 conversion periods, or any other number). P/Y is the 2nd function of the I/Y key; it allows you to set both the number of payments per year (P/Y) and (after using the DOWN arrow) the number of interest conversion periods per year (C/Y). **We advise against using this feature**, since it can be confusing. Also, and very importantly, it is easy to forget to reset P/Y and C/Y after finishing a calculation, and that can lead to errors on later problems. All of the calculator solutions in this manual will be based on setting your calculator for P/Y=1 and C/Y=1.

The I/Y key stands for "Interest per Year." However, you should think of it as "Interest per *Period*." For example, if we invest 1,000 for 5 years at a nominal interest rate of 6% convertible quarterly, we can analyze it as an effective interest rate of 1.5% *per period* for 20 *periods* (where a period is a quarter-year). Set N=20, I/Y=1.5, PV=-1,000, and CPT FV=1,346.86.

Exercise (1.62)

You deposit 1,000 into an account earning 6.5% convertible quarterly. How much will you have after 4.5 years?

Answer: 1,336.63

Example (1.63)

You make an investment where you pay 1,000 now and get 1,500 back in 5 years. What nominal rate of interest convertible quarterly did you earn?

Solution.

In 5 years, there are 20 quarterly periods. We will first calculate the effective interest rate per period and then convert it to a nominal annual rate convertible quarterly.

Let j be the quarterly effective interest rate $\left(j = \dfrac{i^{(4)}}{4}\right)$.

Then the equation of value (with a valuation date of time 5) is:

$$1{,}000 \cdot (1 + j)^{20} = 1{,}500$$

Solving for j, we have: $$j = \left(\frac{1{,}500}{1{,}000}\right)^{\frac{1}{20}} - 1 = 0.02048$$

Finally, the nominal rate convertible quarterly is:

$$i^{(4)} = 4 \cdot j = 8.192\%$$

To find the quarterly effective interest rate using the BA II Plus, the keystrokes are:

20 [N] 1000 [+|-] [PV] 1500 [FV] [CPT] [I/Y]

The resulting quarterly effective rate is 2.048%.
The nominal rate is $4 \times 2.048\% = 8.192\%$ convertible quarterly.

Exercise (1.64)

An investment of 1,000 accumulates to 2,000 in 12 years. What nominal rate of interest convertible semi-annually did the investment earn?

Answer: 5.860%

Other problems may have more than one future payment, or may have an unknown amount at some point. We see this in the next two examples.

Example (1.65)

How much should you deposit now in a bank account earning a 5% annual effective rate to be able to withdraw 1,000 in 2 years and 2,000 in 4 years?

Solution.
The equation of value (with a valuation date of time 0) is:

$$PV = 1{,}000v^2 + 2{,}000v^4 = \frac{1{,}000}{1.05^2} + \frac{2{,}000}{1.05^4} = 2{,}552.43$$

Exercise (1.66)

An amount of 3,000 is deposited into an account at time 0. The account earns a 6% annual effective rate. At the end of 2 years, 500 is withdrawn from the account. What is the account balance at the end of 3 years?

Answer: 3,043.05

Example (1.67)

You deposit 1,000 into an account now and an amount X in one year. The account earns an annual effective rate of 6%. What value of X will result in an account balance of 2,000 at the end of two years?

Solution.
This problem could be solved in multiple steps using the TVM functions.
However, it is much easier to write and solve the equation of value.
Using time 2 for our valuation date, we have:

$$1{,}000(1.06)^2 + X(1.06) = 2{,}000 \quad \rightarrow \quad X = 826.79$$

Exercise (1.68)

You deposit an amount X into an account at time 0 and $2X$ into the same account at time 3. The account balance at time 5 is 5,000. If the account has earned a 4% annual effective rate, what is the value of X?

Answer: 1,479.35

Another type of problem that requires more thought is one in which the interest rate changes over time.

Example (1.69)

You deposit 5,000 into an account that earns interest at an annual effective rate of 5% during the first two years, and 7% in all subsequent years. What is the account balance at the end of 5 years?

Solution.

$$FV = 5{,}000(1.05)^2(1.07)^3 = 6{,}753.05$$

You could also do the problem on the BA II Plus in two steps.

Amount in 2 years:

2 [N] 5 [I/Y] 5000 [+|-] [PV] [CPT] [FV]

(Answer: 5,512.50)

Amount in 5 years:

First hit [+|-] and [PV] . (Since 5,512.50, the accumulated value at time 2, is already in your calculator's display and is the amount you "deposit" for the last 3 years, we make it negative and enter it as PV.)

Then press: 3 [N] 7 [I/Y] [CPT] [FV]

Answer: 6,753.05

Exercise (1.70)

You deposit 2,000 into an account that earns an annual effective rate of 4% during the first 2.5 years, and 6% in all subsequent years. What is the account balance at the end of 4 years?

Answer: 2,407.53

Example (1.71)

What constant annual effective interest rate would have produced the same 5-year accumulation as in Example **(1.69)**?

Solution.
The BA II Plus can be used to solve this quickly.
We accumulated FV = 6,753.05 in 5 years from an initial investment of 5,000. Solve for the interest rate as follows:

5 [N] -5,000 [+|-] [PV] 6,753.05 [FV] [CPT] [I/Y]

Answer: 6.20%

To solve mathematically, denote the unknown interest rate by i.
Then:

$$(1+i)^5 = 1.05^2\left(1.07^3\right) = 1.3506$$
$$1+i = 1.3506^{1/5} = 1.0620$$
$$i = 6.20\%$$

Exercise (1.72)

What constant annual effective interest rate would have produced the same 4-year accumulation as in Exercise **(1.70)**?

Answer: 4.75%

Example (1.73)

You deposit 5,000 at $t=0$ into an account that earns interest at an annual effective rate of 5% for two years, and earns continuously compounded interest at a varying force of interest $\delta(t) = \dfrac{2}{(t+1)}$ in all subsequent years. What is the account balance at the end of 5 years?

Solution.
At the end of two years the account contains 5,512.50. [See Example **(1.69)**, above]. The force of interest must be applied from time 2 through time 5. The accumulated value in 5 years is:

$$5{,}512.50\left(e^{\int_2^5 \frac{2}{t+1}dt}\right)$$

Note the limits on the integral. A common mistake is to integrate from 0 to 3. That produces a 3-year accumulation factor, but it is based on the growth rates $(\delta(t))$ for the period *t* = 0 to *t* = 3, rather than the correct period (*t* = 2 to *t* = 5).

We now calculate:

$$\int_2^5 \frac{2}{t+1}dt = 2\ln(t+1)\Big|_2^5 = 2\left[\ln(6)-\ln(3)\right]$$

$$e^{\int_2^5 \frac{2}{t+1}dt} = e^{2\ln(6)-2\ln(3)} = \frac{6^2}{3^2} = 4$$

The final answer is:

$$5{,}512.50\left(e^{\int_2^5 \frac{2}{t+1}dt}\right) = 5{,}512.50(4) = 22{,}050$$

Exercise (1.74)

You deposit 4,000 at $t=0$ into an account that earns continuously compounded interest at a varying force of interest $\delta(t) = \dfrac{1}{4t+4}$.
What is the account balance at the end of 4 years?

Answer: 5,981.40

Section 1.13

The Rule of 72

The **Rule of 72** is a simple method for estimating how long it will take an investment to double in value.

(1.75)

> **Rule of 72:**
>
> If an investment earns an annual effective interest rate of $x\%$, the investment will double in value in approximately $\frac{72}{x}$ years.
>
> (Similarly, if the investment earns an effective rate of *$x\%$ per period*, it will double in about $\frac{72}{x}$ *periods*.)

As an example, an investment at an annual effective rate of 8% will double in value in approximately 9 years $(72/8=9)$. (Actually, the 9-year accumulation factor is 1.9990.)

The Rule of 72 is most accurate for interest rates close to 8%, but it produces values within 2% of the exact answer for interest rates from 2% to 14%. At interest rates lower than 8%, the resulting accumulation factor is somewhat more than 2 (e.g., $1.01^{72}=2.047$). At interest rates greater than 8%, the accumulation factor is somewhat less than 2 (e.g., $1.18^{4}=1.939$).

This rule can be very useful for making quick estimates or for checking whether your answer to a problem is reasonable. To see how this works, consider Example **(1.51)** in Section 1.12, which asks for the amount you would need to invest at 6% in order to have 120,000 after 18 years. We now know that an investment earning an annual effective rate of 6% will double in value in approximately 12 years. That means that in 18 years it will double approximately one-and-a-half times (since $18/12=1.5$). So in 18 years the amount invested will increase by a factor of $2^{1.5}=2\sqrt{2}$. If you recall that $\sqrt{2}\approx 1.414$, then you know that the investment will grow by a factor of about 2.828. In other words, it will not quite triple in value, so you will need to invest a bit more than 40,000 in order to have 120,000 in 18 years. If you had tried to solve the problem and your answer was less than 40,000, or significantly more than 40,000, you would know that you had made an error, since the answer has to be a little more than 40,000. The value we calculated (42,041.45) appears to be reasonable.

Next, consider Example **(1.55)**, where we calculated how long it would take an investment to double at a force of interest of 0.06. If it were an annual effective interest rate of 6%, the answer would be very close to 12 years. Since a force of interest of 0.06 represents a slightly larger growth rate, we can expect the answer to be a little less than 12. Our answer of 11.5525 for that example appears reasonable, whereas an answer of 11 would be too small, and anything over 12 would be too large.

Exercise **(1.60)** provides a very obvious case where the Rule of 72 can be applied. It describes an investment that doubles in 12 years (from 1,000 to 2,000) and asks what annual effective interest rate was earned. Just as we would divide 72 by the interest rate to find the number of years for the investment to double, we can divide 72 by the number of years to get the interest rate (as a percent). Since $72/12=6$, the answer is about 6% (or a little *less* than 6%, since the Rule of 72 overstates the doubling time for interest rates less than 8%). Based on that reasoning, our calculated answer of 5.9463% is reasonable.

Exercise **(1.64)** is very similar to Exercise **(1.60)** (again, an investment doubles in 12 years), but we are asked to find the *nominal rate convertible semi-annually*. There are 24 semi-annual periods in 12 years, so our estimate of the semi-annual effective rate is 3% (since $72/24=3$). This corresponds to a nominal rate of 6% convertible semi-annually, so it appears that our calculated answer of 5.86% is reasonable.

Section 1.14

Formula Sheet

a(t): accumulated value at time *t* of an initial investment of 1 made at time 0

A(t): accumulated value at time *t* of an initial investment of *A*(0) made at time 0

For simple interest at a constant rate *i*:

$$a(t) = 1 + t \cdot i \qquad A(t) = A(0) \cdot \left(1 + t \cdot i\right)$$

For compound interest at a constant rate *i*:

$$a(t) = (1+i)^t = e^{t \cdot \ln(1+i)} \qquad A(t) = A(0)\left(1+i\right)^t = A(0)e^{t \cdot \ln(1+i)}$$

$$FV = PV\left(1+i\right)^n \qquad PV = \frac{FV}{(1+i)^n}$$

$$v = \frac{1}{1+i} \qquad d = 1 - v \qquad v = 1 - d$$

$$d = \frac{i}{1+i} \qquad d = iv \qquad i = \frac{d}{1-d} \qquad i - d = id$$

$$\delta = \ln\left(1+i\right) \qquad e^{\delta} = 1 + i \qquad a(t) = e^{\delta t}$$

$$(1+i)^n = e^{n\delta} \qquad v^n = (1+i)^{-n} = e^{-n\delta}$$

For a variable force of interest: $\delta(t) = \dfrac{a'(t)}{a(t)}$ $\qquad a(t) = e^{\int_0^t \delta(u)du}$

Equivalent Rates:

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \qquad 1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m \qquad \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}$$

Note the negative exponent, -p.

$$\delta = \lim_{m\to\infty}\left(i^{(m)}\right) = \lim_{m\to\infty}\left(d^{(m)}\right) \qquad d < d^{(m)} < \delta < i^{(m)} < i$$

(given that $i>0$ and $m>1$)

Quoted Rates for Treasury bills:

U.S. T-bills: $\text{Quoted Rate} = \dfrac{360}{\text{Days to Maturity}} \times \dfrac{\text{Amount of Interest}}{\text{Maturity Value}}$

Canadian T-bills: $\text{Quoted Rate} = \dfrac{365}{\text{Days to Maturity}} \times \dfrac{\text{Amount of Interest}}{\text{Current Price}}$

Rule of 72: At an effective interest rate of *x*% per period, an investment will double in about $72/x$ periods.

Section 1.15

Basic Review Problems

1. Find the effective interest rate over the interval [3, 4], and also over the interval [3.5, 4] if interest is accumulating at a 5% annual rate of:
   (a) compound interest (b) simple interest

2. (a) Given $i^{(2)} = 5\%$, find the annual effective rate i.
   (b) Given an annual effective rate of $i = 5.26\%$, find $i^{(6)}$.

3. Given $d = 0.056$, find v and i.

4. Given $i^{(4)} = 0.07$, find $d^{(2)}$.

5. Find the annual effective rate of discount that is equivalent to a nominal rate of discount of 9% convertible monthly.

6. Find the nominal rate of interest convertible quarterly that is equivalent to a nominal rate of interest of 6% convertible semi-annually.

7. Write an expression for $\delta(t)$ (a variable force of interest) such that its accumulation function is the same as a 5% rate of simple interest. What is the value of $d(4)$?

8. Let $a(t) = (t+1)^3$. Find $\delta(t)$.

9. Let $\delta(t) = \dfrac{4}{(t+3)}$. Find $a(t)$.

10. You deposit 1,800 into an account earning a force of interest of 0.05. How long will it take for the account balance to reach 2,700?

11. You make an investment where you pay 10,500 now and receive 12,500 in 3 years. What nominal rate of interest convertible monthly did you earn?

12. You deposit 1,500 into an account that earns a nominal annual rate of 6% convertible monthly for the first year, then a nominal rate of 8% convertible quarterly for the next two years. a) What is the account balance at the end of 3 years? b) What is the equivalent level nominal rate convertible semi-annually for this account over the 3-year period?

13. A Treasury bill that matures in 100 days has a quoted rate of 3.00%. Find its annual effective yield to maturity if it is issued by:
    (a) the United States Treasury (b) the Government of Canada

Section 1.16

Basic Review Problem Solutions

Calculator solutions will be given whenever possible.

1. (a) $a(t) = 1.05^t$ $\quad i[3,4] = \dfrac{a(4) - a(3)}{a(3)} = \dfrac{1.05^4 - 1.05^3}{1.05^3} = 0.05$

   Note that for compound interest at a constant rate, the annual effective rate is the same for all periods.

   $$i[3.5,4] = \frac{a(4) - a(3.5)}{a(3.5)} = \frac{1.05^4 - 1.05^{3.5}}{1.05^{3.5}} = 0.02470$$

   Note that this 6-month effective rate is equal to $\dfrac{i^{(2)}}{2}$ $\left(= 1.05^{0.5} - 1\right)$.

   (b) $a(t) = 1 + 0.05t$. $\quad i[3,4] = \dfrac{a(4) - a(3)}{a(3)} = \dfrac{1.20 - 1.15}{1.15} = 0.04348$

   $$i[3.5,4] = \frac{a(4) - a(3.5)}{a(3.5)} = \frac{1.20 - 1.175}{1.175} = 0.02128$$

2. By formula:

   (a) $\quad 1 + i = \left(1 + \dfrac{0.05}{2}\right)^2 \rightarrow i = 1.025^2 - 1 = 0.050625$

   (b) $\quad 1.0526 = \left(1 + \dfrac{i^{(6)}}{6}\right)^6 \rightarrow i^{(6)} = 6 \cdot \left(1.0526^{1/6} - 1\right) = 0.051483$

   By calculator (using the ICONV worksheet):
   (a) Set NOM = 5 and C/Y = 2. CPT EFF = 5.0625
   (b) Set EFF = 5.26 and C/Y = 6. CPT NOM = 5.1483

3. $v = 1 - d = 0.944$

   $\dfrac{1}{v} = 1 + i = 1.0593 \rightarrow i = 0.0593$

4. $\left(1+\frac{0.07}{4}\right)^4 = 1.07186 = \left(1-\frac{d^{(2)}}{2}\right)^{-2} \rightarrow 0.9659 = \left(1-\frac{d^{(2)}}{2}\right) \rightarrow d^{(2)} = 0.06820$

   By calculator:
   In the ICONV worksheet, set NOM = 7 and C/Y = 4.
   CPT EFF = 7.186 $i = 7.186\%$

   With 7.186 in the display, and *without exiting the ICONV worksheet*:
   Press the % key (changing 7.186 to 0.07186)
   To calculate d (which is $\frac{i}{1+i}$), press: ÷ (+ 1 =
   Note: Each of the last 5 symbols in the preceding sentence is a keystroke, including "(".
   The calculator display now shows 0.06704, which is the value of d.
   Now press × 100 = +|- ENTER (which sets EFF equal to -6.704).
   Finally, set C/Y = 2 and CPT NOM = -6.820.
   $d^{(2)} = 6.820\%$

5. By formula:

$$1-d = \left[1-\frac{0.09}{12}\right]^{12}$$

$$d = 1-\left[1-\frac{0.09}{12}\right]^{12} = 0.086379$$

$$d = 8.6379\%$$

   By calculator (using the ICONV worksheet):
   Set $\text{NOM} = -9$ and C/Y = 12. CPT EFF $= -8.6379$. Answer: d = 8.6379%.

6. By formula:

$$\left(1+\frac{i^{(4)}}{4}\right)^4 = \left(1+\frac{0.06}{2}\right)^2 \qquad i^{(4)} = 4\times\left(\left[(1.03)^2\right]^{\frac{1}{4}} - 1\right) = 0.059557$$

   You can use the calculator's ICONV worksheet to solve this problem in two steps:

   First find the annual effective rate using the given nominal rate:
   Set NOM = 6, C/Y = 2 and CPT EFF = 6.09.
   Then use this effective rate to find the quarterly nominal rate:
   You already have EFF = 6.09.
   Set C/Y = 4 and CPT NOM = 5.9557. Answer 5.9557%.

7. The accumulation function is: $a(t) = 1 + 0.05t$.

$$\delta(t) = \frac{a'(t)}{a(t)} = \frac{0.05}{1+0.05t}$$

$$\delta(4) = \frac{0.05}{1+0.05\cdot(4)} = 0.04167$$

8. $\delta(t) = \frac{a'(t)}{a(t)} = \frac{3(t+1)^2}{(t+1)^3} = \frac{3}{t+1}$

9. $\int_0^t \delta(u)du = \int_0^t \frac{4}{(u+3)} du = 4\ln(u+3)\Big|_0^t = 4\ln\left(\frac{t+3}{3}\right)$

$$a(t) = e^{\int_0^t \delta(u)du} = e^{4\ln((t+3)/3)} = \left(\frac{t+3}{3}\right)^4$$

10. $2{,}700 = 1{,}800e^{0.05t} \rightarrow e^{0.05t} = 1.5 \rightarrow 0.05t = \ln(1.5)$

$$t = \frac{\ln(1.5)}{0.05} = 8.1093$$

11. The investment grows from 10,500 to 12,500 in 3 years (36 months). We will first solve for the monthly effective rate (recall that we do all calculations using *effective* rates), then find the equivalent nominal annual rate convertible monthly.
    Let j be the monthly effective rate. Then:

    $$10{,}500\cdot(1+j)^{36} = 12{,}500 \rightarrow j = 0.0048549$$

    $$i^{(12)} = 12\cdot j = 5.826\%$$

    Using the calculator's TVM worksheet, we could solve first for the monthly effective rate and multiply by 12 (as was done above). But here is a different approach, first calculating the annual effective rate, then converting it to a nominal rate convertible monthly:

    3 [N]   10500 [+|-] [PV] 12500 [FV]   [CPT] [I/Y]

    Answer: 5.984
    The annual effective rate for the 3-year period is 5.984%.

    Now use ICONV to find the nominal rate convertible monthly.
    Set EFF = 5.984 , C/Y = 12, and CPT NOM = 5.826.
    Answer: 5.826%

12. (a) $1500\left(1+\frac{0.06}{12}\right)^{12}\left(1+\frac{0.08}{4}\right)^{4\cdot 2}=1,865.89$

(b) We need to find $i^{(2)}$.

$$\frac{1,865.89}{1,500}=1.2439=\left(1+\frac{i^{(2)}}{2}\right)^{6}$$

$$1.2439^{1/6}=1.03705=\left(1+\frac{i^{(2)}}{2}\right)$$

$$i^{(2)}=0.07410$$

13. (a) Assume a face amount of 1,000.

$$3.00\%=\frac{360}{100}\times\frac{\text{Amt. of Int.}}{1,000}$$

$$\text{Amt. of Int.}=0.03\cdot\frac{100}{360}\cdot 1,000=8.33$$

$$\text{Price}=1,000-8.33333=991.67$$

$$\text{Annual effective yield}=\left(\frac{1,000}{991.67}\right)^{\frac{365}{100}}-1=3.10\%$$

(b) Again assume a 1,000 face amount.

$$3.00\%=\frac{365}{100}\times\frac{\text{Amt. of Int.}}{1,000-\text{Amt. of Int.}}$$

$$\text{Amt. of Int.}=0.03\cdot\frac{100}{365}\cdot(1,000-\text{Amt. of Int.})=8.2192-0.0082192\times\text{Amt. of Int.}$$

$$1.0082192\times\text{Amt. of Int.}=8.2192$$

$$\text{Amt. of Int.}=\frac{8.2192}{1.0082192}=8.1522 \qquad \text{Price}=1,000-8.1522=991.85$$

$$\text{Annual effective yield}=\left(\frac{1,000}{991.85}\right)^{\frac{365}{100}}-1=3.03\%$$

Section 1.17

Sample Exam Problems

1. (2005 Exam FM Sample Questions #1)

Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest of 4% convertible semi-annually.

At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at a force of interest of δ.

After 7.25 years, the value of each account is the same. Calculate δ.

(A) 0.0388 (B) 0.0392 (C) 0.0396 (D) 0.0404 (E) 0.0414

2. (2005 Exam FM Sample Questions #3)

Eric deposits 100 into a savings account at time 0, which pays interest at a nominal rate of i, compounded semi-annually.

Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of i.

Eric and Mike earn the same amount of interest during the last 6 months of the 8th year.
Calculate i.

(A) 9.06% (B) 9.26% (C) 9.46% (D) 9.66% (E) 9.86%

3. (2005 Exam FM Sample Questions #12)

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semi-annually thereafter. The accumulated balance in the fund at the end of 30 years is 100.

Calculate d.

(A) 4.33% (B) 4.43% (C) 4.53% (D) 4.63% (E) 4.73%

4. (2005 Exam FM Sample Questions #13)

Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest

$$\delta_t = \frac{t^2}{100} \qquad t > 0$$

The amount of interest earned from time 3 to time 6 is also X. Calculate X.

(A) 385 (B) 485 (C) 585 (D) 685 (E) 785

5. (2005 Exam FM Sample Questions #20)

David can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n, and 300 at time $2n$
(ii) 600 at time 10

At an annual effective interest rate of i, the present values of the two streams are equal.
Given $v^n = 0.76$, determine i.

(A) 3.5% (B) 4.0% (C) 4.5% (D) 5.0% (E) 5.5%

6. (2005 Exam FM Sample Questions #27)

Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns the same annual effective interest rate.

The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is also equal to X. Calculate X.

(A) 28.0 (B) 31.3 (C) 34.6 (D) 36.7 (E) 38.9

7. (May 05, #13)

At a nominal interest rate of i convertible semi-annually, an investment of 1000 immediately and 1500 at the end of the first year will accumulate to 2600 at the end of the second year. Calculate i.

(A) 2.75% (B) 2.77% (C) 2.79% (D) 2.81% (E) 2.83%

8. (May 05, #18)

A store is running a promotion during which customers have two options for payment.

Option one is to pay 90% of the purchase price two months after the date of sale.

Option two is to deduct *X*% off the purchase price and pay cash on the date of sale.

A customer wishes to determine *X* such that he is indifferent between the two options when valuing them using an effective annual interest rate of 8%.

Which of the following equations of value would the customer need to solve?

A) $\left(\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=.90$

B) $\left(1-\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right)=.90$

C) $\left(\frac{X}{100}\right)(1.08)^{1/6}=.90$

D) $\left(\frac{X}{100}\right)\left(\frac{1.08}{1.06}\right)=.90$

E) $\left(1-\frac{X}{100}\right)(1.08)^{1/6}=.90$

9. (May 05, #19)

Calculate the nominal rate of discount convertible monthly that is equivalent to a nominal rate of interest of 18.9% per year convertible monthly.

(A) 18.0% (B) 18.3% (C) 18.6% (D) 18.9% (E) 19.2%

10. (Nov 05, #7)

A bank offers the following choices for certificates of deposit:

Term (in years)	Nominal annual interest rate convertible quarterly
1	4.00%
3	5.00%
5	5.65%

The certificates mature at the end of the term. The bank does NOT permit early withdrawals. During the next 6 years the bank will continue to offer certificates of deposit with the same terms and interest rates.

An investor initially deposits 10,000 in the bank and withdraws both principal and interest at the end of 6 years. Calculate the maximum annual effective rate of interest the investor can earn over the 6-year period.

(A) 5.09% (B) 5.22% (C) 5.35% (D) 5.48% (E) 5.61%

11. (Nov 05, #25)

The parents of three children, ages 1, 3, and 6, wish to set up a trust fund that will pay X to each child upon attainment of age 18, and Y to each child upon attainment of age 21.

They will establish the trust fund with a single investment of Z.

Which of the following is the correct equation of value for Z ?

(A) $\dfrac{X}{v^{17}+v^{15}+v^{12}}+\dfrac{Y}{v^{20}+v^{18}+v^{15}}$

(B) $3\left[Xv^{18}+Yv^{21}\right]$

(C) $3Xv^{3}+Y\left[v^{20}+v^{18}+v^{15}\right]$

(D) $\left(X+Y\right)\dfrac{v^{20}+v^{18}+v^{15}}{v^{3}}$

(E) $X\left[v^{17}+v^{15}+v^{12}\right]+Y\left[v^{20}+v^{18}+v^{15}\right]$

Section 1.18

Sample Exam Problem Solutions

1.

Since both accounts begin with the same balance (100) at the same time (time 0) and each grows with compound interest at a level rate, and each has the same value at a later date (7.25 years), it follows that they are earning interest at the same rate.

Bruce's account earns interest at a nominal rate of $i^{(2)} = 4\%$, which is equivalent to an annual effective rate of $\left(1.02^2 - 1\right) = 4.04\%$.

Peter's account grows at a force of interest δ, which must be equivalent to an annual effective rate of 4.04%.

$$e^{\delta} = 1.0404 \quad \rightarrow \quad \delta = \ln 1.0404 = 0.0363$$

Answer C

Note: One might incorrectly assume that Bruce's account balance at 7.25 years is the same as it was at 7 years. Its 4% nominal annual interest rate is "convertible semi-annually." This means that no interest has been "converted to principal" between time 7 and time 7.25, which might imply that Bruce's account has not earned any interest after time 7. If we make that assumption, we would write the following equation:

$$1.02^{14} = e^{7.25\delta}$$

This results in a value of 0.0382 for δ, which is not one of the answer choices.

As was mentioned in this module, you should assume that interest is earned continuously, unless the problem specifically states otherwise. In the current problem, even though the interest earned after time 7 has not been "paid" or "converted to principal," the question refers to the value of each account, and it should be clear that Bruce's account has a larger value at time 7.25 than at time 7, even though that additional value has not yet been paid out or converted to principal. If we assume that no interest is earned after time 7, we calculate an answer that does not match any of the multiple choice answers, which is a further indication that this assumption is not valid.

2.
The last 6 months of the 8th year are the period from time 7.5 to time 8. For each of the two savers we will find the interest earned in that half-year interval.

Eric: At time 7.5 he has a balance of $100\left(1+\frac{i}{2}\right)^{15}$. His interest on this balance over the next half year is $100\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right)$.

Mike: Since Mike earns simple interest on only his original principal of 200, his interest earned in *any* half year is $200\left(\frac{i}{2}\right)$.

Setting these two interest amounts equal to each other, we have:

$$100\left(1+\frac{i}{2}\right)^{15}\left(\frac{i}{2}\right)=200\left(\frac{i}{2}\right)$$

$$\left(1+\frac{i}{2}\right)^{15}=2 \rightarrow \quad \frac{i}{2}=0.0473 \quad \rightarrow \quad i=0.0946$$

Another way of analyzing this is:
During the half-year period from 7.5 to 8, each account earns interest equal to $\frac{i}{2}$ times the principal (not necessarily the balance) at the beginning of the period. If both accounts earn the same amount of interest during the period, then they both had the same amount of principal at the beginning of the period.

For Eric (earning compound interest), the principal at time 7.5 is $100\left(1+\frac{i}{2}\right)^{15}$.

For Mike (earning simple interest), the principal at time 7.5 is 200 (the same amount he deposited at $t = 0$).

The resulting equation is:

$$100\left(1+\frac{i}{2}\right)^{15}=200 \quad \rightarrow \quad i=0.0946$$

Answer C

3.

The problem asks us to find the value of d, which is described as a nominal discount rate compounded quarterly. We ordinarily use the symbol $d^{(4)}$ for this function. To avoid confusion, we will use $d^{(4)}$ in the following formulas, and will solve for the value of $d^{(4)}$.

First we will deal with the initial 10 years, during which a discount rate of $d^{(4)}$ applies. For each of these 10 years the relevant values of v and i are:

$$v=\left(1-\frac{d^{(4)}}{4}\right)^{4} \text{ and } 1+i=\frac{1}{v}=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

Thus after 10 years the initial deposit of 10 has grown to:

$$10(1+i)^{10}=10\left(1-\frac{d^{(4)}}{4}\right)^{-40}$$

The accumulated balance from the initial deposit after 20 more years (40 semi-annual periods) at 3% per semi-annual period is:

$$10\left(1-\frac{d^{(4)}}{4}\right)^{-40}(1.03)^{40}=32.62\left(1-\frac{d^{(4)}}{4}\right)^{-40}$$

The second deposit of 20 accumulates (after 30 semi-annual periods at an effective rate of 3% per period) to a value of:

$$20(1.03)^{30}=48.55$$

The total accumulated balance is:

$$100=32.62\left(1-\frac{d^{(4)}}{4}\right)^{-40}+48.55$$

Solving for $d^{(4)}$, we have:

$$\left(1-\frac{d^{(4)}}{4}\right)^{-40}=1.57738$$

$$\left(1-\frac{d^{(4)}}{4}\right)^{40}=0.63396$$

$$\left(1-\frac{d^{(4)}}{4}\right)=0.98867$$

$$d^{(4)}=0.0453$$

Answer C

4.

The amount of interest earned from time 3 to time 6 is the difference between the ending amount at time 6 and the starting amount in the account at time 3.

We will begin by looking at the original deposit of 100. At time 3 it has grown to:

$$100e^{\int_0^3 \delta_t dt} = 100e^{\int_0^3 \frac{t^2}{100} dt} = 100e^{\left.\frac{t^3}{300}\right|_0^3} = 100e^{\frac{27}{300}} = 109.42$$

A deposit of X is made at time 3, so the beginning amount at time 3 is:

$$A(3) = 109.42 + X$$

At time 6 the account has grown to:

$$A(6) = (109.42 + X)e^{\int_3^6 \frac{t^2}{100} dt} = (109.42 + X)e^{\left.\frac{t^3}{300}\right|_3^6}$$
$$= (109.42 + X)1.8776 = 205.45 + 1.8776X$$

The interest earned between time 3 and time 6 is:

$$A(6) - A(3) = (205.45 + 1.8776X) - (109.42 + X) = 96.03 + 0.8776X\,.$$

This amount of interest must equal X:

$$X = 96.03 + .8776X \;\rightarrow\; X = 784.56$$

Answer E

5.
<u>Present value of stream (i):</u>

$$100+200v^n+300v^{2n}=100+200(.76)+300\left(.76^2\right)=425.28$$

<u>Present value of stream (ii):</u>

$$600v^{10}$$

Since the present values are equal

$$600v^{10}=425.28 \rightarrow v=0.9662 \rightarrow 1+i=1.035$$

Answer A

6.
The interest earned during a year equals:

(Balance at beginning of year) × (Interest Rate)

Let i denote the unknown interest rate.

For Bruce, the interest earned during year 11 is $X=i\cdot 100(1+i)^{10}$

For Robbie, the interest earned during year 17 is $X=i\cdot 50(1+i)^{16}$

It follows that:

$$i\cdot 50(1+i)^{16}=i\cdot 100(1+i)^{10} \quad \rightarrow \quad 50(1+i)^{16}=100(1+i)^{10}$$
$$(1+i)^6=2 \quad \rightarrow \quad (1+i)=1.12246$$
$$X=i\cdot 100(1+i)^{10}=0.12246(100)(1.12246)^{10}=38.88$$

Answer E

7.
The equation of value is

$$1{,}000\left(1+\frac{i^{(2)}}{2}\right)^4+1{,}500\left(1+\frac{i^{(2)}}{2}\right)^2=2{,}600$$

This is a problem that can be reduced to a quadratic (a common Exam FM technique).

Let $x=\left(1+\frac{i^{(2)}}{2}\right)^2$. Then the equation becomes:

$$1{,}000x^2+1{,}500x=2{,}600 \text{ or } x^2+1.5x-2.6=0$$

The positive root of this equation (by the quadratic formula) is:

$$x=1.02834=\left(1+\frac{i^{(2)}}{2}\right)^2$$

Thus: $\left(1+\frac{i^{(2)}}{2}\right)=(1.02834)^{\frac{1}{2}}\rightarrow i^{(2)}=0.0281$

Answer D

8.
Let P be the purchase price. The customer has two options:

1) Pay cash on the date of sale, with X% taken off the price.
   An amount of $P\left(1-\frac{X}{100}\right)$ is paid immediately.

2) Pay 90% of the purchase price in two months.
   The amount paid in two months (1/6 of a year) is $0.90P$.
   The present value of this amount on the date of sale is: $\frac{0.90P}{1.08^{1/6}}$.

The equation of value is $P\left(1-\frac{X}{100}\right)=\frac{0.90P}{1.08^{1/6}}$.

Dividing by P and multiplying by $1.08^{1/6}$, we have:

$$\left(1-\frac{X}{100}\right)\cdot(1.08)^{1/6}=0.90$$

Answer E

Note: $X=100\cdot\left(1-\frac{0.90}{1.08^{1/6}}\right)=88.85$

9.

$$\left(1+\frac{0.189}{12}\right)^{12}=1.2063=\left(1-\frac{d^{(12)}}{12}\right)^{-12}$$

$$\left(1-\frac{d^{(12)}}{12}\right)=1.2063^{-\frac{1}{12}}=0.9845$$

$$d^{(12)}=12\cdot(1-0.9845)=0.186$$

Answer C

10.

Since the one-year rate of 4% is the lowest and the 3-year rate is second lowest, we can immediately eliminate the possibility of investing in six consecutive one-year CDs or three consecutive one-year CDs coupled with a three-year CD. The two possible choices for maximum yield are A) two consecutive 3-year CDs or B) a 5-year CD coupled with a one-year CD. The CDs earn the following quarterly effective rates:

Term	Nominal Annual Rate	Quarterly Eff. Rate
1	4.00%	1.0000%
3	5.00%	1.2500%
5	5.65%	1.4125%

The total accumulation factors under the two options are:

Option A) 2 consecutive 3-year CDs for n=12 quarters each:

$$(1.0125)^{24}=1.34735$$

Option B) A 5-year CD coupled with a one-year CD:

$$(1.014125)^{20}(1.01)^{4}=1.377575$$

Option B) is better. It gives an accumulation factor of 1.377575 over six years. Therefore, the maximum annual effective rate an investor can earn over the six-year period is:

$$1.377575^{1/6}-1=0.0548$$

Answer D

11.

Below we tabulate the years remaining to ages 18 and 21 for each child.

Age now	1	3	6
Years to age 18	17	15	12
Years to age 21	20	18	15

The present value of the payments of X at age 18 is: $X\left(v^{17}+v^{15}+v^{12}\right)$

The present value of the payments of Y at age 21 is: $Y\left(v^{20}+v^{18}+v^{15}\right)$

The present value of the total fund required is:

$$Z = X\left(v^{17}+v^{15}+v^{12}\right) + Y\left(v^{20}+v^{18}+v^{15}\right)$$

Answer E

Section 1.19

Supplemental Exercises

1. Given $d^{(4)} = 0.05$, find $i^{(6)}$, v, and δ.

2. A deposit at time 0 earns simple interest at a rate of 6.5% per year. Find the effective interest rates for the intervals [4, 5] and [4.5, 4.75].

3. If $a(t) = (2t + 1)^4$, find $\delta(t)$.

4. A deposit is made into a fund. For the first 5 years interest is credited at a nominal annual rate of 6% convertible quarterly. For the next 5 years interest is credited at a nominal annual rate of discount of 7% convertible semi-annually. What is the equivalent constant force of interest for the 10-year period?

5. A man deposits 500 into an account. At the end of 5 years the account has grown to 650. Find the nominal annual rate of interest convertible quarterly for this account.

6. Tom and Jerry deposit money into accounts at the same time. Tom's account earns interest at an annual effective rate of r. Jerry's account earns simple interest at an annual rate of r. Tom's effective rate of interest for year 8 is 1.5 times Jerry's effective rate for year 8. Find r.

7. At time 0 an amount X is deposited into an account that earns 8% simple interest. Also at time 0, an amount $\frac{X}{2}$ is deposited into an account that accumulates at a constant force of interest δ. The total amount of interest earned in each account as of time 10 years is the same. Find δ.

8. An account pays an annual effective interest rate of i. A man deposits 1,000 at time 0, and 1,500 one year later. At the end of two years his account balance is 2,800. Find i.

9. A woman makes two deposits into an account: 100 at time 0, and 300 twelve years later. For the first 12 years interest is credited at a nominal annual rate of 6% convertible quarterly. For the next 8 years the account accumulates at a force of interest δ. At the end of 20 years the account balance is 802. Find δ.

10. Elmer deposits 1,000 into a bank account. The bank credits interest at a nominal annual rate of i convertible quarterly for the first 8 years, and a nominal annual rate of $1.5 \cdot i$ convertible bimonthly (every 2 months) thereafter. The amount in Elmer's account at the end of 5 years is 1,516. What is the amount in his account at the end of 10 years?

Section 1.20

Supplemental Exercise Solutions

1. We have the following equivalences:
   $(1 - d^{(4)}/4)^{-4} = (1 + i^{(6)}/6)^{6} = 1 + i = v^{-1} = e^{\delta}$
   Since $d^{(4)}$ is given as 0.05, the common value of these expressions is:
   $(1 - 0.05/4)^{-4} = 1.05160 - 1 + i$.
   From this, we can solve for the other variables:
   $i^{(6)} = 6(1.0516^{1/6} - 1) = 0.0505$.
   $v = 1/1.0516 = 0.9509$
   $\delta = \ln(1.0516) = 0.0503$

2. $i[4, 5] = [a(5) - a(4)]/a(4) \quad a(t) = 1 + 0.065t \quad a(4) = 1.260 \quad a(5) = 1.325$.
   $i[4, 5] = (1.325 - 1.26)/1.26 = 0.0516$

   $i[4.5, 4.75] = [a(4.75) - a(4.5)]/a(4.5) \quad a(4.5) = 1.2925 \quad a(4.75) = 1.30875$.
   $i[4.5, 4.75] = (1.30875 - 1.2925)/1.2925 = 0.01257$

3. $\delta(t) = a'(t)/a(t) = 4(2t + 1)^3(2)/(2t + 1)^4 = 8/(2t + 1)$

4. The accumulation factor $a(10)$ for the 10-year period is:
   $$\left(1+\frac{0.06}{4}\right)^{(4\times 5)}\left(1-\frac{0.07}{2}\right)^{-(2\times 5)} = e^{10\cdot\delta}$$
   $$(1.015)^{20}(1- 0.035)^{-10} = 1.9233 = e^{10\delta}$$

   $$\delta = \frac{\ln(1.9233)}{10} = 0.0654$$

5. $$500\left(1+\frac{i^{(4)}}{4}\right)^{20} = 650$$
   $$i^{(4)} = 4\left[\left(\frac{650}{500}\right)^{1/20} - 1\right] = 0.0528$$

6. Tom's effective rate of interest for year 8 (and every other year) is r.
   Jerry's effective rate for year 8 is $[(1 + 8r) - (1 + 7r)]/(1 + 7r) = r/(1 + 7r)$.

   Therefore, $r = 1.5\dfrac{r}{(1+7r)} \rightarrow 1+7r = 1.5 \rightarrow r = \dfrac{0.5}{7} = 0.0714$

7. The total interest earned on the first account is $(0.08)(10)X = 0.8X$.The total interest earned on the second account is $(X/2)(e^{10\delta} - 1)$

$$0.8X = \frac{X}{2}(e^{10\delta} - 1) \;\rightarrow\; 1.6 = e^{10\delta} - 1$$
$$10\delta = \ln(2.6) \;\rightarrow\; \delta = 0.0956$$

8. The account balance at time 2 is $1{,}000(1+i)^2 + 1{,}500(1+i) = 2{,}800$.
   Let $x = 1 + i$.

   This yields the quadratic equation $10x^2 + 15x - 28 = 0$.
   The positive root of the equation is $x = 1.084 = 1 + i$. $\Rightarrow i = 0.084$

9. The account balance is $\left[100(1.015)^{48} + 300\right]e^{8\delta} = 802$.

   Thus $504.35e^{8\delta} = 802 \rightarrow 8 \cdot \delta = \ln(1.59) \rightarrow \delta = 0.058$

10. At the end of 5 years the account balance is $1{,}000\left(1+\frac{i}{4}\right)^{20} = 1{,}516$.

    At the end of 10 years the accumulation is:
    $$1{,}000\left(1+\frac{i}{4}\right)^{32}\left(1+\frac{1.5i}{6}\right)^{12} = 1{,}000\left(1+\frac{i}{4}\right)^{44} \quad \text{(Note that } i/4 = 1.5i/6.\text{)}$$
    $$\left(1+\frac{i}{4}\right)^{44} = \left[\left(1+\frac{i}{4}\right)^{20}\right]^{\frac{44}{20}} = \left(\frac{1{,}516}{1{,}000}\right)^{44/20} = 2.49769$$
    The account value at 10 years is $1{,}000 \times 2.49769 = 2{,}497.69$
    By this approach, we found the answer without solving for i.

    Alternatively, we can start by solving for i, and then compute the balance as of time 10:
    $$1{,}000\left(1+\frac{i}{4}\right)^{20} = 1{,}516 \;\rightarrow\; i = 4\left[\left(\frac{1{,}516}{1{,}000}\right)^{1/20} - 1\right] = 0.08409$$
    $$1{,}000\left(1+\frac{i}{4}\right)^{32}\left(1+\frac{1.5i}{6}\right)^{12} = 1{,}000\left(1+\frac{0.08409}{4}\right)^{32}\left(1+\frac{1.5\ (0.08409)}{6}\right)^{12} = 2{,}497.69$$

Module 2

Annuities

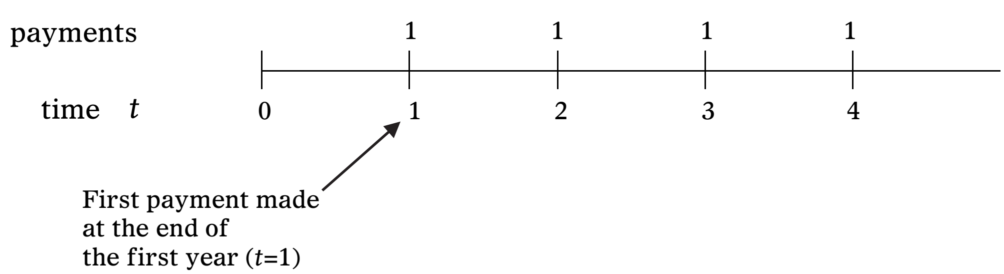
Section 2.1

Introduction

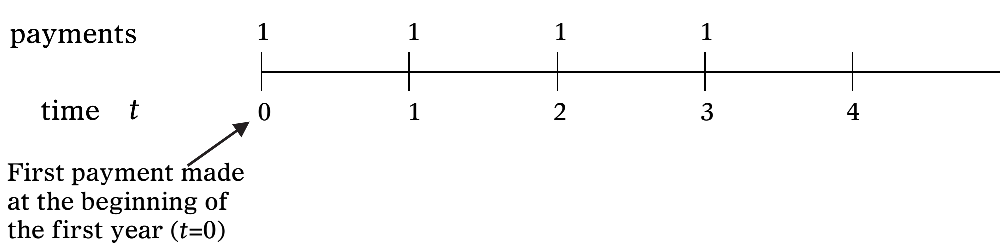
Many financial obligations require a regular series of periodic payments. Mortgage and car loan payments are usually made monthly. A retiree's pension plan typically pays a set amount at the beginning of every month. Premiums for an insurance policy might be paid monthly, quarterly, semi-annually, or annually. Series of regular payments such as these are called **annuities**. If the payments continue for a fixed period (e.g., 10 years), the annuity is called an **annuity-certain**. If the payment period is *not* fixed (e.g., a pension plan that makes monthly payments only as long as the retiree survives), it is a *contingent* annuity. (In the case of the pension, it is a *life*-contingent annuity, or simply a life annuity.) Exam FM deals with annuities-certain. Contingent annuities are covered in later exams.

A **unit annuity** is one for which each regular payment is 1. Annuity payments can be made at the beginning or the end of each time period. If an annuity's payments occur at the end of each period, it is called an **annuity-immediate**. If the payments are made at the beginning of each period, it is an **annuity-due**. The diagrams below illustrate the payment patterns for unit annuities with four annual payments.

Annuity-Immediate



Annuity-Due



The preceding diagrams are called **timelines**. You will find them to be very useful in visualizing payment patterns and solving annuity problems.

Geometric Series

To find the present value or future value of an annuity, we will need to use the formula for the sum of a geometric series. Geometric series are very important for Exam FM. Consider the geometric series with n terms where the first term is 1 and the common ratio is r:

(2.1)

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}=\frac{r^n-1}{r-1},\ r\neq 1$$

*Note that if $|r|<1$, n can be infinite, because then the **infinite geometric series** converges:*

(2.2)

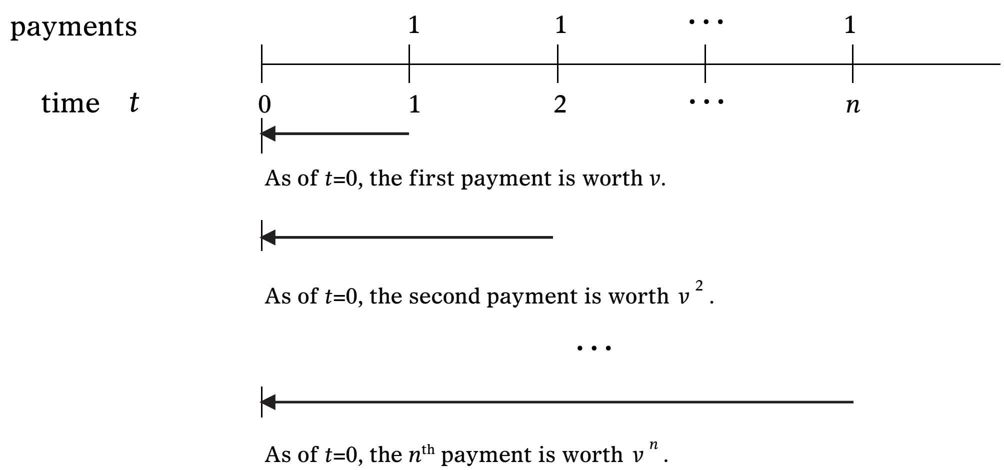
$$1+r+r^2+\ldots=\frac{1}{1-r} \qquad \text{for } |r|<1$$

Section 2.2

Annuity-Immediate Calculations

The present value of an annuity-immediate with n payments of 1 and interest rate i is denoted by $a_{\overline{n}|i}$, or we can simply write $a_{\overline{n}|}$ if the value of the interest rate is clear and does not need to be specified. When referring to $a_{\overline{n}|}$, we say "***a*-angle-*n***." The basic formula for $a_{\overline{n}|i}$ is so important that we will derive it here:

The present value of the n-year unit annuity-immediate is the sum of the individual present values of the n payments of 1.



$$
\begin{aligned}
\text{Present value } &= a_{\overline{n}|} \\
&= v + v^2 + \ldots + v^n \\
&= v\left(1 + v + \ldots + v^{n-1}\right) \\
&= v\,\frac{\left(1 - v^n\right)}{1 - v} = v\,\frac{\left(1 - v^n\right)}{d} \\
&= v\,\frac{\left(1 - v^n\right)}{iv} \\
&= \frac{1 - v^n}{i}
\end{aligned}
$$

Thus we obtain the important formula:

(2.3)
$$
\boxed{a_{\overline{n}|i} = \frac{1 - v^n}{i}}
$$

Example (2.4)

If $i = 0.05$ and $n = 10$:

$$a_{\overline{10}|5\%} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05} = 7.7217$$

Calculator Note

Not surprisingly, your calculator's TVM worksheet can be used to solve the above problem. The PMT key is used for the periodic payment of 1. On the BAII Plus calculator, the following entries give the result PV = – 7.7217:

10	N
5	I/Y
1	PMT
0	FV
CPT	PV

Note the sign convention. Positive amounts represent money paid to you, and negative amounts represent cash that you must pay out. If the applicable interest rate is 5%, you would need to pay 7.7217 now (i.e., you would have a cash flow of -7.7217) to receive ten subsequent payments of +1.

On exams most students use the calculator functions instead of formulas whenever possible, because it saves time. You must still know the formulas, since formula knowledge is required to solve the problems, and some questions are designed so that the calculator cannot be used directly.

Exercise (2.5)

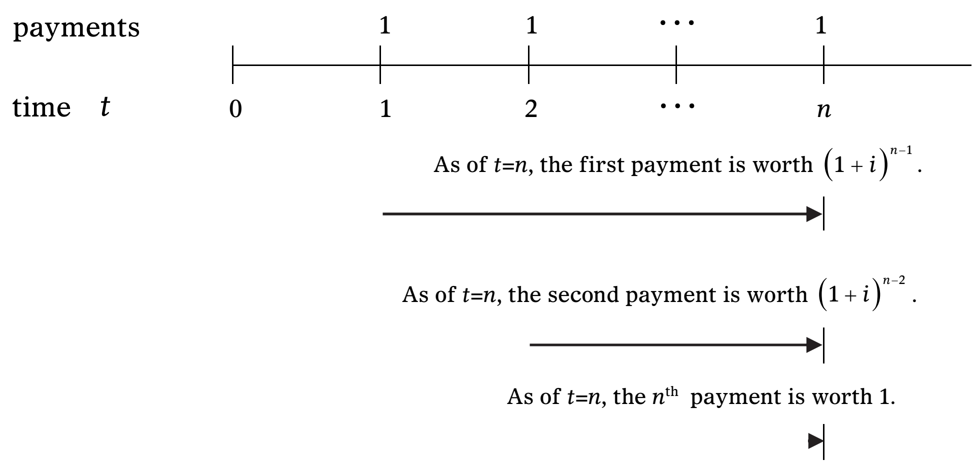
Find the value of $a_{\overline{20}|0.05}$ using the annuity formula, and then check it using the calculator.

Answer: 12.4622

The *future* value of the unit annuity-immediate with n payments is denoted by $s_{\overline{n}|}$, which is pronounced "**s-angle-*n*.**" It is the sum of the future values (at time n) of the n individual payments of 1.

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + ... + (1+i) + 1$$

Note that since the annuity-immediate has year-end payments, the first payment (made at time 1) earns interest for only n-1 periods (from time 1 to time n), and the last payment of 1 (made at time n) earns no interest. It is important to understand that these are the same n payments that had a present value of $a_{\overline{n}|}$ at time 0. What is different is the *valuation date*. The present value ($a_{\overline{n}|}$) of these payments has a valuation date of time 0. The future value ($s_{\overline{n}|}$) of these same payments has a valuation date of time n.



We could use geometric series summation to develop a formula for $s_{\overline{n}|}$, but we can also find it quickly from the formula for $a_{\overline{n}|}$. Since $a_{\overline{n}|}$ is the value of this series of n payments as of time 0, and $s_{\overline{n}|}$ is the value of the same payments n periods later (at time n), we multiply $a_{\overline{n}|}$ by $(1+i)^n$ to find $s_{\overline{n}|}$:

(2.6)
$$s_{\overline{n}|i} = (1+i)^n \cdot a_{\overline{n}|} = (1+i)^n \cdot \frac{1-v^n}{i} = \frac{(1+i)^n - 1}{i}$$

You can use this approach to avoid excessive memorization.
If you know the formula for $a_{\overline{n}|}$, you can easily find $s_{\overline{n}|}$.

Example (2.7)

If i = 5% and n = 10,

$$s_{\overline{10}|5\%} = (1.05)^{10}(7.7217) = \frac{1.05^{10} - 1}{0.05} = 12.5779$$

This can also be done on the financial calculator:
Set N=10, I/Y = 5, PMT =1, and CPT FV.
Naturally, PMT and FV have opposite signs.

Exercise (2.8)

If $n = 15$ and $i = 6\%$, find $a_{\overline{15|}}$ and $s_{\overline{15|}}$.

Answers: $a_{\overline{15|}} = 9.712$, $s_{\overline{15|}} = 23.276$

To get another very useful relationship, divide both sides of **(2.6)** by $(1+i)^n$:

(2.9)

$$a_{\overline{n|}} = v^n \cdot s_{\overline{n|}}$$

The relationships between $a_{\overline{n|}}$ and $s_{\overline{n|}}$ in **(2.6)** and **(2.9)** are intuitive. They represent the value of n payments at time 0 and time n, respectively. So their values differ by a factor of $(1+i)^n$, or by $(1+i)^{-n} = v^n$.

Section 2.3

Perpetuities

A perpetuity is an annuity with payments that continue forever. The present value of a perpetuity-immediate that pays 1 per period is denoted by $a_{\overline{\infty}|}$.

(2.10)

$$a_{\overline{\infty}|} = v + v^2 + v^3 + \ldots$$

If we write $a_{\overline{\infty}|}$ as a limit, we obtain the following formula:

(2.11)

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

Example (2.12)

If $i = 5\%$,

$$a_{\overline{\infty}|} = \frac{1}{0.05} = 20$$

Exercise (2.13)

Find the present value of a unit perpetuity-immediate with $i = 8\%$.

Answer: 12.50

Note that we cannot write a formula for the *future* value of a perpetuity (which would be $s_{\overline{\infty}|}$), since this would be the value as of the date of the *last* payment, and there is no "last payment" under a perpetuity.

Section 2.4

Annuities with Level Payments Other Than 1

Note that the present value or future value of any annuity-immediate with level payments can be found using $a_{\overline{n}|}$ and $s_{\overline{n}|}$. If an annuity-immediate has payments of amount P, its present value and future value are given by:

$$PV = P \cdot a_{\overline{n}|}$$
$$FV = P \cdot s_{\overline{n}|}$$

Example (2.14)

Find the present value of an annuity-immediate with $n = 10$ and payments of $P = 100$ if $i = 5\%$:

$$100 \cdot a_{\overline{10}|} = 100 \cdot \frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05} = 100(7.7217) = 772.17$$

Before the electronic computing age, mathematicians compiled tables of values of $a_{\overline{n}|}$ for a range of values for n and i. Present values of annuities were calculated by multiplying tabular values of $a_{\overline{n}|}$ by the relevant P, as above. Today the problem in Example **(2.14)** is more likely to be solved using a financial calculator and computing PV with [N] = 10, [I/Y] = 5, [PMT] = 100, and [FV] = 0.

Exercise (2.15)

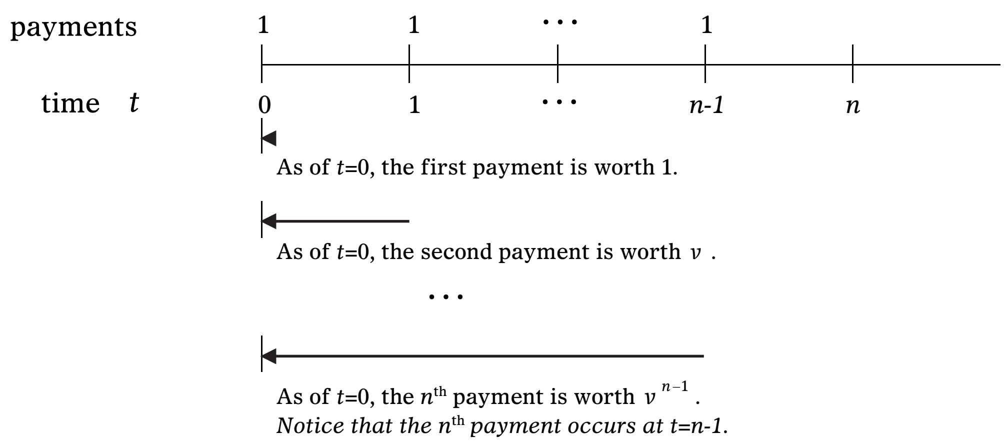
Find the present value of an annuity with $n = 30$ payments of $P = 500$ if $i = 8\%$.

Answer: 5,628.89

Section 2.5

Annuity-Due Calculations

The present value of an n-period unit **annuity-due** is denoted by $\ddot{a}_{\overline{n}|}$, which is pronounced "***a*-double-dot-angle-*n*.**"



Since payments are made at the beginning of the period, we have:

$$\ddot{a}_{\overline{n}|} = 1 + v + \ldots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

Thus:

(2.16)

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

Another way to develop this formula is to recognize that each of the n payments in this annuity-due occurs *one period earlier* than the corresponding payment under an n-year annuity-immediate. As a result, $\ddot{a}_{\overline{n}|}$ has a value that is larger than $a_{\overline{n}|}$ by a factor of $(1+i)$:

(2.17)

$$\ddot{a}_{\overline{n}|} = (1+i)\cdot a_{\overline{n}|} = (1+i)\cdot\frac{1-v^n}{i} = \frac{1-v^n}{i/(1+i)} = \frac{1-v^n}{d}$$

The formula for $\ddot{a}_{\overline{n}|}$ is easy to remember, since it is obtained by taking the equation for $a_{\overline{n}|}$ and replacing the i in the denominator by d. As a memory aid, you might use the fact that the words "immediate" and "due" begin with the letters "i" and "d," and that i and d are their respective denominators. This pattern of denominators also applies to the formulas for the *future* value of an annuity-due and the present value of a perpetuity-due:

(2.18)

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

(2.19)

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$$

Example(2.20)

Given i = 5% and n = 10, find $\ddot{a}_{\overline{n}|}$ directly and check it using **(2.17)**.

Solution.

$$\ddot{a}_{\overline{10}|} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{\left(\frac{0.05}{1.05}\right)} = 8.1078$$

Check:
From Example (2.4), $a_{\overline{10}|} = 7.7217$

$$\ddot{a}_{\overline{10}|} = (1+i)\cdot a_{\overline{10}|} = 1.05(7.7217) = 8.1078$$

Calculator Note

Annuity-due calculations are done with the calculator set to the BGN (begin) mode for payments made at the *beginning* of the payment period. The letters BGN appear above the PMT key. If you key in 2ND BGN you will see either BGN or END in the calculator's display. You can then change to the other mode by pressing 2ND and SET (the 2nd function of the ENTER key). *Remember that you can leave the* BGN/END *menu by pressing the CE/C key.*

You can tell whether your calculator is set to BGN or END mode at any time by looking at the upper right of the screen. If BGN appears in small letters at the upper right, the calculator is in BGN (annuity-due) mode. If BGN does *not* appear on the screen, it is in END (annuity-immediate) mode.

It is most important on actuarial exams to be aware of your calculator's BGN/END mode. The majority of problems require END mode. If you do a BGN mode problem and do not set the mode back to END, you will have trouble on subsequent problems. Many students avoid this difficulty by keeping their calculators set to END at all times. When they need to find the value of an annuity-due, they calculate the value of the corresponding annuity-immediate, and then multiply by (1+*i*).

Exercise (2.21)

If n = 15 and i = 6%, find $\ddot{a}_{\overline{15}|}$ and $\ddot{s}_{\overline{15}|}$.

Answers: $\ddot{a}_{\overline{15}|} = 10.295$, $\ddot{s}_{\overline{15}|} = 24.673$

Section 2.6

Continuously Payable Annuities

A continuous unit annuity pays a total of 1 per year, but spreads the payment evenly throughout the year by making a continuous stream of payments at a constant rate of 1 per year. We can think of this continuously payable annuity as making a payment of $(1 \cdot dt)$ in each infinitesimal time interval of length dt. The present value (at time 0) of the payment of $(1 \cdot dt)$ made at time t is $(v^t \cdot dt)$.

An n-year continuous unit annuity, which makes continuous payments at a rate of 1 per year from time 0 to time n, is denoted by $\bar{a}_{\overline{n}|}$ (“***a*-bar-angle-*n***”). Its present value is found by integrating the present value of its payments $(v^t \cdot dt)$ from time 0 to time n:

$$\bar{a}_{\overline{n}|} = \int_{t=0}^{n} v^t \cdot dt = \left. \frac{v^t}{\ln(v)} \right|_{t=0}^{n}$$

$$= \frac{v^n - v^0}{\ln\left(\frac{1}{1+i}\right)} = \frac{v^n - 1}{-\delta} = \frac{1 - v^n}{\delta}$$

In this derivation, we used the relation $\ln(v) = \ln\left(\frac{1}{1+i}\right) = -\ln(1+i) = -\delta$.

The final result is:

(2.22)

$$\boxed{\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{i}{\delta} \cdot a_{\overline{n}|}}$$

Note that this formula follows the pattern we observed in the formula for $\ddot{a}_{\overline{n}|}$. We can find $\bar{a}_{\overline{n}|}$ by changing the denominator of $a_{\overline{n}|}$ from i to δ (just as we changed it to d for $\ddot{a}_{\overline{n}|}$). This is equivalent to multiplying $a_{\overline{n}|}$ by $\frac{i}{\delta}$.

Similarly, for the *future* value of a continuously payable annuity, we have:

(2.23)

$$\boxed{\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n}|}}$$

For a continuously payable perpetuity:

(2.24)

$$\boxed{\bar{a}_{\overline{\infty}|} = \frac{1}{\delta} = \frac{i}{\delta} a_{\overline{\infty}|}}$$

Example (2.25)

If $i = 5\%$, find $\bar{a}_{\overline{10}|}$ and $\bar{s}_{\overline{10}|}$.

Solution.

$$\bar{a}_{\overline{10}|} = \frac{1 - v^{10}}{\delta} = \frac{1 - \left(\frac{1}{1.05}\right)^{10}}{\ln(1.05)} = 7.9132$$

This can be checked by using the $\frac{i}{\delta}$ relationship and the fact that $a_{\overline{10}|} = 7.7217$:

$$\bar{a}_{\overline{10}|} = \frac{i}{\delta} \cdot a_{\overline{10}|} = \frac{0.05}{\ln(1.05)}(7.7217) = 7.9132$$

For $\bar{s}_{\overline{10}|}$, we have:

$$\bar{s}_{\overline{10}|} = \frac{1.05^{10} - 1}{\ln(1.05)} = 12.8898$$

We can check this value by accumulating $\bar{a}_{\overline{10}|}$ for 10 years (moving the valuation date from $t = 0$ to $t = 10$):

$$\bar{s}_{\overline{10}|} = 1.05^{10} \cdot \bar{a}_{\overline{10}|} = 1.6289 \cdot (7.9132) = 12.8898$$

Calculator Note

The BA II Plus has 10 memories (in addition the memories associated with the TVM functions and the various worksheets). These 10 memories are numbered 0 to 9. To store the currently displayed value in (for example) Memory 2, press STO 2. Then to recall that value for use in a calculation, press RCL 2.

When solving a problem, you will frequently calculate an intermediate result that will be needed later in solving that problem. It is best to use your calculator's memories to store such values, in order to maintain full calculator precision. It may also be useful to write down a 3- or 4-digit approximation of the intermediate result as a record of your work. But re-entering that approximate value into the calculator in place of the original number wastes time and loses accuracy.

Exercise (2.26)

If $i = 6\%$, find $\bar{a}_{\overline{15|}}$ and $\bar{s}_{\overline{15|}}$.

Answers: $\bar{a}_{\overline{15|}} = 10.0008$ $\bar{s}_{\overline{15|}} = 23.9675$

Example (2.27)

A 20-year continuous stream of payments consists of payments at a rate of 3,000 per year for the first 10 years, then at a rate of 2,000 per year from $t = 10$ to $t = 20$. At an interest rate of 6% convertible monthly, what is the present value of this payment stream?

Solution.
The payment stream can be broken into two parts: continuous payments at a rate of 2,000 per year for the full 20-year period, and continuous payments of 1,000 per year for just the first 10 years. The total present value is:

$$2{,}000 \cdot \bar{a}_{\overline{20|}} + 1{,}000 \cdot \bar{a}_{\overline{10|}} = 2{,}000 \cdot \frac{1-v^{20}}{\delta} + 1{,}000 \cdot \frac{1-v^{10}}{\delta}$$

In order to evaluate this expression, we need values for v and δ:

$$v = \frac{1}{1+i} = \frac{1}{\left(1+\frac{i^{(12)}}{12}\right)^{12}} = \frac{1}{\left(1+\frac{0.06}{12}\right)^{12}} = \frac{1}{1.06168} = 0.9419$$

$$\delta = \ln(1+i) = \ln(1.06168) = 0.05985$$

Using these values, the present value of this payment stream is:

$$2{,}000 \cdot \frac{1-v^{20}}{\delta} + 1{,}000 \cdot \frac{1-v^{10}}{\delta} = 2{,}000 \cdot \frac{1-0.9419^{20}}{0.05985} + 1{,}000 \cdot \frac{1-0.9419^{10}}{0.05985} = 30{,}846.44$$

Exercise (2.28)

An account pays interest at a continuously compounded rate of 0.05 per year. Continuous deposits are made to the account at a rate of 1,000 per year for 6 years, and then at a rate of 2,000 per year for the next 4 years. What is the account balance at the end of 10 years?

Answer: 17,402.48

Section 2.7

Basic Annuity Problems for Calculator Practice

Note that we can solve for each of the variables PMT, PV, FV, I/Y, and N using the BA II Plus. In this section we give an example of solving for each of these.

Example (2.29)

Solving for PMT
A loan for 20,000 must be repaid by 5 year-end payments with interest at an annual effective rate of 12%. What is the annual payment?

Solution.
The 5 payments must have a present value equal to the amount of the loan:

$$P = \frac{20{,}000}{a_{\overline{5}|12\%}}$$

Set N=5, I/Y = 12, PV = 20,000, and CPT PMT = -5,548.19.
The annual payment is 5,548.19.

Exercise (2.30)

A loan for 20,000 must be repaid by 5 year-end payments with interest at an annual effective rate of 10%. What is the annual payment?

Answer: 5,275.95

Example (2.31)

Solving for PMT
You have a 5,000 balance in an account earning a 4.5% annual effective rate. You want to increase your balance to 20,000 at the end of 12 years by making a level deposit at the beginning of each of the next 12 years. Find the required level payment.

Solution.

$$5{,}000 \cdot 1.045^{12} + D \cdot \ddot{s}_{\overline{12}|4.5\%} = 20{,}000$$

$$D = \frac{20{,}000 - 5{,}000 \cdot 1.045^{12}}{\ddot{s}_{\overline{12}|4.5\%}}$$

Put the calculator in BGN mode, set N = 12, I/Y = 4.5, PV = −5,000, FV = 20,000, and CPT PMT = −1,237.63
The level payment is 1,237.63

The problem of Example **(2.31)** could also have been solved with the calculator in END mode. In that case, you would enter the same values:

N = 12, I/Y = 4.5, PV = −5,000, FV = 20,000, and CPT PMT = −1,293.32

1,293.32 is the amount you would need to deposit at the *end* of each year. Since this problem involves deposits made one year earlier (at the beginning of each year), the deposits should be smaller by a factor of $\frac{1}{1+i}$:

$$\frac{1,293.32}{1.045} = 1,237.63$$

Exercise (2.32)

What is the required level deposit in **(2.31)** if the current balance is 4,000 and the annual effective interest rate is 6%?

Answer: 708.43

If you have changed your calculator setting to BGN mode, be sure to reset it to END mode.

Example (2.33)

Solving for PV

You wish to make a deposit now in an account earning a 5% annual effective rate so that that you can get a payment of 1,000 at the end of each of the next 15 years. How much should you deposit today?

Solution.

$D = 1,000 \cdot a_{\overline{15}|5\%}$

Set N=15, I/Y = 5, PMT=1,000, and CPT PV = −10,379.66.

You should deposit 10,379.66.

Exercise (2.34)

What would be the required deposit in Example **(2.33)** if you wanted 20 years of payments instead of 15?

Answer: 12,462.21

Example (2.35)

Solving for FV

An account earning a 5% annual effective rate has a current balance of 6,000. If a deposit of 1,500 is made at the end of each year for 20 years, what will be the balance in the account at the end of 20 years?

Solution.

$Bal_{20} = 6,000 \cdot 1.05^{20} + 1,500 \cdot s_{\overline{20}|5\%}$

Set N = 20, I/Y = 5, PV = -6,000, PMT = -1,500, and CPT FV = 65,518.72.

The balance at the end of 20 years will be 65,518.72.

Exercise (2.36)

What would be the ending balance in Example **(2.33)** if the interest rate were 6% instead of 5%?

Answer: 74,421.20

Example (2.37)

Solving for I/Y

You have borrowed 15,000 and agreed to repay the loan with 5 level payments of 4,000, with the first payment occurring one year from today. What annual effective interest rate are you paying?

Solution.

$15,000 = 4,000 \cdot a_{\overline{5}|i}$

Set N=5, PV=15,000, PMT = −4,000 and CPT I/Y = 10.42

You are paying 10.42% interest per year.

Note that the PV is positive since it represents cash you received, and the PMT is negative because it is cash that you must pay. If you forget the minus sign, the BA II Plus will give an error message when you hit CPT I/Y.

Exercise (2.38)

What would the interest rate be in **(2.37)** if the annual loan payment were 4,300?

Answer: 13.34%

Note: The equations of Example (2.37) and Exercise (2.38) involve 5th degree polynomials in v or (1+i). There is no formula to solve for i directly, so the calculator iterates until it finds a value for i that satisfies the equation within a small margin of error.

Example (2.39)

Solving for N

You want to accumulate at least 20,000 in an account earning a 5% annual effective rate. You will make a level deposit of 1,000 at the beginning of each year for n years. What is the value of n? What is the account balance after n years?

Solution.

The equation of value is $20{,}000 = 1{,}000 \cdot \ddot{s}_{\overline{n}|5\%}$.

We can find n using algebra, as follows:

$$20{,}000 = 1{,}000 \cdot \ddot{s}_{\overline{n}|5\%} = 1{,}000 \cdot \frac{1.05^n - 1}{0.05/1.05}$$

$$20 \cdot (0.05/1.05) + 1 = 1.05^n$$

$$n = \frac{\ln\left[20 \cdot (0.05/1.05) + 1\right]}{\ln 1.05} = 13.71$$

The *computed* value for n is 13.71. This answer means that 13 payments are not enough, and a 14th payment is required to reach 20,000. Thus n = 14.

$$1{,}000 \cdot \ddot{s}_{\overline{14}|5\%} = 1{,}000 \cdot \frac{1.05^{14} - 1}{0.05/1.05} = 20{,}578.56$$

The account balance after 14 years is 20,578.56.

Alternatively, put the calculator into BGN mode and set I/Y = 5, PMT = -1,000, FV=20,000, and CPT N = 13.71.
Now enter 14 for N and CPT FV = 20,578.56,.
(Or, to solve the problem in END mode, make the same entries, except enter -1,050 for PMT $(1{,}050 = 1{,}000 \times 1.05)$ to reflect that a payment of 1,050 at year-end is equivalent to a payment of 1,000 at the beginning of the year.)

A note about the answer to Example **(2.39)**:
The calculated value of 13.71 for n indicates (because it is greater than 13) that the account value at time 13 will be less than 20,000. However, it also indicates (because it is less than 14) that the balance at time 14 will be *greater* than 20,000. We can calculate the amount of the <u>partial</u> *payment* needed at the beginning of the 14th year to produce a balance of *exactly* 20,000 at the end of 14 years. We have already calculated that the balance at time 14 will be 20,578.56. This suggests that we could have deposited $\frac{578.56}{1.05} = 551.01$ *less* at the beginning of the 14th year (at time 13). Thus a final payment of 448.99 $(= 1{,}000 - 551.01)$ would have produced a balance of exactly 20,000 at time 14.

It is important to understand that the annuity formula $a_{\overline{n|}} = \dfrac{1-v^n}{i}$ is not valid if n is not an integer, because it is based on the geometric series formula, which requires n to be an integer. The calculated value of 13.71 in Example **(2.39)** does *not* mean that the account balance is 20,000 at time 13.71 years. Rather, it simply tells us that the balance will be less than 20,000 at time 13 (after 13 payments), and it will be more than 20,000 at time 14 (after 14 payments).

Exercise (2.40)

How many payments (n) would be needed in Example **(2.39)** if the interest rate were 6%? What payment amount at the beginning of the n^{th} year would produce a balance of exactly 20,000 at time n?

Answers: 13 985.79

Section 2.8

Annuities with Varying Payments

Not all series of payments are level. In practice, it is quite common to encounter series with varying payment amounts, such as the following:

Series of Payments:	*Payments Made:*	*Type of Annuity Sequence:*
500, 0, 200, 300	At end of period	--
1, 2, 3, 4	At end of period	Arithmetic increasing
4, 3, 2, 1	At end of period	Arithmetic decreasing
1, 1.05, 1.1025=$(1.05)^2$	Beginning of period	Geometric

In interest theory, there are formulas for the last three sequences presented here, and they will be covered in the following sections. But if there are only four or five terms to input, you can calculate the annuity's value more quickly by using your calculator's Cash Flow worksheet.

For example, if $i = 0.05$, you can use the BA II Plus to find the present value of the increasing annuity {1, 2, 3, 4} using the CF (Cash Flow) and NPV (Net Present Value) keys:

Hit the [CF] key to activate the Cash Flow worksheet. Then hit [2ND] [CE|C] to clear any cash flow values that were previously entered. You will see a prompt for the value of CF_0, the cash flow at time 0. In this case, there is no payment until time 1, so leave the CF_0 value at 0 and use the down-arrow key to scroll down. You will see the prompt "C01," requesting the cash flow at time 1. Press 1 and hit "ENTER." Scroll down again, and there will be a new prompt: "F01." This is a request for the number of times (frequency) that this value is repeated. The default value is 1, and if you scroll down again, the value of 1 will be assumed with no entry. After you scroll down, you will be prompted for the value of C02. Enter 2. Repeat this process until all four cash flow values have been entered. Then calculate the NPV at 5% with the keystrokes:

[NPV]		
I =	5	[ENTER]
[↓]		
NPV =	[CPT]	

The display will show the answer: "NPV = 8.6488"

Another example: If i = 0.05, we can use the BA II Plus to find the present value of the first series {500, 0, 200, 300} by entering the cash flows provided and using the NPV function with I=5.
The present value (NPV) is 895.77.

Note that CF_0=0, because CF_0 is the initial payment at t=0. The first payment (of 500) is at the end of the first period (t=1). Thus, we have CF_0=0, C01=500, C02=0, and C03=1,000.

To calculate the present value of the last series in the above table, enter 1 for CF_0 (because the first payment is at $t=0$), 1.05 for C01, and 1.1025 for C02. If the interest rate is 6%, you should calculate an NPV of 2.9718.

Section 2.9

Increasing Annuities with Terms in Arithmetic Progression

An annuity whose n payments are 1, 2, 3, . . ., n is called a **unit increasing annuity**. If payments are made at the *end* of each period, it is an increasing annuity-immediate. The present value of this annuity is denoted by $(Ia)_{\overline{n|}}$ and, of course, it is equal to the present value of its payments:

$$(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \ldots + nv^n$$

To develop a practical formula for this function, first note that the payment pattern is as follows:

payments		1	2	3	$\cdots$	n
time t	0	1	2	3	$\cdots$	n

These same payments can be arranged as shown in the following table. This arrangement allows us to see how we can write a formula for the present value of the payments in each line. We can then sum those present values to create a formula for the value of $(Ia)_{\overline{n|}}$:

time: $t=$	1	2	3	$\cdots$	n	Present Value
Payments:	1	1	1	$\cdots$	1	$\dfrac{1-v^n}{i}$
		1	1	$\cdots$	1	$v \cdot \dfrac{1-v^{n-1}}{i} = \dfrac{v-v^n}{i}$
			1	$\cdots$	1	$v^2\dfrac{1-v^{n-2}}{i} = \dfrac{v^2-v^n}{i}$
					$\cdots$	$\cdots$
					1	$v^{n-1} \cdot \dfrac{1-v}{i} = \dfrac{v^{n-1}-v^n}{i}$
Total	1	2	3		n	$\dfrac{\sum_{t=0}^{n} v^t - n \cdot v^n}{i} = \dfrac{\ddot{a}_{\overline{n|}} - nv^n}{i}$

The result is:

(2.41)

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

Example (2.42)

Let $i = 5\%$ and $n = 4$. Then the annuity payments are 1, 2, 3, 4 and the present value is:

$$(Ia)_{\overline{4}|} = \frac{\ddot{a}_{\overline{4}|} - 4v^4}{0.05} = 8.6488$$

Note that 8.6488 *is the same value we calculated for this sequence of payments in the previous section using the Cash Flow worksheet.*

Exercise (2.43)

Find $(Ia)_{\overline{15}|i=6\%}$.

Answer: 67.2668

As with level annuities, to create a formula for an increasing unit annuity-*due* or a continuously payable annuity, we simply change the denominator:

(2.44) $$(I\ddot{a})_{\overline{n}|} = \frac{i}{d}\cdot(Ia)_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

(2.45) $$(I\bar{a})_{\overline{n}|} = \frac{i}{\delta}\cdot(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

In the case of an increasing perpetuity, the present value is the limit of the appropriate formula as n approaches infinity:

(2.46) $$(Ia)_{\overline{\infty}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{i} = \frac{\frac{1}{d}-0}{i} = \frac{1}{id}$$

(2.47) $$(I\ddot{a})_{\overline{\infty}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{d} = \frac{\frac{1}{d}-0}{d} = \frac{1}{d^2}$$

(2.48) $$(I\bar{a})_{\overline{\infty}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{\delta} = \frac{\frac{1}{d}-0}{\delta} = \frac{1}{\delta d}$$

Example (2.49)

Let $i = 5\%$ and $n = 4$.

Then, $(I\ddot{a})_{\overline{4}|} = 1.05(8.6488) = 9.0812$

Note: 8.6488 is the value calculated for $(Ia)_{\overline{4}|5\%}$ *in Example* **(2.42)**.

Exercise (2.50)

Find $(I\ddot{a})_{\overline{15}|6\%}$.

Answer: 71.3028

The *future* value of an increasing unit annuity-immediate is denoted by $(Is)_{\overline{n}|}$. One can avoid excessive memorization of formulas by using the relationship $(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|}$. (The value as of the date of the n^{th} payment equals the value as of time 0, $(Ia)_{\overline{n}|}$, accumulated n periods to time n.) Below, we show the commonly used expressions for calculating $(Is)_{\overline{n}|}$, $(I\ddot{s})_{\overline{n}|}$, and $(I\bar{s})_{\overline{n}|}$.

(2.51)

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

(2.52)

$$(I\ddot{s})_{\overline{n}|} = (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{i}{d}(Is)_{\overline{n}|}$$

(2.53)

$$(I\bar{s})_{\overline{n}|} = (1+i)^n (I\bar{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta} = \frac{i}{\delta}(Is)_{\overline{n}|}$$

The number of formulas here appears overwhelming, but the situation is relatively simple. If you can calculate $(Ia)_{\overline{n}|}$, then to obtain the value of any other type of increasing annuity, simply do a multiplication:

Multiply by: $(1+i)^n$ to convert $(Ia)_{\overline{n}|}$ to $(Is)_{\overline{n}|}$.

Multiply by i/d to convert to an annuity-due.

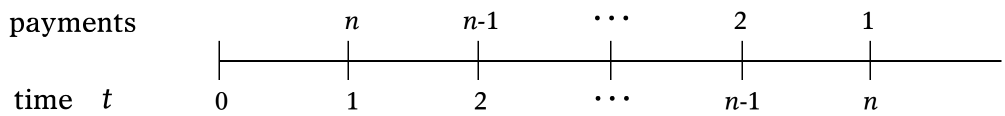
Multiply by i/δ to convert to a continuously-payable annuity.

Extensive memorization is not required!

Section 2.10

Decreasing Annuities with Terms in Arithmetic Progression

The n-year **unit decreasing annuity-immediate** has n payments: n, n-1, . . . , 1, payable at the end of each year (at times 1, 2, 3, . . . , n). Its present value is denoted by $(Da)_{\overline{n}|}$.



To develop a formula for $(Da)_{\overline{n}|}$, first note that $(Da)_{\overline{n}|} + (Ia)_{\overline{n}|} = (n+1)\cdot a_{\overline{n}|}$.
(At time 1, $(Da)_{\overline{n}|}$ pays n and $(Ia)_{\overline{n}|}$ pays 1, for a total of n+1; at time 2, $(Da)_{\overline{n}|}$ pays n-1 and $(Ia)_{\overline{n}|}$ pays 2, for a total of n+1, etc. So in each of the n years, the total payment is n+1.)

We can rearrange that formula and solve for $(Da)_{\overline{n}|}$:

$$\begin{aligned}(Da)_{\overline{n}|} &= (n+1)\cdot a_{\overline{n}|} - (Ia)_{\overline{n}|} = (n+1)\cdot\frac{1-v^n}{i} - \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \\ &= \frac{(n+1) - \ddot{a}_{\overline{n}|} - v^n}{i} = \frac{n - \left(\ddot{a}_{\overline{n}|} - 1 + v^n\right)}{i} \\ &= \frac{n - a_{\overline{n}|}}{i}\end{aligned}$$

As with arithmetic increasing annuities, you really need only one formula for arithmetic decreasing annuities:

(2.54)
$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

There are also formulas for the present value of a decreasing annuity-*due* and a decreasing continuously-payable annuity, as well as for the *future* values of these annuities. But, as shown here, their values can also be found by simple adjustments to the formula for $(Da)_{\overline{n}|}$:

(2.55)
$$(D\ddot{a})_{\overline{n}|} = (1+i)(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

(2.56)
$$(D\bar{a})_{\overline{n}|} = \frac{i}{\delta}(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{\delta}$$

(2.57)

$$\left(Ds\right)_{\overline{n}|} = (1+i)^n \left(Da\right)_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{i}$$

(2.58)

$$\left(D\ddot{s}\right)_{\overline{n}|} = (1+i)^n \left(D\ddot{a}\right)_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{d}$$

(2.59)

$$\left(D\bar{s}\right)_{\overline{n}|} = (1+i)^n \left(D\bar{a}\right)_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{\delta}$$

Example (2.60)

Given $i = 5\%$ and $n = 4$, find $(Da)_{\overline{4}|}$.

$$(Da)_{\overline{4}|} = \frac{4 - a_{\overline{4}|}}{0.05} = \frac{4 - 3.546}{0.05} = 9.08$$

This can be checked on the BA II Plus using the NPV function on the sequence 4, 3, 2, 1 with I=5.

Exercise (2.61)

Find $\left(Da\right)_{\overline{15}|}$ for $i = 0.06$.

Answer: 88.1292

Section 2.11

A Single Formula for Annuities with Terms in Arithmetic Progression

Suppose the first payment in an annuity-immediate is P and the subsequent payments change by Q per period, where Q can be either positive or negative. If the annuity has n payments, the sequence of payments is:

$$P,\ P+Q,\ P+2Q,\ \ldots,\ P+(n-1)Q\,.$$

The present value of this annuity is:

(2.62)

$$\boxed{PV = P\cdot a_{\overline{n}|} + Q\cdot\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)}$$

We can analyze this formula in two steps:
Because there is a payment of P at the end of each year for n years, the present value of the P component is $P\cdot a_{\overline{n}|}$.
The Q-related payments in years 1 through n are 0, 1, 2, …, (n-1). This amounts to an (n-1)-year increasing annuity beginning at time 1 (with its first payment at time 2), so the value of these payments is:

$$v\cdot Q\cdot(Ia)_{\overline{n-1}|} = Q\cdot v\cdot\frac{\ddot{a}_{\overline{n-1}|} - (n-1)\cdot v^{n-1}}{i} = Q\left(\frac{a_{\overline{n-1}|} - (n-1)\cdot v^n}{i}\right) = Q\left(\frac{a_{\overline{n}|} - n\cdot v^n}{i}\right)$$

(The last step in this development uses the relationship $a_{\overline{n-1}|} + v^n = a_{\overline{n}|}$.)

The total of the P and Q components combined is $P\cdot a_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$, as in **(2.62)**, above. Note that the formula's coefficient for Q, $\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$, is very similar to the formula for $(Ia)_{\overline{n}|}$, but it uses $a_{\overline{n}|}$ instead of $\ddot{a}_{\overline{n}|}$. This makes it relatively easy to memorize the PQ formula.

Note also that $(Ia)_{\overline{n}|}$ is the special case where P=1 and Q=1. Similarly, $(Da)_{\overline{n}|}$ is the special case where $P = n$ and Q = -1. The PQ formula thus provides an alternative way to find these values.

We can multiply **(2.62)** by $(1+i)^n$ to develop a formula for the future value of the annuity at time n:

(2.63)

$$\boxed{FV = P\cdot s_{\overline{n}|} + Q\cdot\left(\frac{(s_{\overline{n}|} - n)}{i}\right)}$$

Note that the limit of **(2.62)** as n becomes infinite gives the present value of an increasing perpetuity-immediate of the form $P,\ P+Q,\ P+2Q,\ \ldots$:

(2.64)
$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

This formula has been used in past exam problems. For example, see Sample Exam Problem #19 at the end of this module.

In the case of an annuity-due or perpetuity-due with payments that follow the PQ pattern, the PV and FV can be found by multiplying the above formulas by $1+i$ (or by $\frac{i}{d}$).

Some students prefer to solve increasing and decreasing annuity problems using *only* the PQ formula, because it can be applied to *any* arithmetic annuity. We recommend that you learn the PQ formula *in addition to* (Ia) and (Da), since each formula has time-saving advantages in different problems.

Example (2.65)

A 10-year annuity-immediate has a first-year payment of 500. The subsequent payments increase by 100 each year. Find the present value of this annuity based on an annual effective rate of 5%.

$$PV = P \cdot a_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right), \text{ where } n=10,\ P=500 \text{ and } Q=100$$

$$PV = 500 \cdot \frac{1-1.05^{-10}}{0.05} + 100 \cdot \left(\frac{\frac{1-1.05^{-10}}{0.05} - \frac{10}{1.05^{10}}}{0.05}\right) = 7{,}026.07$$

Exercise (2.66)

A 5-year annuity-immediate has a first-year payment of 1,000. The subsequent payments *decrease* by 100 each year. Find the present value of this annuity based on an annual effective rate of 6%.

Answer: 3,418.91

Section 2.12

Annuities with Terms in Geometric Progression

Consider the sequence of three payments 1, 1.05, 1.1025 = $(1.05)^2$ made at the end of the year:

payments		1	1.05	$(1.05)^2$
time t	0	1	2	3

The payments increase geometrically with a common ratio of 1.05. This is a **geometric annuity** with a growth rate g= 0.05.

Suppose that we wish to find the present value of this geometric annuity at an annual effective interest rate of i = 10%. Applying the formula for the sum of a geometric series, we have:

$$PV = \frac{1}{1.10} + \frac{1.05}{(1.10)^2} + \frac{(1.05)^2}{(1.10)^3} = \frac{1}{1.10}\underbrace{\left[1 + \frac{1.05}{1.10} + \left(\frac{1.05}{1.10}\right)^2\right]}_{\text{This is a geometric series with } n=3 \text{ and } r=\frac{1.05}{1.10}}$$

$$= \frac{1}{1.10}\left[\frac{1-\left(\frac{1.05}{1.10}\right)^3}{1-\left(\frac{1.05}{1.10}\right)}\right] = 2.6052$$

More generally, we can consider an n-year geometric annuity-immediate with growth rate g. Its payments are $1, (1+g), (1+g)^2, \ldots, (1+g)^{n-1}$, and its present value is represented by the symbol $a^{g}_{\overline{n}|i}$ or $a^{g}_{\overline{n}|}$. (Note: This actuarial symbol for geometric annuities is not widely used, but it is descriptive and will be used in this text.) The first payment in the series represented by $a^{g}_{\overline{n}|}$ is 1, and the annual rate of change, g, can be either positive or negative, reflecting payments that increase or decrease geometrically.

There are 3 standard methods for calculating the value of a geometric annuity. We will present all three in this module. You may choose to learn only one of the three, or you might prefer to learn all three, so that you can apply the one that is best suited to a particular problem.

Geometric Series Method

Using the same approach as above, we can express the present value of a geometric annuity-immediate as follows:

$$a_{\overline{n}|i}^{g} = \frac{1}{(1+i)} + \frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^n}$$
$$= \frac{1}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \ldots + \left(\frac{1+g}{1+i}\right)^{n-1}\right]$$

The quantity in parentheses is a geometric series with ratio $r = \frac{(1+g)}{(1+i)}$, so we can apply the formula for the sum of a geometric series to find its value:

(2.67)

$$a_{\overline{n}|i}^{g} = \left(\frac{1}{1+i}\right)\frac{1-\left(\frac{1+g}{1+i}\right)^n}{1-\left(\frac{1+g}{1+i}\right)}$$

In a similar manner, the geometric series formula can be applied to find the value of any geometric annuity (including present and future values of geometric annuities-immediate and annuities-due). Of course, g and i must each be constant throughout the term of the annuity.

Note that if $g = i$, the formula can't be applied (because the denominator is 0), but in that case all of the terms of the geometric series are equal, so the sum equals n times the value of the first term:

(2.68)

$$\text{If } g = i: \quad a_{\overline{n}|i}^{g} = \frac{1}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \ldots + \left(\frac{1+g}{1+i}\right)^{n-1}\right]$$
$$= \frac{1}{1+i}\left[1+1+1+\ldots+1\right] = \frac{n}{1+i}$$

In the case of a perpetuity (where n is infinite), the present value can be determined, provided that $i > g$, because then we have $\lim_{n\to\infty}\left(\frac{1+g}{1+i}\right)^n = 0$, so the numerator becomes simply 1:

(2.69)

$$a_{\overline{\infty}|i}^{g} = \left(\frac{1}{1+i}\right)\frac{1}{1-\left(\frac{1+g}{1+i}\right)} \qquad i > g$$

Geometric Annuity Formula Method

It is also possible to develop *standard* formulas for the present value and future value of a geometric annuity-immediate or annuity-due. For the present value of an annuity-immediate, we can simplify the formula we developed above, giving the following formula:

(2.70)

$$a^{g}_{\overline{n}|i} = \left(\frac{1}{1+i}\right)\frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{1-\left(\frac{1+g}{1+i}\right)} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{i-g}$$

Note that when g is 0, the above formula simplifies to $\frac{1-v^{n}}{i}$, which is the standard formula for $a_{\overline{n}|}$.

The perpetuity version of **(2.70)** is easy to remember. If i is the interest rate, g is the rate of growth, and $i > g$, the present value is simply:

(2.71)

$$a^{g}_{\overline{\infty}|} = \frac{1}{i-g} \qquad i > g$$

If $g \geq i$ the present value of the perpetuity is infinite.

To modify Formula **(2.70)** for geometric annuities-due or for the future values of geometric annuities, you can either multiply by the appropriate accumulation factors, or you can memorize the formulas for these values, which are:

(2.72)

$$\ddot{a}^{g}_{\overline{n}|} = (1+i)\cdot a^{g}_{\overline{n}|} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{\frac{i-g}{1+i}} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{d-g\cdot v}$$

(2.73)

$$s^{g}_{\overline{n}|} = (1+i)^{n}\cdot a^{g}_{\overline{n}|} = \frac{(1+i)^{n}-(1+g)^{n}}{i-g}$$

(2.74)

$$\ddot{s}^{g}_{\overline{n}|} = (1+i)^{n+1}\cdot a^{g}_{\overline{n}|} = \frac{(1+i)^{n}-(1+g)^{n}}{\frac{i-g}{1+i}} = \frac{(1+i)^{n}-(1+g)^{n}}{d-g\cdot v}$$

For a perpetuity-due, we have:

(2.75)

$$\ddot{a}^{g}_{\overline{\infty}|} = \frac{1}{d - g \cdot v} = \frac{1+i}{i-g} \qquad i > g$$

If you choose to use the Geometric Annuity Formula method, we recommend that you memorize Formula **(2.70)** for the present value of a geometric annuity-immediate, and modify it as necessary for other situations (as shown in the above formulas), rather than memorizing all of these formulas.

Artificial Interest Rate Method

We will again begin with the formula we developed from the geometric series, and will modify it to find the present value of a geometric annuity-due:

$$a^{g}_{\overline{n}|i} = \left(\frac{1}{1+i}\right)\frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{1-\left(\frac{1+g}{1+i}\right)} \qquad \ddot{a}^{g}_{\overline{n}|i} = (1+i)\cdot a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{1-\left(\frac{1+g}{1+i}\right)}$$

Now define an "artificial interest rate" j, such that $1+j = \frac{1+i}{1+g}$.

The formula for a geometric annuity-*due* then becomes:

(2.76)

$$\ddot{a}^{g}_{\overline{n}|i} = \frac{1-(1+j)^{-n}}{1-(1+j)^{-1}} = \frac{1-v_j^{n}}{d_j} = \ddot{a}_{\overline{n}|j} \quad \text{where } 1+j = \frac{1+i}{1+g}$$

Note that v_j and d_j are used to represent the present value factor and rate of discount based on the artificial interest rate j.

In a way, this is the simplest of the 3 methods, because the formula is just the familiar formula for an annuity-due, but with an interest rate j that has been calculated from $1+j = \frac{1+i}{1+g}$. However, it is important to remember that *this formula applies only to the* <u>*present value*</u> *of an annuity-*<u>*due*</u>. For an annuity-immediate or for future values, you must first calculate the present value of the annuity-due, and then adjust it by the appropriate interest factor, using $(1+i)$, *not* $(1+j)$:

$$a^{g}_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|j}}{1+i} \qquad s^{g}_{\overline{n}|i} = (1+i)^{n-1}\,\ddot{a}_{\overline{n}|j} \qquad \ddot{s}^{g}_{\overline{n}|i} = (1+i)^{n}\,\ddot{a}_{\overline{n}|j}$$

Example (2.77)

Given i = 10%, find the present value of the sequence of payments:

$$1.05, (1.05)^2, \dots, (1.05)^{10}$$

Payments are made at the beginning of each year.

Solution.

Note that this series starts with 1.05, not 1, and that payments are made at the beginning of each period.

By geometric series:

$$1.05 \cdot \ddot{a}^{0.05}_{\overline{10}|0.10} = 1.05 \cdot \left[1 + \frac{1.05}{1.10} + \left(\frac{1.05}{1.10}\right)^2 + \dots + + \left(\frac{1.05}{1.10}\right)^9\right] = 1.05 \cdot \frac{1 - \left(\frac{1.05}{1.10}\right)^{10}}{1 - \left(\frac{1.05}{1.10}\right)} = 8.59$$

By the geometric annuity formula:

$$1.05 \cdot \ddot{a}^{0.05}_{\overline{10}|0.10} = 1.05 \cdot \frac{1 - \left(\frac{1.05}{1.10}\right)^{10}}{d - g \cdot v} = 1.05 \cdot \frac{1 - \left(\frac{1.05}{1.10}\right)^{10}}{\frac{0.10}{1.10} - \frac{0.05}{1.10}} = 8.59$$

By the artificial interest rate method:

$$1 + j = \frac{1+i}{1+g} = \frac{1.10}{1.05} = 1.047619 \qquad j = 4.7619\%$$

$$1.05 \cdot \ddot{a}^{0.05}_{\overline{10}|0.10} = 1.05 \cdot \ddot{a}_{\overline{10}|0.047619} = 1.05 \cdot \frac{1 - 1.047619^{-10}}{0.047619 / 1.047619} = 8.59$$

Exercise (2.78)

Given i = 8%, find the present value of the following sequence of payments:

$$1, 1.06, (1.06)^2, \dots, (1.06)^9$$

Payments are made at the end of each year.

Answer: 8.5246

Example (2.79)

Given i = 10%, find the present value of a perpetuity with payments:

$$1.05, (1.05)^2, \ldots, (1.05)^n, \ldots$$

if a) payments are made at the end of the period, and
b) payments are made at the beginning of the period.

Solution.

By **geometric series**:

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \dfrac{1}{1.10} \cdot \left[1 + \dfrac{1.05}{1.10} + \left(\dfrac{1.05}{1.10}\right)^2 + \ldots\right] = \dfrac{1.05}{1.10} \cdot \dfrac{1}{1 - \dfrac{1.05}{1.10}} = 21$

b) $1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = (1+i) \cdot 1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.10(21) = 23.1$

By the **geometric annuity formula**:

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \dfrac{1}{i-g} = \dfrac{1.05}{0.10 - 0.05} = 21$

b) $1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \dfrac{1}{d - v \cdot g} = 1.05 \cdot \dfrac{1}{\dfrac{0.10}{1.10} - \dfrac{0.05}{1.10}} = 23.1$

By the **artificial interest rate** method, doing part b) first:

b) $1 + j = \dfrac{1+i}{1+g} = \dfrac{1.10}{1.05} = 1.047619$ $\qquad j = 4.7619\%$

$$1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \ddot{a}_{\overline{\infty}|4.7619\%} = 1.05 \cdot \frac{1}{0.047619 / 1.047619} = 23.1$$

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = \dfrac{1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10}}{1+i} = \dfrac{23.1}{1.10} = 21$

Exercise (2.80)

Given i = 8%, find the present value of a perpetuity with payments:

$$1.075, (1.075)^2, \ldots, (1.075)^n, \ldots$$

if a) payments are made at the end of the period, and
b) payments are made at the beginning of the period.

Answers: a) 215 b) 232.20

Example (2.81)

You want to save 1,000,000 for retirement. You plan to make annual deposits at the beginning of each year into an account that earns an annual effective rate of 7.5%. You will increase the amount of your deposit each year by 4%. If you plan to retire in 40 years, what should be the amount of your first deposit?

Solution.
Let D be the amount of the first deposit.

By **geometric series**:

$$1{,}000{,}000 = D\cdot\left(1.075^{40} + 1.04\cdot 1.075^{39} + 1.04^{2}\cdot 1.075^{38} + \ldots + 1.04^{39}\cdot 1.075\right)$$

$$= D\cdot 1.075^{40}\cdot\left[1 + \frac{1.04}{1.075} + \left(\frac{1.04}{1.075}\right)^{2} + \ldots + \left(\frac{1.04}{1.075}\right)^{39}\right]$$

$$= D\cdot 1.075^{40}\cdot\frac{1-\left(\dfrac{1.04}{1.075}\right)^{40}}{1-\left(\dfrac{1.04}{1.075}\right)} = 406.756\cdot D$$

$$D = \frac{1{,}000{,}000}{406.756} = 2{,}458.48$$

By the **geometric annuity formula**:

$$1{,}000{,}000 = D\cdot \ddot{s}^{\,0.04}_{\overline{40}|7.5\%} = D\cdot\frac{(1+i)^{40}-(1+g)^{40}}{d-g\cdot v} = D\cdot\frac{1.075^{40}-1.04^{40}}{\dfrac{0.075}{1.075}-\dfrac{0.04}{1.075}} = 406.756\cdot D$$

$$D = \frac{1{,}000{,}000}{406.756} = 2{,}458.48$$

By the **artificial interest rate** method:

$$1+j = \frac{1+i}{1+g} = \frac{1.075}{1.04} = 1.033654 \qquad j = 3.3654\%$$

$$1{,}000{,}000 = D\cdot \ddot{s}^{\,0.04}_{\overline{40}|7.5\%} = D\cdot \ddot{a}^{\,0.04}_{\overline{40}|7.5\%}\cdot 1.075^{40} = D\cdot \ddot{a}_{\overline{40}|j}\cdot 1.075^{40}$$

$$= D\cdot\frac{1-1.033654^{-40}}{\dfrac{0.033654}{1.033654}}\cdot 1.075^{40} = 406.756\cdot D$$

$$D = \frac{1{,}000{,}000}{406.756} = 2{,}458.48$$

Exercise (2.82)

If the deposits in Example **(2.81)** increase by 5% each year, what is the amount of the first deposit?

Answer: 2,286.96

Section 2.13

Equations of Value and Loan Payments

We have already used equations of value to solve annuity problems. In Section 2.7 we solved problems that involved finding the periodic payment for a loan or determining the amount (or the number) of level deposits needed to accumulate a targeted amount. We also solved those problems using the BA II Plus's TVM worksheet. Now we will discuss in more detail the use of equations of value to solve annuity problems.

Suppose that you borrow 10,000 at an interest rate i = 8% with level payments at the end of each year for 10 years. How do you find the annual payment P?

The principle that is used to find P is that the present value of the borrower's loan payments must equal the present value of the loan: $10,000 = P \cdot a_{\overline{10|}}$.

Thus, $P = \dfrac{10,000}{a_{\overline{10|}}} = \dfrac{10,000}{6.7101} = 1,490.29$.

Our valuation date for this equation is $t = 0$, the date of the loan. We could have chosen a different valuation date (such as $t = 10$), but $t = 0$ is convenient because we already know the value of the loan on that date (10,000).

> The computation is simple, but the key point here is the principle involved: the two sets of payments (those made by the lender, and those made by the borrower) must have the same value as of the valuation date. In other words, the value received by the borrower must equal the value repaid to the lender.

Also note that the loan payment amount could easily be calculated on a financial calculator: N = 10, I/Y = 8, PV =10,000, and CPT PMT.

In order to understand the calculator's TVM calculations, it is useful to know that it uses the following equation of value:

$$\text{PV} \cdot \left(1 + \frac{\text{I/Y}}{100}\right)^{\text{N}} + \text{PMT} \cdot \frac{\left(1 + \frac{\text{I/Y}}{100}\right)^{\text{N}} - 1}{\frac{\text{I/Y}}{100}} + \text{FV} = 0$$

In actuarial notation, this would be:

$$\text{PV} \cdot (1 + i)^{n} + \text{PMT} \cdot s_{\overline{n|}} + \text{FV} = 0$$

Since the accumulated values of PV, PMT, and FV are all added together (on the left side of this equation), it is clear that at least one must be positive and at least one must be negative in order for the total to be 0. In terms of a loan arrangement, we would say that the lender must pay something to the borrower (the loan amount) and the borrower must pay something to the lender (the repayment(s)).

The following example demonstrates that the valuation date does not have to be the date of the first or last cash flow.

Example (2.83)

You will make 3 deposits of 10,000 each into a bank account, at the beginning of this year and the following two years. At the end of two years, you will retire and want to withdraw a level payment P starting at the beginning of year 4 and continuing for five years. The account pays interest at an annual effective rate $i = 8\%$. What is the amount of the level payment P?

Solution.
The following timeline illustrates the cash flows in this problem.

Deposits	**10,000**	**10,000**	**10,000**					
time t	0	1	2	3	4	5	6	7
Withdrawals				***P***	***P***	***P***	***P***	***P***

Note that the first withdrawal occurs at the *beginning* of year 4, which is $t = 3$. (Year 4 begins at $t = 3$ and ends at $t = 4$.)

We will determine the values of the deposits and the withdrawals as of time $t = 3$. In other words, our *valuation date* is $t = 3$. (We could have chosen other dates, such as $t = 0$, for the valuation date, but the formulas are simpler when we use $t = 3$. Note, however, that $t = 2$ also works out nicely.)

[value of deposits as of $t = 3$] = [value of withdrawals as of t=3]

$$10{,}000(\ddot{s}_{\overline{3|}}) = P\ddot{a}_{\overline{5|}}$$

$$P = 10{,}000\left(\frac{\ddot{s}_{\overline{3|}}}{\ddot{a}_{\overline{5|}}}\right) = 10{,}000\left(\frac{3.5061}{4.3121}\right) = 8{,}130.82$$

Exercise (2.84)

If the withdrawals in Example **(2.83)** begin at the *end* of the 4th year and continue for *ten* years, what is the amount of each withdrawal?

Answer: 5,225.14

Section 2.14

Deferred Annuities

There are cases in which you may want to find the present value of an annuity with payments that begin in some future period. For example, you might plan to retire in 5 years and want to purchase an annuity-immediate that pays 10,000 per year for ten years starting 5 years from now. The present value of this annuity would be $v^5(10{,}000)a_{\overline{10}|}$. (Note that in this example the first payment would occur at the end of the 6th year.)

An annuity like this is called a **deferred annuity**. In general, the present value of an ***n*-year unit annuity-immediate deferred for *k* years** is $v^k a_{\overline{n}|}$. One form of notation for such an annuity is ${}_{k|}a_{\overline{n}|}$, so we have:

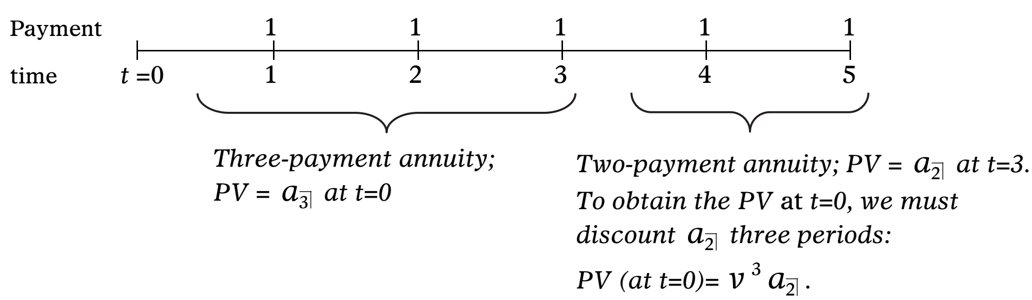
(2.85)

$$ {}_{k|}a_{\overline{n}|} = v^k \cdot a_{\overline{n}|} $$

There is an identity that breaks down the present value of an annuity-immediate into the sum of two shorter annuities, one that begins now and one that is deferred. We will illustrate this with an example.

The present value of a five-period annuity-immediate can be expressed as the sum of a 3-year annuity-immediate and a 2-year annuity-immediate deferred for 3 years:

$$ a_{\overline{5}|} = v + v^2 + v^3 + v^4 + v^5 = \left(v + v^2 + v^3\right) + v^3\left(v + v^2\right) = a_{\overline{3}|} + {}_{3|}a_{\overline{2}|} = a_{\overline{3}|} + v^3 a_{\overline{2}|} $$



Total *PV* at *t=0:* $a_{\overline{3}|} + v^3 a_{\overline{2}|}$

Thus the present value of a 5-period unit annuity-immediate can be broken down into the present value of a 3-period annuity and the present value of a 2-period annuity that begins in 3 periods.

This reasoning works in general, so we have:

$$a_{\overline{n+k}|} = a_{\overline{k}|} + {}_{k|}a_{\overline{n}|} = a_{\overline{k}|} + v^k \cdot a_{\overline{n}|}$$

The present value of an annuity-immediate for *n*+*k* periods is the sum of the present value of a *k*-period annuity-immediate and an *n*-period annuity deferred for *k* periods.

We can rewrite this identity as:

(2.86)

$$ {}_{k|}a_{\overline{n}|} = a_{\overline{n+k}|} - a_{\overline{k}|}$$

It is typical for actuarial examination questions to give some pieces of this identity when you really need other pieces:

Example (2.87)

Based on a 5% annual interest rate, find the present value of a 10-year annuity with level payments of 100 each, with the first payment occurring 4 years from now.

Solution.
This is a 3-year-deferred annuity immediate (no payments during the first 3 years; first payment at the *end* of the 4th year).

$$100 \cdot {}_{3|}a_{\overline{10}|} = 100 \cdot v^3 \cdot \frac{1-v^{10}}{i} = 100 \cdot 1.05^{-3} \cdot \frac{1-1.05^{-10}}{0.05} = 667.03$$

Alternatively, $100 \cdot \left(a_{\overline{13}|} - a_{\overline{3}|}\right) = 100 \cdot (9.3936 - 2.7232) = 667.03$

Exercise (2.88)

Based on a 5% annual interest rate, find the present value of a 10-year annuity with level payments of 100 each, with the first payment occurring at the beginning of the 6th year.

Answer: 635.27

Section 2.15

Annuities With More Complex Payment Patterns

We pointed out previously that the BA II Plus's NPV function could be used to evaluate increasing and decreasing annuities. In some problems, the calculator approach may require more than one step, as we see in the next problem.

Example (2.89)

An annuity pays 1 at the end of each of the next four years and 2 at the end of each of the four following years. Based on a 5% annual effective rate, what is the present value of this annuity?

Solution.
This sequence of payments can be broken down into a 4-year annuity-immediate with payments of 1, plus a 4-year-deferred annuity-immediate with 4 payments of 2 each:

$$a_{\overline{4}|} + 2 \cdot {}_{4|}a_{\overline{4}|} = \frac{1-1.05^{-4}}{0.05} + 2 \cdot 1.05^{-4} \cdot \frac{1-1.05^{-4}}{0.05} = 9.38$$

This can be done on the calculator by calculating $a_{\overline{4}|}$ (N=4, I/Y=5, PMT=-1, and CPT PV), and then multiplying $a_{\overline{4}|}$ by $\left(1+2(1.05)^{-4}\right)$

Another approach is to analyze this series of payments as an 8-year annuity-immediate with payments of 1, plus a 4-year-deferred annuity-immediate with 4 payments of 1 each:

$$a_{\overline{8}|} + {}_{4|}a_{\overline{4}|} = \frac{1-1.05^{-8}}{0.05} + 1.05^{-4} \cdot \frac{1-1.05^{-4}}{0.05} = 9.38$$

In this case, you can use the calculator to find $a_{\overline{4}|}$ (as above), multiply by $(1.05)^{-4}$ and store the result in a memory (e.g., STO 1), then change N to 8, CPT PV (to find $a_{\overline{8}|}$), and add the stored value (+ RCL 1 =).

Exercise (2.90)

An annuity pays 100 at the end of each of the next 10 years and 200 at the end of each of the five subsequent years. If $i = 0.08$, find the present value of the annuity.

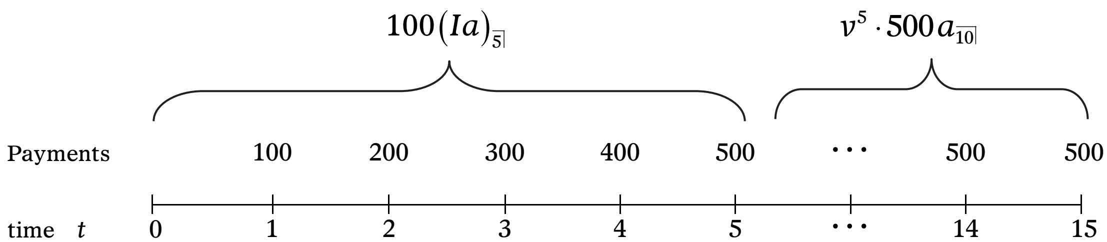
Answer: 1,040.89

An annuity's periodic payments can vary in many different patterns, which you will see as you look at the examination problems at the end of this module. The next two examples illustrate this.

Example (2.91)

An annuity-immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. There are 10 further payments of 500. Find the present value of this annuity at 6.5%.

Solution.
The equation of value is
$PV = 100(Ia)_{\overline{5}|} + v^5 \cdot 500a_{\overline{10}|} = 100(11.9445) + 0.7299(500)(7.1888) = 3{,}817.95$
(The following diagram shows how the payments were grouped to develop this formula.)



In many cases, there is more than one way to break an annuity's payments into components that we can value. In this example, we found a way to express the present value in terms of $(Ia)_{\overline{5}|}$ and ${}_{5|}a_{\overline{10}|}$. Since the 5th-year payment of 500 can be either the 5th payment of the increasing annuity or the first payment of the level annuity, we could also have calculated the answer as:

$$PV = 100(Ia)_{\overline{4}|} + v^4 \cdot 500a_{\overline{11}|} = 100 \cdot 8.2951 + 0.77732 \cdot 500 \cdot 7.6890 = 3{,}817.95$$

Yet another way to write the present value of this annuity involves a decreasing arithmetic annuity:

$$500 \cdot a_{\overline{15}|} - 100(Da)_{\overline{4}|}$$

The payment pattern is equivalent to a level 500 in all 15 years, reduced in the first 4 years by 400, 300, 200, and 100.
This expression can be evaluated as:

$$500\frac{1-v^4}{i} - 100\frac{4-a_{\overline{4}|}}{i} = 4{,}701.3344 - 883.3867 = 3{,}817.95$$

Example (2.92)

An annuity-immediate has 5 annual payments of 100, followed by a perpetuity of 200 starting in the 6th year. Find the present value at 8%.

Solution.
There are a number of ways to attack this problem. Perhaps the simplest is to think of this annuity as a perpetuity-immediate of 100 starting now, augmented by a second perpetuity-immediate of 100 starting in 5 years:

$$100 \cdot a_{\overline{\infty}|} + 100v^5 \cdot a_{\overline{\infty}|}$$

Payments:

PV	100 / 0.08	100	100	100	100	100	100	100	...
PV	$v^5(100/0.08)$	0	0	0	0	0	100	100	...
time	t 0	1	2	3	4	5	6	7	...

The present value of a single perpetuity of 100 is $\frac{100}{0.08} = 1250$.

Thus the total present value is $1250 + v^5 \cdot 1250 = 2{,}100.73$.

Alternatively, we could treat this payment pattern as a perpetuity-immediate of 200, minus a 5-year annuity-immediate of 100:

$$200 \cdot a_{\overline{\infty}|} - 100 \cdot a_{\overline{5}|} = \frac{200}{0.08} - 100 \cdot \frac{1 - 1.08^{-5}}{0.08} = 2{,}100.73$$

Exercise (2.93)

An annuity-immediate has a first payment of 100, and its payments increase by 100 each year until they reach 500. The remaining payments are a perpetuity-immediate of 500 beginning in year 6. Find the present value at 6.5%.

Answer: 6,808.92

Section 2.16

Annuities with Payments More Frequent than Annual

Thus far we have considered annuities with annual payments and continuous payments. However, many annuities, such as loan payments or pensions, have more frequent payments (e.g., monthly or quarterly). In this section, we will develop methods for annuities whose payments are more frequent than annual.

Interest rates for loans, including mortgage loans, are typically quoted as nominal rates. If a lender offers you a mortgage rate of 6%, it is probably a nominal annual rate of 6% convertible monthly, which is a monthly effective rate of $6\% \div 12 = 0.5\%$. Knowing the monthly effective rate makes it easy to calculate the monthly payment amount.

Example (2.94)

Find the level monthly payment for a 30-year mortgage loan of 300,000 at an interest rate of 6% convertible monthly.

Solution.
The monthly effective interest rate is 0.5%, so we can simply use the usual annuity functions, recognizing that the period is 1 month instead of 1 year:

$$P = \frac{300,000}{a_{\overline{360}|0.5\%}} = \frac{300,000}{166.7916} = 1,798.65$$

The calculator solution is direct. Note that mortgage payments are made at the end of the month, so that your calculator should be in END mode. The loan is for 360 months at a 0.5% monthly effective rate. Set N=360, I/Y=0.5, PV=300,000, and CPT PMT = −1,798.65.

Exercise (2.95)

Find the monthly payment for the loan in Example **(2.94)** if the term is 15 years.

Answer: 2,531.57

Example (2.96)

An annuity-immediate has 20 initial quarterly payments of 25 each, followed by a perpetuity of quarterly payments of 50 starting in the sixth year. Find the present value at 8% convertible quarterly.

Solution.
We can think of this annuity as a quarterly perpetuity-immediate of 25 starting now, augmented by a second quarterly perpetuity-immediate of 25 starting in 5 years.

Payments:

PV	25 / 0.02		25	25	25	···	25	25	25	···
PV	$v^{20}(25/0.02)$		0	0	0	0	0	0	25	···
Quarters		$t=0$	1	2	3	···	19	20	21	···

The quarterly effective interest rate is 2%.

The present value of a single perpetuity of 25 is $\dfrac{25}{0.02}=1{,}250$

Thus the total present value is $1{,}250+\dfrac{1{,}250}{1.02^{20}}=2{,}091.21$

There is actuarial notation for the present value of an annuity that makes m payments per year. The symbol $a_{\overline{n}|}^{(m)}$ indicates the present value of an annuity-immediate that pays $1/m$ at the end of each $1/m$ of a year. For example, consider an annuity that makes 12 payments per year. The symbol $a_{\overline{n}|}^{(12)}$ is the present value of an annuity that pays 1/12 at the end of each month for n years. Note that the total amount paid each year is 1. The formula for the present value of an annuity that makes m payments per year can be developed as follows:

$$a_{\overline{n}|}^{(m)}=\frac{1}{m}\cdot a_{\overline{m\cdot n}|\frac{i^{(m)}}{m}}=\frac{1}{m}\cdot\frac{1-\left(1+\frac{i^{(m)}}{m}\right)^{-m\cdot n}}{\frac{i^{(m)}}{m}}=\frac{1-\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right]^{-n}}{i^{(m)}}=\frac{1-v^n}{i^{(m)}}=\frac{i}{i^{(m)}}a_{\overline{n}|}$$

This gives us the following formula for the present value of an n-year unit annuity-immediate that makes "$m$thly" payments totaling 1 per year:

(2.97)

$$a_{\overline{n}|}^{(m)}=\frac{1-v^n}{i^{(m)}}=\frac{i}{i^{(m)}}\cdot a_{\overline{n}|}$$

Note: The expression $\frac{1-v^n}{i^{(m)}}$ *in Formula* **(2.97)** *may* *<u>appear</u>* *to violate the rule that we never do calculations with nominal interest rates. (Clearly,* $i^{(m)}$ *is a nominal interest rate.) However, note that in the derivation of this formula (i.e., the equations just above Formula* **(2.97)** *), the expression after the second equals sign has a fraction whose denominator is* $\frac{i^{(m)}}{m}$*. This is the mthly* *<u>effective</u>* *interest rate. This is, of course, proper. That fraction with* $\frac{i^{(m)}}{m}$ *in the denominator is then multiplied by* $\frac{1}{m}$*, which cancels with the m in* $\frac{i^{(m)}}{m}$*, leaving just* $i^{(m)}$ *in the denominator. So we are not actually performing a calculation with a nominal interest rate, but rather with an mthly* *<u>effective</u>* *rate that has been multiplied by m. An example of an* *<u>improper</u>* *use of a nominal rate is the accumulation factor* $\left(1+i^{(4)}\right)^n$*, which is* *<u>never</u>* *appropriate.*

By analysis similar to the above, we can develop the following additional formulas for annuities with $m$thly payments:

(2.98)

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} \cdot a_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{a}_{\overline{n}|}$$

(2.99)

$$s_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} \cdot s_{\overline{n}|}$$

(2.100)

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{s}_{\overline{n}|}$$

These appear to be a lot of formulas to memorize, but if you see the pattern, you will find them easier to absorb. In fact, you can think of the annual-payment formulas $\left(\frac{1-v^n}{i}, \frac{1-v^n}{d}, \text{etc.}\right)$ as being *special cases* of the above formulas with $m=1$. (By definition $i^{(1)}=i$ and $d^{(1)}=d$, so the i or d in the denominator of an annual-payment annuity is actually $i^{(1)}$ or $d^{(1)}$.) At the opposite extreme, when payments are made continuously, m is infinite and the denominator is $i^{(\infty)}=d^{(\infty)}=\delta$. So the above formulas for $m$thly annuities cover *all* payment frequencies, from annual $(m=1)$ to continuous $(m=\infty)$.

The present value of any annuity with m payments per year that are equally-spaced and equal in amount can be found by first calculating the value of an *annual*-payment annuity-immediate, and then multiplying that value by a factor of $\frac{i}{i^{(m)}}$ or $\frac{i}{d^{(m)}}$, depending on whether it is an m-thly annuity-immediate or annuity-due. The requirements are:

1) the m payments *within each year* must be *level*, and
2) the *total* of the m payments during each year must equal the *annual* payment in that year under the annual-payment annuity

This adjustment methodology is particularly useful when working with arithmetic or geometric increasing or decreasing annuities where the increases or decreases occur annually, but the payments *within* each year are level. In the case of geometric annuities, the denominator of the annual-payment geometric annuity-immediate is not simply i; nonetheless, we can still apply an adjustment factor of $\frac{i}{i^{(m)}}$ or $\frac{i}{d^{(m)}}$ to the value calculated using annual payments.

The above technique can also be used with level annuities, but it is generally easier to calculate their values by finding the *effective interest rate per payment period* and then applying the standard annuity formulas (with one payment per interest conversion period). For example, if a 10-year level-payment annuity has quarterly payments, find the quarterly effective rate and treat the annuity as a 40-period annuity at that interest rate. The examples in the next section include both level and non-level annuities. However, the following example provides an opportunity to practice using the formulas for $m$thly annuities.

Example (2.101)

A saver deposits 100 into a bank account at the end of every month for 10 years. If the account earns interest at an annual effective rate of 6%, what is the saver's balance at the end of 10 years?

Solution.
We need to find the accumulated value of a 10-year annuity-immediate with monthly payments:

$$s_{\overline{10}|6\%}^{(12)} = \frac{(1+i)^{10}-1}{i^{(12)}} = \frac{1.06^{10}-1}{12\cdot\left(1.06^{1/12}-1\right)} = 13.539$$

This is the value at time 10 of deposits of 1 each year for 10 years. But each year's deposits consist of 12 payments of $\frac{1}{12}$ each, made at the end of each month. In order to match the facts of this problem, we have to multiply by 1,200, because the saver is depositing a total of 1,200 each year:

$$Bal_{10} = 1,200\cdot(13.539) = 16,247.34$$

We can also solve this problem by calculating the monthly effective rate and treating it as a 120-period annuity-immediate with payments of 100 per period:

$$\text{monthly effective rate} = 1.06^{1/12}-1 = 0.004868 = 0.4868\%$$

$$Bal_{10} = 100\cdot s_{\overline{120}|0.4868\%} = 100\cdot\frac{1.004868^{120}-1}{0.004868} = 16,247.34$$

Exercise (2.102)

A car loan of 20,000 is to be repaid by monthly payments over a 4-year period. Payments are made at the end of each month, with the first payment one month after the loan amount was received. If the loan is based on a nominal rate of 6.6% convertible monthly, what is the amount of each monthly payment?

Answer: 475.22

Section 2.17

Payment Periods that Don't Match the Interest Conversion Period

The next examples involve annuities where interest rate conversion is either more or less frequent than the annuity's payments. We use examples to demonstrate how to solve these problems, since the basic approach is intuitively obvious.

Example (2.103)

An annuity-immediate has semi-annual payments of 100 for 10 years. Find its present value at an interest rate of 6% convertible monthly.

Solution.
There are 20 semi-annual payments of 100. We are given a monthly effective rate of $6\% \div 12 = 0.5\%$, but we need a semi-annual rate. Compound the monthly rate 6 times to find the semi-annual effective rate:

$$i = (1.005)^6 - 1 = .0304\,.$$

Now we can use our calculator's TVM functions.
Set N=20, I/Y=3.04, PMT=100, and CPT PV = −1,482.57

Note: If you enter the rounded value of the interest rate (3.04%) for I/Y, your answer will not exactly match the value shown here. You should instead take the result of your interest rate calculation $(1.005\,^\wedge 6 - 1 = 0.030377509)$, *multiply it by 100 (to make it a percent), and then press* I/Y. *This retains the full accuracy of the calculated value.*

Exercise (2.104)

An annuity-immediate has quarterly payments of 200 for 15 years. Find its present value at an interest rate of 9% convertible monthly.

Answer: 6,523.84

Example (2.105)

An annuity-immediate has monthly payments of 100 for 10 years. Find its present value at a rate of 6% convertible semi-annually.

Solution.
There are 120 monthly payments of 100. We are given a semi-annual effective rate of $6\% \div 2 = 3\%$, but we need a monthly effective rate. Take the 6th root of the semi-annual interest factor (1.03) to obtain a monthly rate:

$$i = (1.03)^{1/6} - 1 = 0.004939 .$$

Using the TVM functions:
Set N=120, I/Y=0.4939, PMT=100, and CPT PV = −9,037.42

Exercise (2.106)

An annuity has quarterly payments of 200 for 15 years. Find its present value at a rate of 9% convertible semi-annually if it is:

a) an annuity-immediate
b) an annuity-due

Answer: a) 6,588.05 b) 6,734.65

Note: The value of an mthly-payment annuity-due equals the value of an mthly-payment annuity-immediate times $\left(1+\frac{i^{(m)}}{m}\right)$, *not* $(1+i)$, *since its payments occur only 1 period (*$\frac{1}{m}$ *years) earlier, not a full year earlier.*

Example (2.107)

A 15-year annuity-immediate makes monthly payments of 10 in year 1, 20 in year 2, increasing to 150 per month in year 15. Find its present value at an interest rate of 8% convertible quarterly.

Solution.
Because payments are monthly and the payment increases are annual, we can't treat this as a 180-period increasing annuity. Instead, we will calculate the value for an *annual-payment* increasing annuity, and then apply an adjustment factor to reflect that payments are made monthly.
The quarterly effective rate is $8\% / 4 = 2\%$, so the annual effective rate is:

$$1.02^4 - 1 = 8.24\% .$$

(Save the computed value in a calculator memory, so that you can access it with full accuracy, not the 3-digit approximation shown here.)
This increasing annuity's total payments in the first year are $120\left(=10\times 12\right)$, so the present value based on annual payments is:

$$120\left(Ia\right)_{\overline{15}|8.24\%} = 120\cdot\frac{\ddot{a}_{\overline{15}|8.24\%} - \dfrac{15}{1.0824^{15}}}{0.0824} = 6,634.26$$

To reflect monthly payments, we need to adjust this result by a factor of $\dfrac{i}{i^{(12)}}$. First, we calculate the value of $i^{(12)}$:

$$\left(1+\frac{i^{(12)}}{12}\right)^{12} = 1+i = 1.0824$$

$$i^{(12)} = 12\cdot\left(1.0824^{1/12}-1\right) = 0.07947$$

The value of this increasing annuity with monthly payments is:

$$120\left(Ia\right)^{(12)}_{\overline{15}|8.24\%} = \left[120\left(Ia\right)_{\overline{15}|8.24\%}\right]\cdot\frac{i}{i^{(12)}} = 6,634.26\cdot\frac{0.0824}{0.07947} = 6,881.32$$

Note: If this had been a continuously payable annuity (paying at a rate of 120 per year the first year, 240 per year the second year, etc.), the adjustment factor would have been i/δ *:*

$$120\left(I\bar{a}\right)_{\overline{15}|8.24\%} = \left[120\left(Ia\right)_{\overline{15}|8.24\%}\right]\cdot\frac{i}{\delta} = 6,634.26\cdot\frac{0.0824}{\ln\left(1.0824\right)} = 6,904.08$$

This value is larger than the answer based on monthly payments, because the continuous payments occur, not at the end of each month, but throughout the month. This means that, on average, they occur one-half month earlier. Not surprisingly, the value of the continuous payments is larger than that of the monthly payments by a factor of about $1.0824^{1/24}$.

Exercise (2.108)

A 10-year annuity-immediate makes quarterly payments of 100 in year 1, 90 in year 2, decreasing to 10 per quarter in year 10. Find the present value of this annuity at an interest rate of 6% convertible monthly.

Answer: 1,789.75

Example (2.109)

A 12-year annuity-due makes monthly payments of 100 in year 1, 98 in year 2, and the monthly payments continue to decrease by 2% each year during the 12-year term of the annuity. Find the present value of this annuity at an interest rate of 5% convertible semi-annually.

Solution.

Because payments are monthly and the decreases are annual, we will first calculate the annuity's value based on annual payments, and then adjust that value to reflect monthly payments.

The semi-annual effective rate is $5\% / 2 = 2.5\%$, so the annual effective rate is:

$$1.025^2 - 1 = 5.0625\%$$

The present value for a geometric annuity-due based on annual payments (1,200 in the first year, decreasing 2% each year) is calculated here using the artificial interest rate method:

$$1 + j = \frac{1+i}{1+g} = \frac{1.050625}{0.98} = 1.072066$$

$$1200 \cdot \ddot{a}^{-2\%}_{\overline{12}|5.0625\%} = 1200 \cdot \ddot{a}_{\overline{12}|7.2066} = 1200 \cdot \frac{1 - 1.072066^{-12}}{\frac{0.072066}{1.072066}} = 10{,}106.52$$

To reflect monthly payments, we adjust this result by a factor of $d / d^{(12)}$.

(Note that we could have calculated the value of an annuity-*immediate* with annual payments and multiplied by $i / d^{(12)}$. Because we have calculated the value of an annual-payment annuity-*due*, we apply a factor of $d / d^{(12)}$ to adjust to a *monthly* annuity-due.)

$$\left(1 - \frac{d^{(12)}}{12}\right)^{-12} = 1 + i = 1.050625 \qquad d^{(12)} = 0.049284$$

$$d = \frac{i}{1+i} = \frac{0.050625}{1.050625} = 0.048186$$

The value of this geometric decreasing annuity-due with monthly payments is:

$$1{,}200 \cdot \ddot{a}^{-2\%}_{\overline{12}|5.0625\%} \cdot \frac{d}{d^{(12)}} = 10{,}106.52 \cdot \frac{0.048186}{0.049284} = 9{,}881.33$$

Note: Changing from annual payments to monthly payments decreases the value of an annuity-due, because the payments are being spread throughout the year instead of occurring at the beginning of the year. With an annuity-immediate, the opposite occurs: spreading the payments throughout the year increases the annuity's value compared to an annuity that doesn't make a payment until the end of the year.

Exercise (2.110)

A 10-year annuity-immediate makes semi-annual payments of 1,000 in year 1, 1,050 in year 2, and the semi-annual payments continue to increase by 5% each year during the 10-year term of the annuity. Find the *future* value of this annuity (at time 10) at an interest rate of 6% convertible quarterly.

Answer: 33,074.43

Section 2.18

Continuously Payable Annuities With Continuously Varying Payments

In the previous section, we solved problems that involved annuities with payments that increase or decrease annually, but where the payments were made more frequently than annual (e.g., Example **(2.107)** had monthly payments that increase once a year). Example **(2.107)** included a note regarding the adjustment to convert from annual to continuous payments (i.e., multiply by i/δ). This adjustment factor handles annuities with continuous payments where the payment rate is adjusted *annually*. Now we will address situations where a continuously payable annuity's rate of payment changes *continuously*.

At the end of this section, we will examine the situation where both the rate of payment and the interest rate (the force of interest) are changing continuously. This is the most complex form of annuity that we will consider. While such annuities are neither common nor practical, they have appeared on exams (such as Sample Exam Problem number 8 at the end of this module).

Arithmetic Annuities

The first situation we will consider is the continuously increasing annuity. The symbol for the present value of this annuity is $(\bar{I}\bar{a})_{\overline{n|}}$. The bar over the a indicates that payments are continuous. The bar over the I indicates that the *increases* in the rate of payment also occur continuously. The payment rate at the end of the first year is 1 per year, but at the beginning of the first year (at time 0), the payment rate is 0. It builds up from 0 at time 0 to 1 at time 1, and continues to increase, reaching a rate of n per year at time n. At any time t, the payment rate is t per year, so the amount paid during a small time interval dt is $(t \cdot dt)$, and the present value of that amount at time 0 is $(t \cdot v^t \cdot dt)$. To find the present value of all the payments from time 0 to time n, we integrate this value from 0 to n:

$$
\begin{aligned}
(\bar{I}\bar{a})_{\overline{n|}} &= \int_{t=0}^{n} t \cdot v^t \cdot dt = \left[\frac{t \cdot v^t}{-\delta} - \frac{v^t}{-\delta^2} \right]\Bigg|_{t=0}^{n} \\
&= \left[\frac{n \cdot v^n}{-\delta} + \frac{v^n}{-\delta^2} \right] - \left[\frac{0 \cdot v^0}{-\delta} + \frac{v^0}{-\delta^2} \right] \\
&= \frac{\dfrac{1 - v^n}{\delta} - n \cdot v^n}{\delta} = \frac{\bar{a}_{\overline{n|}} - n \cdot v^n}{\delta}
\end{aligned}
$$

This gives us the formula for the present value of a continuously increasing arithmetic annuity:

(2.111)

$$\left(\bar{I}\bar{a}\right)_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n \cdot v^n}{\delta}$$

This formula is very similar to Formula (2.41) for an arithmetic increasing annuity-immediate: $\left(Ia\right)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i}$. The differences are that the continuously increasing annuity's denominator is δ (because *payments* are continuous rather than annual) and the numerator has $\bar{a}_{\overline{n}|}$ instead of $\ddot{a}_{\overline{n}|}$ (because the *increases* are continuous rather than annual). Recognizing the similarity to the formula for $\left(Ia\right)_{\overline{n}|}$ makes this formula easier to remember. In the case of a continuously increasing perpetuity, we have:

(2.112)

$$\left(\bar{I}\bar{a}\right)_{\overline{\infty}|} = \frac{1}{\delta^2}$$

A continuously decreasing annuity $\left(\bar{D}\bar{a}\right)_{\overline{n}|}$ makes continuous payments at a rate that decreases linearly from n per year at time 0 to a rate of 0 at time n. We can develop a formula for $\left(\bar{D}\bar{a}\right)_{\overline{n}|}$ using the fact that $\left(\bar{I}\bar{a}\right)_{\overline{n}|} + \left(\bar{D}\bar{a}\right)_{\overline{n}|} = n \cdot \bar{a}_{\overline{n}|}$:

(2.113)

$$\left(\bar{D}\bar{a}\right)_{\overline{n}|} = n \cdot \bar{a}_{\overline{n}|} - \left(\bar{I}\bar{a}\right)_{\overline{n}|} = n \cdot \frac{1 - v^n}{\delta} - \frac{\bar{a}_{\overline{n}|} - n \cdot v^n}{\delta} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$$

Again, we can see that this is very similar to Formula (2.54) for a decreasing annuity-immediate: $\left(Da\right)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$ but with the continuous functions $\bar{a}_{\overline{n}|}$ and δ.

Formulas for the future values of these annuities are also similar to the corresponding formulas for annual-payment increasing and decreasing annuities:

(2.114)

$$\left(\bar{I}\bar{s}\right)_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}$$

(2.115)

$$\left(\bar{D}\bar{s}\right)_{\overline{n}|} = \frac{n \cdot \left(1 + i\right)^n - \bar{s}_{\overline{n}|}}{\delta}$$

Example (2.116)

An annuity provides continuous payments at the rate of 500 per year at time 0, and its payment rate increases continuously at a rate of 100 per year each year for 10 years. (For example, at time 0.5, the payment rate is 550 per year (the original 500 per year, plus an increase of 50 per year for the half-year that has elapsed).) Calculate the accumulated value of this annuity at the end of 10 years, assuming an annual effective interest rate of 10%.

Solution.
We will treat the original payment rate of 500 per year as a level continuously payable annuity, and treat the increasing portion as a separate continuously increasing annuity. The accumulated value is then:

$$500 \cdot \bar{s}_{\overline{10}|} + 100 \cdot \left(\bar{I}\bar{s}\right)_{\overline{10}|} = 500 \cdot \frac{1.10^{10} - 1}{\ln 1.10} + 100 \cdot \frac{\frac{1.10^{10} - 1}{\ln 1.10} - 10}{\ln 1.10} = 15,413.20$$

Exercise (2.117)

Calculate the present value at an annual effective rate of 8% of a 10-year continuously increasing arithmetic annuity. The annuity's payment rate begins at 0 and increases continuously at a rate of 400 per year.

Answer: 12,178.15

Note: The continuously increasing unit annuity makes total payments of only 0.5 during the first year, 1.5 during the second year, etc. Its annual payment rate is $(t-1)$ *at the beginning of year t*, and t at the end of year t. So its average rate during year *t is* $(t-0.5)$, *and the total amount of payments during year t is also* $(t-0.5)$. *For an increasing annuity-immediate or annuity-due, the payment amount during year t would be t, so their values are generally larger than those of a continuously increasing annuity with the same term.*

Geometric Annuities

A continuously payable geometric annuity can also have a continuously changing rate of payment. The present value of such an annuity can be represented by the symbol $\bar{a}_{\overline{n}|}^{\bar{g}}$, where $\bar{g}$ is the *continuously compounded* rate of change in the payment rate. This means that the payment rate at time t is $e^{\bar{g}t}$. We can develop a formula for $\bar{a}_{\overline{n}|}^{\bar{g}}$ by integration:

(2.118)

$$\bar{a}_{\overline{n}|}^{\bar{g}} = \int_{t=0}^{n} e^{\bar{g}t} \cdot e^{-\delta t} \cdot dt = \int_{t=0}^{n} e^{-(\delta-\bar{g})t} \cdot dt = \bar{a}_{\overline{n}|(\delta-\bar{g})} = \frac{1-e^{-n\cdot(\delta-\bar{g})}}{\delta-\bar{g}}$$

The present value of the annuity is equal to the present value of a level continuously payable annuity, where the present value is calculated at a force of interest equal to $(\delta - \bar{g})$. This is similar to the "artificial interest rate" method for finding the value of a geometric annuity-due.

In the case of a continuously increasing (or decreasing) perpetuity, we have:

(2.119)

$$\bar{a}^{\bar{g}}_{\overline{\infty}|} = \frac{1}{\delta - \bar{g}} \qquad \delta > \bar{g}$$

The *future* value of a continuously varying geometric annuity is found by accumulating the present value to time *n* at the force of interest, δ:

(2.120)

$$\bar{s}^{\bar{g}}_{\overline{n}|} = \bar{a}^{\bar{g}}_{\overline{n}|} \cdot e^{n\delta} = \frac{e^{n\delta} - e^{n\cdot\bar{g}}}{\delta - \bar{g}}$$

Continuously Varying Payments at a Varying Force of Interest

In some exam problems, the pattern of continuously varying payments is not arithmetic or geometric, and the interest rate also varies continuously. In this situation, the present value of the annuity must be expressed as an integral:

(2.121)

For an annuity with a continuously varying rate of payment, $\rho(t)$, and a continuously varying force of interest, $\delta(t)$:

$$PV = \int_{t=0}^{n} \rho(t) \cdot e^{-\int_{u=0}^{t} \delta(u)\cdot du} \cdot dt$$

Here $\rho(t)$ is a rate of payment per year that varies as a function of t. For example, if the payment rate begins at 100/year and increases (continuously) at a rate of 50/year per year (technically, that would be 50/year^2), then $\rho(t) = 100 + 50 \cdot t$.

There is no general solution for the integral in **(2.121)**, as it depends on the forms of $\rho(t)$ and $\delta(t)$. The following example illustrates how problems of this type are solved.

Example (2.122)

A 5-year annuity makes continuous payments at a rate of $(1{,}000+50t)$ per year. Calculate the present value of this annuity at a continuously varying force of interest $\delta(t)=\dfrac{1}{20+t}$.

Solution.
The present value of this annuity can be written as an integral:

$$PV=\int_{t=0}^{n}\rho(t)\cdot e^{-\int_{u=0}^{t}\delta(u)\cdot du}\cdot dt=\int_{t=0}^{5}(1{,}000+50t)\cdot e^{-\int_{u=0}^{t}\frac{1}{20+u}\cdot du}\cdot dt$$

The present value factor in this expression is $e^{-\int_{u=0}^{t}\frac{1}{20+u}\cdot du}$. The integral can be evaluated as follows:

$$-\int_{u=0}^{t}\frac{1}{20+u}\cdot du=-\ln(20+u)\Big|_{u=0}^{t}=-\ln\left(\frac{20+t}{20}\right)=-\ln(1+0.05t)$$

The present value factor is $e^{-\ln(1+0.05t)}=\dfrac{1}{1+0.05t}$, and the present value of the annuity is:

$$PV=\int_{t=0}^{5}(1{,}000+50t)\cdot\frac{1}{1+0.05t}\cdot dt=\int_{t=0}^{5}1{,}000\cdot dt=5{,}000$$

Note: This example is typical of the type of problem that would appear on an exam, in that it initially appears very complex, but it simplifies nicely.

Section 2.19

Reinvestment Problems

In some cases where an investment is providing interest payments, the investor might reinvest the payments as they are received. The account in which the money is reinvested might earn a different interest rate than the original investment. The following examples illustrate first a basic reinvestment problem, and then a more complex problem of a type that has appeared on exams.

Example (2.123)

You lend a relative 1,000 and he agrees to pay you 6% interest on the original 1,000 at the end of every year for 10 years and then return the 1,000. You can reinvest the interest payments at 5%. How much will you have in total in 10 years? What overall rate of interest will you have earned on your investment of 1,000?

Solution.

At the end of 10 years you will have:

a) the return of the original 1,000
b) the future value of 10 payments of 60
(60 is one year's interest at 6% on 1,000).
Set N=10, I/Y=5, PMT=-60, and CPT FV = 754.67

The total is 1,000+754.67 = 1,754.67.

To find the overall interest rate earned, note that you invested 1,000 and have a total of 1,754.67 after 10 years.
Set N=10, PV = -1,000, FV = 1754.67, and CPT I/Y = 5.78.
Or simply calculate that your investment has grown by a factor of:
$\frac{1,754.67}{1,000} = 1.75467$, and calculate the 10th root: $1.75467^{0.1} = 1.0578$.
You have earned an interest rate of 5.78%.
It is reasonable that your rate is between the interest rates for the two components of your investment (5% and 6%). The overall rate (5.78%) is closer to the 6% rate at which most of the money was invested for the entire 10 years.

Exercise (2.124)

How much would you have in **(2.123)** if your reinvestment rate was 4%, and what would be your overall interest rate for the 10 years?

Answer: 1,720.37 and 5.58%

Example (2.125)

You deposit 1,000 into Account A at the beginning of each year for 5 years, earning an annual effective rate of 10%. However, the interest earned in Account A each year must be reinvested in Account B at 8%. How much will you have at the end of 5 years?

Solution.
The following table shows the relevant payments.

Time	0	1	2	3	4	5
Payment deposited to Account A	1,000	1,000	1,000	1,000	1,000	0
Total deposits to date into Account A	1,000	2,000	3,000	4,000	5,000	5,000
Deposits into Account B (= Interest on Account A)	0	100	200	300	400	500

Note that the numbers in the last row (the deposits to the 8% account), represent an arithmetic annuity-immediate. At the end of 5 years you will have 5,000 in Account A, plus the amount in Account B. The total will be:

$$5,000 + 100(Is)_{\overline{5}|0.08} = 5,000 + 100(16.6991) = 6,669.91$$

Note that the pattern for the formula in this type of problem is:
(# of pmts)(amt per pmt)+(amt per pmt)(int rate on pmts)$(Is)_{\overline{n}|}$, where $(Is)_{\overline{n}|}$ is calculated at the reinvestment rate.

(This is based on beginning-of-year deposits to the first account. If deposits are made at the end of each year, $(Is)_{\overline{n}|}$ must be changed to $(Is)_{\overline{n-1}|}$, since the deposits to the second account don't begin until the end of the second year.)

Exercise (2.126)

How much would you have at the end of 5 years in **(2.125)** if Account A earned 9% and Account B earned 6%?

Answer: 6,642.98

Section 2.20

Inflation

Price inflation is a frequent topic in our daily news. People often worry about increases in the price of gasoline and food. Inflation also affects the interest rates lenders charge; if lenders expect high inflation, they will raise the interest rates charged to borrowers. In Module 8 we will study how market interest rates are affected by inflation. In this module, we will simply introduce the basic concepts, including the idea of "real" and "nominal" rates of return.

We will start our discussion of inflation with a simple example of the effect of inflation on purchasing power. Suppose that you like to have wine with dinner, and you buy an annual supply of 52 liters of wine (one per week) each January. If the price today is $10 per liter, you will spend $520 this January. If you want to put money aside for next year's purchase, you might decide to invest another $520 to provide for a wine purchase next year. If the current interest rate is 5%, in one year you will have.

$$\$520(1.05) = \$546$$

You hope that this investment will enable you to buy more wine next year than this year, but inflation has to be considered. Suppose that next January, due to inflation, the price of wine has increased by 3% to $10.30 per liter. Then the number of liters you can buy next January is:

$$\frac{546}{10.30} \approx 53.01 \text{ liters}$$

This certainly gives you more wine, but not 5% more. The increase in the amount of wine purchased is:

$$\frac{53.01}{52} - 1 \approx 0.0194$$

Thus your 5% investment gives you only a 1.94% "real" increase in purchasing power.

We will denote the 5% rate at which you can invest by i. This is called the **nominal rate**. It is important to recognize that this use of the word "nominal" is different from our other use of that word when we speak of "nominal annual rates" of interest or discount. In each case, the word "nominal" means that this is *not the actual rate* that is being earned. In the case of a nominal annual interest rate convertible quarterly (for example), it is not the actual annual rate because it doesn't reflect the compounding of interest within the year. In this section, we are talking about a nominal rate that is affected by inflation. The result is that our "nominal dollars" grow at the nominal interest rate, but our *purchasing power* grows more slowly due to the effect of inflation. So the nominal interest rate is not the *real* rate at which our purchasing power increases.

We will denote the annual **inflation rate** by r. The rate at which the purchasing power of an investment increases is referred to as the **real rate** of interest, and it will be denoted by j. In general:

$$(1+i)=(1+j)(1+r) \quad \text{or} \quad (1+j)=\frac{(1+i)}{(1+r)}$$

In practice, you will generally know i and r and will solve for the real rate of return j using the second equation above.

Note: In Module 8, our in-depth examination of inflation and interest rates will use different notation that matches the notation in the study note on Determination of Interest Rates. In that notation, "i" is a rate of inflation and "r" is a rate of interest (Both are continuously compounded rates.) Since "i" is an interest rate in all modules other than Module 8, "i" is used here to represent an annual effective interest rate.

The first equation gives us the relation $j=i-r-j\cdot r$. It is common to omit the $j\cdot r$ term and approximate the real rate of return by $j\approx i-r$.

In our first example above, an analyst might say that the real rate of return is $5\%-3\%=2\%$, but the real rate is truly 1.94%. The difference between 1.94% and 2% is relatively small, but the approximation can work badly in countries that experience hyperinflation and high interest rates. For example if $i=0.50$ and $r=0.20$, then:

$$i-r=0.30, \text{ but } j=\frac{1.5}{1.2}-1\approx 0.25 .$$

Inflation rates are used in two different ways:

1. The first is the *historical* inflation rate, **reflecting what has already happened**. In the wine-buying example, we found that with a nominal interest rate of 5% and an actual past inflation rate of 3%, we had a real increase in purchasing power of 1.94%.

2. The second is a *projected* inflation rate, which is **used to make decisions about the future**.

   Suppose that you are a lender who wants to earn a real rate of return of 3% over the next year and you believe that inflation will be 2%. Then you will want to lend at the nominal rate i defined by:

   $$1+i=1.03(1.02)=1.0506 \text{ or } i=5.06\% .$$

   In one year you can look back at the actual inflation rate and see whether you actually did earn the intended 3% real rate of return.

Section 2.21

Formula Sheet

Geometric Series

$$1+r+r^2+\ldots+r^{n-1}=\frac{1-r^n}{1-r}=\frac{r^n-1}{r-1} \quad r\neq 1 \qquad 1+r+r^2+\ldots=\frac{1}{1-r} \quad |r|<1$$

Annuities

Level Annuities:

Immediate $\quad a_{\overline{n}|}=\dfrac{1-v^n}{i} \qquad s_{\overline{n}|}=(1+i)^n\cdot a_{\overline{n}|}=\dfrac{(1+i)^n-1}{i} \qquad a_{\overline{n}|}=v^n\cdot s_{\overline{n}|}$

Due $\quad \ddot{a}_{\overline{n}|}=\dfrac{1-v^n}{d} \qquad \ddot{s}_{\overline{n}|}=(1+i)^n\cdot \ddot{a}_{\overline{n}|}=\dfrac{(1+i)^n-1}{d} \qquad \ddot{a}_{\overline{n}|}=v^n\cdot \ddot{s}_{\overline{n}|}$

Payable m-thly $\quad a_{\overline{n}|}^{(m)}=\dfrac{1-v^n}{i^{(m)}}=\dfrac{i}{i^{(m)}}\cdot a_{\overline{n}|}$

$$s_{\overline{n}|}^{(m)}=(1+i)^n\cdot a_{\overline{n}|}^{(m)}=\frac{(1+i)^n-1}{i^{(m)}}=\frac{i}{i^{(m)}}\cdot s_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|}^{(m)}=\frac{1-v^n}{d^{(m)}}=\frac{i}{d^{(m)}}\cdot a_{\overline{n}|}=\left(1+\frac{i^{(m)}}{m}\right)\cdot a_{\overline{n}|}^{(m)}$$

$$\ddot{s}_{\overline{n}|}^{(m)}=\frac{(1+i)^n-1}{d^{(m)}}=\frac{i}{d^{(m)}}\cdot s_{\overline{n}|}=\left(1+\frac{i^{(m)}}{m}\right)\cdot s_{\overline{n}|}^{(m)}$$

Continuously payable $\quad \bar{a}_{\overline{n}|}=\dfrac{1-v^n}{\delta}=\dfrac{i}{\delta}\cdot a_{\overline{n}|} \qquad \bar{s}_{\overline{n}|}=\dfrac{(1+i)^n-1}{\delta}=\dfrac{i}{\delta}\cdot s_{\overline{n}|}$

Perpetuities $\quad a_{\overline{\infty}|}=v+v^2+v^3+\ldots=\dfrac{1}{i} \qquad \ddot{a}_{\overline{\infty}|}=1+v+v^2+\ldots=\dfrac{1}{d} \qquad \bar{a}_{\overline{\infty}|}=\dfrac{1}{\delta}$

Deferred $\quad {}_{k|}a_{\overline{n}|}=v^k\cdot a_{\overline{n}|} \qquad {}_{k|}a_{\overline{n}|}=a_{\overline{n+k}|}-a_{\overline{k}|}$

Relations $\quad \ddot{a}_{\overline{n}|}=\dfrac{i}{d}\cdot a_{\overline{n}|}=(1+i)a_{\overline{n}|} \qquad \ddot{s}_{\overline{n}|}=\dfrac{i}{d}\cdot s_{\overline{n}|}=(1+i)s_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|}=a_{\overline{n-1}|}+1 \qquad s_{\overline{n}|}=\ddot{s}_{\overline{n-1}|}+1$$

$$\ddot{a}_{\overline{\infty}|}=a_{\overline{\infty}|}+1=(1+i)\cdot a_{\overline{\infty}|}$$

$$a_{\overline{k}|}=a_{\overline{n+k}|}-{}_{k|}a_{\overline{n}|} \qquad a_{\overline{n}|}=a_{\overline{\infty}|}-{}_{n|}a_{\overline{\infty}|}=\frac{1}{i}-v^n\cdot\frac{1}{i}=\frac{1-v^n}{i}$$

Arithmetic Annuities:

Increasing: Payments are 1, 2, . . ., n

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (I\ddot{a})_{\overline{n}|} = (1+i)\cdot(Ia)_{\overline{n}|} = \frac{i}{d}\cdot(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

$$(Is)_{\overline{n}|} = (1+i)^n\cdot(Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} \qquad (I\ddot{s})_{\overline{n}|} = (1+i)^n\cdot(I\ddot{a})_{\overline{n}|} = \frac{i}{d}\cdot(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d}$$

$$(I\bar{a})_{\overline{n}|} = \frac{i}{\delta}\cdot(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\cdot v^n}{\delta} \qquad (I\bar{s})_{\overline{n}|} = (1+i)^n\cdot(I\bar{a})_{\overline{n}|} = \frac{i}{\delta}\cdot(Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta}$$

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n\cdot v^n}{\delta} \qquad (\bar{I}\bar{s})_{\overline{n}|} = (1+i)^n\cdot(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta}$$

Increasing perpetuity $\quad (Ia)_{\overline{\infty}|} = \frac{1}{id} \quad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2} \quad (I\bar{a})_{\overline{\infty}|} = \frac{1}{\delta d} \quad (\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$

Decreasing: Payments are n, n-1,…, 2, 1

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} = (1+i)(Da)_{\overline{n}|}$$

$$(Ds)_{\overline{n}|} = (1+i)^n(Da)_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{i}$$

$$(D\ddot{s})_{\overline{n}|} = (1+i)^n(D\ddot{a})_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{d} = \frac{i}{d}\cdot(Ds)_{\overline{n}|}$$

$$(D\bar{a})_{\overline{n}|} = \frac{i}{\delta}\cdot(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{\delta} \qquad (D\bar{s})_{\overline{n}|} = (1+i)^n(D\bar{a})_{\overline{n}|} = \frac{n\cdot(1+i)^n - s_{\overline{n}|}}{\delta}$$

$$(\bar{D}\bar{a})_{\overline{n}|} = \frac{n - \bar{a}_{\overline{n}|}}{\delta} \qquad (\bar{D}\bar{s})_{\overline{n}|} = (1+i)^n(\bar{D}\bar{a})_{\overline{n}|} = \frac{n\cdot(1+i)^n - \bar{s}_{\overline{n}|}}{\delta}$$

***PQ* Formula** for Arithmetic Annuities-Immediate:

Payments are $P, (P+Q), (P+2Q), \ldots, (P+(n-1)Q)$

$$PV = P\cdot a_{\overline{n}|} + Q\cdot\left(\frac{a_{\overline{n}|} - nv^n}{i}\right) \qquad FV = P\cdot s_{\overline{n}|} + Q\cdot\left(\frac{s_{\overline{n}|} - n}{i}\right)$$

For perpetuities-immediate (infinite n): $\quad PV = \frac{P}{i} + \frac{Q}{i^2}$

Geometric Annuities:

Payments are $1, (1+g), (1+g)^2, \ldots, (1+g)^{n-1}$ $\qquad g \neq i$

$$a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \qquad \ddot{a}^{g}_{\overline{n}|i} = (1+i)\cdot a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{d-v\cdot g} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{\frac{i-g}{1+i}}$$

$$s^{g}_{\overline{n}|i} = \frac{(1+i)^n-(1+g)^n}{i-g} \qquad \ddot{s}^{g}_{\overline{n}|i} = (1+i)\cdot s^{g}_{\overline{n}|i} = \frac{(1+i)^n-(1+g)^n}{d-v\cdot g} = \frac{(1+i)^n-(1+g)^n}{\frac{i-g}{1+i}}$$

$$\bar{a}^{\bar{g}}_{\overline{n}|} = \frac{1-e^{-n\cdot(\delta-\bar{g})}}{\delta-\bar{g}} \qquad \bar{s}^{\bar{g}}_{\overline{n}|} = \bar{a}^{\bar{g}}_{\overline{n}|}\cdot e^{n\delta} = \frac{e^{n\delta}-e^{n\cdot\bar{g}}}{\delta-\bar{g}}$$

$$\ddot{a}^{g}_{\overline{n}|i} = \ddot{a}_{\overline{n}|j} \quad \text{where } 1+j = \frac{1+i}{1+g}$$

Geometric perpetuity $\quad (i>g)$:

$$a^{g}_{\overline{\infty}|i} = \frac{1}{i-g} \qquad \ddot{a}^{g}_{\overline{\infty}|i} = (1+i)\cdot a^{g}_{\overline{\infty}|i} = \frac{1}{d-v\cdot g} = \frac{1+i}{i-g} \qquad \bar{a}^{\bar{g}}_{\overline{\infty}|i} = \frac{1}{\delta-\bar{g}}$$

Varying the Payment Frequency:

Converting from an annuity-immediate with annual payments (at end of year) to an annuity with a different timing or frequency of payments:

To convert to:	Multiply by:	Example
Annuity-due (beginning-of-year pmts)	$\frac{i}{d}$	$\ddot{a}_{\overline{n}|} = \frac{i}{d}\cdot a_{\overline{n}|}$
Continuous payments	$\frac{i}{\delta}$	$\bar{s}_{\overline{n}|} = \frac{i}{\delta}\cdot s_{\overline{n}|}$
m-thly annuity-immediate (payments at end)	$\frac{i}{i^{(m)}}$	$(Ia)^{(m)}_{\overline{\infty}|} = \frac{i}{i^{(m)}}\cdot (Ia)_{\overline{\infty}|}$
m-thly annuity-due (payments at beginning)	$\frac{i}{d^{(m)}}$	$\ddot{s}^{g\,(m)}_{\overline{n}|} = \frac{i}{d^{(m)}}\cdot s^{g}_{\overline{n}|}$

Note: In each case, we begin with a unit annuity-immediate. That is, it is an annuity-immediate that pays 1 at the end of each year (if it is a level annuity), or it pays 1 at the end of the first year (if it is a geometric annuity or an arithmetic increasing annuity) or it pays n at the end of the first year and 1 at the end of the n^{th} year (if it is an arithmetic decreasing annuity). And the total payments each year for the converted annuity are the same as for the original annuity (e.g., for a level unit annuity, it could be monthly payments of 1/12 each month, or it could be continuous payments at a rate of 1 per year).

Section 2.22

Basic Review Problems

1. Find $a_{\overline{25}|}$, $s_{\overline{25}|}$, $\ddot{a}_{\overline{20}|}$, $\bar{a}_{\overline{20}|}$, and $a_{\overline{\infty}|}$, all at a 6% annual effective interest rate.

2. A loan for 8,000 at an annual effective rate of 11% is to be repaid with 6 year-end payments. What is the annual payment?

3. You wish to make a deposit now into an account earning 6% annually so that you can make a withdrawal of 250 at the end of each of the next 8 years. How much should you deposit today?

4. You have an account that earns interest at a 5% annual effective rate. You plan to make a level deposit at the beginning of each of the next 9 years so that your balance at the end of 9 years is 12,000. Find the required level deposit.

5. You have borrowed 10,000 and agreed to repay the loan with 5 annual payments of 2,500. What annual effective interest rate are you paying?

6. For $i = 0.06$, find $(Ia)_{\overline{15}|}$, $(I\bar{a})_{\overline{15}|}$, $(Is)_{\overline{15}|}$, $(\bar{I}\,\bar{s})_{\overline{15}|}$, and $(D\ddot{a})_{\overline{15}|}$.

7. Given $i = 8\%$, find the present value of a perpetuity with annual payments of $1.04, (1.04)^2, \ldots, (1.04)^n, \ldots$:

   a) if it is a perpetuity-immediate, and
   b) if it is a perpetuity-due

8. An annuity pays 100 at the end of each of the next 5 years and 300 at the end of each of the 5 following years. At an annual effective interest rate of 6%, what is the present value of the annuity?

9. An annual-payment annuity-immediate has a first payment of 200, and payments increase by 100 each year until they reach 600. There are then 5 additional level payments of 600. Find the present value of this annuity at a 5.5% annual effective rate.

10. An annuity-immediate has quarterly payments of 20 for 10 years, followed by a perpetuity of quarterly payments of 25 starting in the 11th year. Find the present value at 4% convertible quarterly.

11. An annuity-immediate has semi-annual payments of 1,000 for 25 years. Find its present value at an interest rate of 6% convertible quarterly.

12. An annuity-immediate has quarterly payments of 500 for 6 years. Find its present value at an interest rate of 4% convertible semi-annually.

13. You lend 10,000 and the borrower agrees to pay 8% interest at the end of each year for 5 years and then return the 10,000. You will reinvest the interest payments at 6%.
    a) How much will you have in total in 5 years?
    b) What annual effective rate did you earn over the 5 years on your investment of 10,000?

14. You deposit 2,000 into an account at the beginning of each year for 8 years. The account earns interest at an annual effective rate of 8%, but the interest received from this account must be reinvested at a 5% annual effective rate. How much will you have at the end of 8 years?

Section 2.23

Basic Review Problem Solutions

The solutions provided are based on the formulas and methods found in the text, with only occasional use of the BA II Plus's financial functions. For many of the problems, some or all of the calculations can also be performed using the calculator's TVM worksheet. (The required keystrokes can be found in similar problems in the text.)

1. $a_{\overline{25|}} = \dfrac{1-1.06^{-25}}{0.06} = 12.78$

   $s_{\overline{25|}} = \dfrac{1.06^{25}-1}{0.06} = 54.86$, which equals $1.06^{25} \cdot a_{\overline{25|}}$

   $\ddot{a}_{\overline{20|}} = \dfrac{1-1.06^{-20}}{0.06/1.06} = 12.16$

   $\bar{a}_{\overline{20|}} = \dfrac{1-1.06^{-20}}{\delta} = \dfrac{0.688195}{\ln 1.06} = 11.81$

   $a_{\overline{\infty|}} = \dfrac{1}{0.06} = 16.67$

   Note that the Rule of 72 can be used to check the reasonableness of annuity calculations. For example, $a_{\overline{25|}}$ *involves 25 payments of 1. It is a present value, so it will be less than 25. How much less? Well, on average the payments are made about 13 years from now. Since an investment at 6% doubles in 12 years, a payment of 1 in 13 years is worth about 0.5 now. If there are 25 payments that (on average) are payable in about 13 years, then their present value should be about 12.50. So the answer of 12.78 is reasonable.*

   We could also apply the rule of 72 to the *formula* *for* $a_{\overline{25|}}$: $\dfrac{1-1.06^{-25}}{0.06}$. *In the numerator, the value of* 1.06^{-25} *equals about 0.25 (since an investment at 6% doubles about twice in 25 years), so the value of the annuity is approximately* $\dfrac{1-0.25}{0.06} = 12.50$.

2. Start with the equation of value: $8{,}000 = P \cdot a_{\overline{6|}11\%} = P \cdot \dfrac{1-1.11^{-6}}{0.11}$

   Then solve for *P*. *P* = 1,891.01

3. The equation of value is: $X = 250 \cdot a_{\overline{8}|6\%} = 250 \cdot \dfrac{1-1.06^{-8}}{0.06}$

   $X = 1{,}552.45$

4. $12{,}000 = X \cdot \ddot{s}_{\overline{9}|5\%} = X \cdot \dfrac{1.05^9 - 1}{0.05/1.05}$

   $X = 1{,}036.46$

5. The equation of value for this loan is: $10{,}000 = 2{,}500 \cdot a_{\overline{5}|i} = 2{,}500 \cdot \dfrac{1-(1+i)^{-5}}{i}$.

   However, there is no direct method to solve for i, so we will use the TVM worksheet:

   5 [N] 10,000 [PV] -2500 [PMT] 0 [FV] [CPT] [I/Y] → 7.93

   $i = 7.93\%$

   If you got an error when you computed I/Y, it is probably because you did not make PMT negative. (The total value of the cash flows has to be 0, so PV and PMT can't both be positive.)

6. $$(Ia)_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15 \cdot v^{15}}{i} = \frac{\dfrac{1-1.06^{-15}}{0.06/1.06} - \dfrac{15}{1.06^{15}}}{0.06} = 67.27$$

   $$(I\bar{a})_{\overline{15}|} = (Ia)_{\overline{15}|} \cdot \frac{i}{\delta} = 67.27 \cdot \frac{0.06}{\ln 1.06} = 69.27$$

   $$(Is)_{\overline{15}|} = \frac{\ddot{s}_{\overline{15}|} - 15}{i} = (Ia)_{\overline{15}|} \cdot (1+i)^{15} = 67.27 \cdot 1.06^{15} = 161.21$$

   $$(\overline{Is})_{\overline{15}|} = \frac{\bar{s}_{\overline{15}|} - 15}{\delta} = \frac{\dfrac{(1+i)^{15}-1}{\delta} - 15}{\delta} = \frac{\dfrac{(1.06)^{15}-1}{\ln 1.06} - 15}{\ln 1.06} = 153.90$$

   $$(D\ddot{a})_{\overline{15}|} = \frac{15 - a_{\overline{15}|}}{d} = \frac{15 - \dfrac{1-1.06^{-15}}{0.06}}{0.06/1.06} = 93.42$$

7. a) For the perpetuity-immediate (payments at end of period):

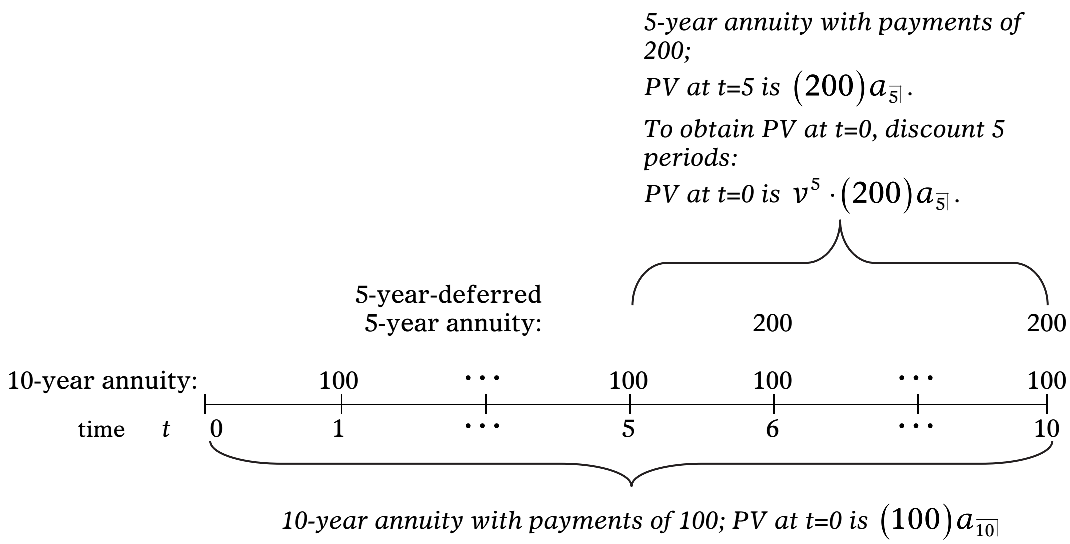
   $$PV = 1.04 \cdot a^{4\%}_{\overline{\infty}|8\%} = 1.04 \cdot \frac{1}{i-g} = 1.04 \cdot \frac{1}{0.08-0.04} = 26$$

   b) For the perpetuity-due (payments at beginning of period):

   $$PV = 1.04 \cdot \ddot{a}^{4\%}_{\overline{\infty}|8\%} = 1.04 \cdot \frac{1}{d - v \cdot g} = 1.04 \cdot \frac{1}{\dfrac{0.08}{1.08} - \dfrac{0.04}{1.08}} = 28.08$$

   More simply, $PV_{DUE} = PV_{IMMEDIATE} \times (1+i) = 26 \times 1.08 = 28.08$

8. We can break this annuity into two parts: a 10-year annuity with annual payments of 100, and a 5-year-deferred 5-year annuity with annual payments of 200.



The total present value is:

$$(100)a_{\overline{10}|} + v^5(200)a_{\overline{5}|} = 736.01 + (0.7473)(842.47) = 1{,}365.59$$

9. Consider the payment streams as three separate annuities:

Total payment:		**200**	**300**	**. . .**	**600**	**600**	**600**	**. . .**	**600**
Which equals:									
$100a_{\overline{5}\vert}$		100	100	. . .	100				
$+100(Ia)_{\overline{5}\vert}$		100	200	. . .	500				
$+v^5(600)a_{\overline{5}\vert}$						600	600	. . .	600
time t	0	1	2	. . .	5	6	7	. . .	10

The present value of this annuity is:

$$PV = 100a_{\overline{5}|} + 100(Ia)_{\overline{5}|} + v^5 600a_{\overline{5}|}$$
$$= 100(4.270) + 100(12.3542) + 0.7651(600)(4.270)$$
$$= 3,622.85$$

Alternatively, we could analyze the first 5 years of payments using the *PQ* formula, with *P* = 200 and *Q* = 100. This would replace the first 2 annuities in the above analysis with:

$$P \cdot a_{\overline{5}|} + Q \cdot \frac{a_{\overline{5}|} - 5 \cdot v^5}{i} = 200 \cdot \frac{1-1.055^{-5}}{0.055} + 100 \cdot \frac{\frac{1-1.055^{-5}}{0.055} - \frac{5}{1.055^5}}{0.055} = 1,662.44$$

When we add this to the value of the third annuity above, the total matches our previous result.

Yet another approach is to treat this as a 10-year annuity-immediate for 600, *minus* a 4-year decreasing arithmetic annuity with payments of 400, 300, 200, 100:

$$PV = 600 \cdot a_{\overline{10}|} - 100(Da)_{\overline{4}|} = 600\frac{1-1.055^{-10}}{0.055} - 100 \cdot \frac{4 - a_{\overline{4}|}}{i} = 3,622.85$$

10. We can think of this annuity as a quarterly perpetuity-immediate of 20 starting now, augmented by a second quarterly perpetuity-immediate of 5 starting 10 years (40 periods) from now. The quarterly effective interest rate is 1%.

The present value of the perpetuity of 20 is $\dfrac{20}{0.01} = 2,000$.

The value at t = 10 of the perpetuity of 5 is $\dfrac{5}{0.01} = 500$.

The total present value is $2,000 + \dfrac{500}{(1.01)^{40}} = 2,335.83$.

11. There are 50 semi-annual payments of 1,000. We are given a quarterly effective rate of $6\% \div 4 = 1.5\%$, but we need a semi-annual rate. The semi-annual effective rate is:

$$i = (1.015)^2 - 1 = 0.0302.$$

Now we can use the TVM worksheet to solve the problem.
Set N=50, I/Y=3.02, PMT=1,000, and CPT PV = −25,620.20.
Note: If you entered 3.02 manually for I/Y, *your answer will not match the above value. To get the exact answer, you need to transfer the value of* i *into* I/Y *with all of the digits in the calculator's register. (Multiply by 100 to make it a percent, then press* I/Y.*)*

12. There are 24 quarterly payments of 500. We are given a semi-annual effective rate of $4\% \div 2 = 2\%$, but we need a quarterly rate. Take the square root of the semi-annual accumulation factor (1.02) to find the quarterly effective rate:

$$i = (1.02)^{1/2} - 1 = 0.00995$$

Then: $$500a_{\overline{24}|0.995\%} = 500 \cdot \frac{1 - 1.00995^{-24}}{0.00995} = 10,627.96$$

Or use the TVM worksheet: N=24, I/Y=0.995, PMT=500; CPT PV = -10,627.96

13. At the end of five years you will have:
- The return of the original 10,000.
- The future value of 5 payments of 800 (annual interest at 8% on 10,000). Set PMT=-800, I/Y=6, N=5 and CPT FV = 4,509.67

  Or compute $800 \cdot s_{\overline{5}|6\%} = 800 \cdot \frac{1.06^5 - 1}{0.06} = 4,509.67$

The total is $10,000 + 4,509.67 = 14,509.67$.

14.

Time	0	1	2	3	4	5	6	7	8
Payment invested	2,000	2,000	2,000	2,000	2,000	2,000	2,000	2,000	
Total payments to date	2,000	4,000	6,000	8,000	10,000	12,000	14,000	16,000	16,000
Interest on payments at 8%		160	160(2)	160(3)	160(4)	160(5)	160(6)	160(7)	160(8)

$$8(2,000) + 160(Is)_{\overline{8}|0.05} = 16,000 + 160 \cdot \frac{\ddot{s}_{\overline{8}|} - 8}{0.05} = 16,000 + 160(40.5313) = 22,485.01$$

Section 2.24

Sample Exam Problems

1. (2005 Exam FM Sample Questions #2)
Kathryn deposits 100 into an account at the beginning of each 4-year period for 40 years. The account credits interest at an annual effective interest rate of i.

The accumulated amount in the account at the end of 40 years is X, which is 5 times the accumulated amount in the account at the end of 20 years. Calculate X.

(A) 4695 (B) 5070 (C) 5445 (D) 5820 (E) 6195

2. (2005 Exam FM Sample Questions #6)

A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3, …., n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.

(A) 17 (B) 18 (C) 19 (D) 20 (E) 21

3. (2005 Exam FM Sample Questions #7)

1000 is deposited into Fund X, which earns an annual effective rate of 6%. At the end of each year, the interest earned plus an additional 100 is withdrawn from the fund. At the end of the tenth year, the fund is depleted. The annual withdrawals of interest and principal are deposited into Fund Y, which earns an annual effective rate of 9%.

Determine the accumulated value of Fund Y at the end of year 10.
(A) 1519 (B) 1819 (C) 2085 (D) 2273 (E) 2431

4. (2005 Exam FM Sample Questions #11)

A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. The annual effective rate of interest is 8%.

Calculate X.

(A) 54 (B) 64 (C) 74 (D) 84 (E) 94

5. (2005 Exam FM Sample Questions #14)

Mike buys a perpetuity-immediate with varying annual payments. During the first 5 years, the payment is constant and equal to 10. Beginning in year 6, the payments start to increase. For year 6 and all future years, the current year's payment is $K\%$ larger than the previous year's payment. At an annual effective interest rate of 9.2%, the perpetuity has a present value of 167.50.

Calculate K, given $K < 9.2$.

(A) 4.0 (B) 4.2 (C) 4.4 (D) 4.6 (E) 4.8

6. (2005 Exam FM Sample Questions #17)

To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years. The annual effective rate of interest is i. You are given $(1+i)^n = 2$. Determine i.

(A) 11.25% (B) 11.75% (C) 12.25% (D) 12.75% (E) 13.25%

7. (2005 Exam FM Sample Questions #18)

Olga buys a 5-year increasing annuity for X. Olga will receive 2 at the end of the first month, 4 at the end of the second month, and for each month thereafter the payment increases by 2. The nominal interest rate is 9% convertible quarterly.

Calculate X.

(A) 2680 (B) 2730 (C) 2780 D) 2830 (E) 2880

8. (2005 Exam FM Sample Questions #21)

Payments are made to an account at a continuous rate of $(8k + tk)$, where $0 \le t \le 10$. Interest is credited at a force of interest $\delta_t = \dfrac{1}{8+t}$. After 10 years, the account is worth 20,000.

Calculate k.

(A) 111 (B) 116 (C) 121 (D) 126 (E) 131

9. (2005 Exam FM Sample Questions #25)

A perpetuity-immediate pays X per year. Brian receives the first n payments, Colleen receives the next n payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and Jeff's share is K.

Calculate K.

(A) 24% (B) 28% (C) 32% (D) 36% (E) 40%

10. (2005 Exam FM Sample Questions #29)

At an annual effective interest rate of i, $i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 3, is 32. At the same annual effective rate of i, the present value of a perpetuity paying 1 at the end of each 4-month period, with first payment at the end of 4 months, is X.

Calculate X.

(A) 31.6 (B) 32.6 (C) 33.6 (D) 34.6 (E) 35.6

11. (2005 Exam FM Sample Questions #31)

An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are $5,000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today.

Find the present value of the obligation if the annual effective interest rate is 5%.

(A) 87,932 (B) 102,514 (C) 114,611
(D) 122,634 (E) Cannot be determined

12. (2005 Exam FM Sample Questions #48)
A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of X to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. On his 65th birthday, each 1000 of the fund will provide 9.65 of income at the beginning of each month starting immediately and continuing as long as he survives.

Calculate X.

(A) 324.73 (B) 326.89 (C) 328.12 (D) 355.45 (E) 450.65

13. (2005 Exam FM Sample Questions #49)

Happy and financially astute parents decide at the birth of their daughter that they will need to provide 50,000 at each of their daughter's 18th, 19th, 20th, and 21st birthdays to fund her college education. They plan to contribute X at each of their daughter's 1st through 17th birthdays to fund the four 50,000 withdrawals. If they anticipate earning a constant 5% annual effective rate on their contributions, which of the following equations of value can be used to determine X, assuming compound interest?

(A) $X\left[v_{.05}^{1}+v_{.05}^{2}+\ldots+v_{.05}^{17}\right]=50{,}000\left[v_{.05}^{1}+\ldots+v_{.05}^{4}\right]$

(B) $X\left[(1.05)^{16}+(1.05)^{15}+\ldots+1.05^{1}\right]=50{,}000\left[1+\ldots+v_{.05}^{3}\right]$

(C) $X\left[(1.05)^{17}+(1.05)^{16}+\ldots+1\right]=50{,}000\left[1+\ldots+v_{.05}^{3}\right]$

(D) $X\left[(1.05)^{17}+(1.05)^{16}+\ldots+(1.05)^{1}\right]=50{,}000\left[1+\ldots+v_{.05}^{3}\right]$

(E) $X\left[v_{.05}^{1}+v_{.05}^{2}+\ldots+v_{.05}^{17}\right]=50{,}000\left[v_{.05}^{18}+\ldots+v_{.05}^{22}\right]$

14. (May 05 #1)

Which of the following expressions does NOT represent a definition for $a_{\overline{n}|}$?

(A) $v^{n}\left[\dfrac{(1+i)^{n}-1}{i}\right]$ (B) $\dfrac{1-v^{n}}{i}$ (C) $v+v^{2}+\ldots+v^{n}$

(D) $v\left[\dfrac{1-v^{n}}{1-v}\right]$ (E) $\dfrac{s_{\overline{n}|}}{(1+i)^{n-1}}$

15. (May 05 #4)

An estate provides a perpetuity with payments of X at the end of each year. Seth, Susan, and Lori share the perpetuity such that Seth receives the payments of X for the first n years and Susan receives the payments of X for the next m years, after which Lori receives all the remaining payments of X. Which of the following represents the difference between the present value of Seth's and Susan's payments using a constant rate of interest?

(A) $X\left[a_{\overline{n}|}-v^{n}a_{\overline{m}|}\right]$ (B) $X\left[\ddot{a}_{\overline{n}|}-v^{n}\ddot{a}_{\overline{m}|}\right]$ (C) $X\left[a_{\overline{n}|}-v^{n+1}a_{\overline{m}|}\right]$

(D) $X\left[a_{\overline{n}|}-v^{n-1}a_{\overline{m}|}\right]$ (E) $X\left[va_{\overline{n}|}-v^{n+1}a_{\overline{m}|}\right]$

16. (May 05 #9)

The present value of a series of 50 payments starting at 100 at the end of the first year and increasing by 1 each year thereafter is equal to X. The annual effective rate of interest is 9%. Calculate X.

(A) 1165 (B) 1180 (C) 1195 (D) 1210 (E) 1225

17. (May 05 #12)

Which of the following are characteristics of all perpetuities?

I. The present value is equal to the first payment divided by the annual effective interest rate.

II. Payments continue forever.

III. Each payment is equal to the interest earned on the principal.

(A) I only
(B) II only
(C) III only
(D) I, II, and III
(E) The correct answer is not given by (A), (B), (C), or (D).

18. (May 05 #14)

An annuity-immediate pays 20 per year for 10 years, then payments decrease by 1 each year for 19 years. At an annual effective interest rate of 6%, the present value is equal to X.
Calculate X.

(A) 200 (B) 205 (C) 210 (D) 215 (E) 220

19. (May 05 #17)

At an annual effective interest rate of i, the present value of a perpetuity-immediate starting with a payment of 200 in the first year and increasing by 50 each year thereafter is 46,530. Calculate i.

(A) 3.25% (B) 3.50% (C) 3.75% (D) 4.00% (E) 4.25%

20. (May 05 #20)

An investor wishes to accumulate 10,000 at the end of 10 years by making level deposits at the beginning of each year. The deposits earn a 12% annual effective rate of interest paid at the end of each year. The interest is immediately reinvested at an annual effective interest rate of 8%.

Calculate the level deposit.

(A) 541 (B) 572 (C) 598 (D) 615 (E) 621

21. (May 05 #21)

A discount electronics store advertises the following financing arrangement: “We don’t offer you confusing interest rates. We’ll just divide your total cost by 10 and you can pay us that amount each month for a year.” The first payment is due on the date of sale and the remaining eleven payments at monthly intervals thereafter.

Calculate the annual effective interest rate the store’s customers are paying on their loans.

(A) 35.1% (B) 41.3% (C) 42.0% (D) 51.2% (E) 54.9%

22. (May 05 #24)

An annuity pays 1 at the end of each year for n years. Using an annual effective interest rate of i, the accumulated value of the annuity at time $(n+1)$ is 13.776 . It is also known that $(1+i)^n = 2.476$.

Calculate n.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

23. (Nov 05 #3)

An investor accumulates a fund by making payments at the beginning of each month for 6 years. Her monthly payment is 50 for the first 2 years, 100 for the next 2 years, and 150 for the last 2 years. At the end of the 7th year the fund is worth 10,000. The annual effective interest rate is i, and the monthly effective interest rate is j.

Which of the following formulas represents the equation of value for this fund accumulation?

(A) $\ddot{s}_{\overline{24}|i}(1+i)\left[(1+i)^4 + 2(1+i)^2 + 3\right] = 200$

(B) $\ddot{s}_{\overline{24}|i}(1+j)\left[(1+j)^4 + 2(1+j)^2 + 3\right] = 200$

(C) $\ddot{s}_{\overline{24}|j}(1+i)\left[(1+i)^4 + 2(1+i)^2 + 3\right] = 200$

(D) $s_{\overline{24}|j}(1+i)\left[(1+i)^4 + 2(1+i)^2 + 3\right] = 200$

(E) $s_{\overline{24}|i}(1+i)\left[(1+j)^4 + 2(1+j)^2 + 3\right] = 200$

24. (Nov 05 #8)

Matthew makes a series of payments at the beginning of each year for 20 years. The first payment is 100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment.

Calculate the present value of these payments at the time the first payment is made, using an annual effective rate of 7%.

(A) 1375 (B) 1385 (C) 1395 (D) 1405 (E) 1415

25. (Nov 05 #9)

A company deposits 1000 at the beginning of the first year and 150 at the beginning of each subsequent year into perpetuity. In return the company receives payments at the end of each year forever. The first payment is 100. Each subsequent payment increases by 5%.

Calculate the company's yield rate for this transaction.

(A) 4.7% (B) 5.7% (C) 6.7% (D) 7.7% (E) 8.7%

26. (Nov 05 #12)

Megan purchases a perpetuity-immediate for 3250 with annual payments of 130. At the same price and interest rate, Chris purchases an annuity-immediate with 20 annual payments that begin at amount P and increase by 15 each year thereafter. Calculate P.

(A) 90 (B) 116 (C) 131 (D) 176 (E) 239

27. (Nov 05 #13)

For 10,000, Kelly purchases an annuity-immediate that pays 400 quarterly for the next 10 years. Calculate the nominal annual interest rate convertible monthly earned by Kelly's investment.

(A) 10.0% (B) 10.3% (C) 10.5% (D) 10.7% (E) 11.0%

28. (Nov 05 #14)

Payments of X are made at the beginning of each year for 20 years. These payments earn interest at the end of each year at an annual effective rate of 8%. The interest is immediately reinvested at an annual effective rate of 6%. At the end of 20 years, the accumulated value of the 20 payments and the reinvested interest is 5600. Calculate X.

(A) 121.67 (B) 123.56 (C) 125.72 (D) 127.18 (E) 128.50

29. (Nov 05 #23)

The present value of a 25-year annuity-immediate with a first payment of 2500 and decreasing by 100 each year thereafter is X. Assuming an annual effective interest rate of 10%, calculate X.

(A) 11,346 (B) 13,615 (C) 15,923 (D) 17,396 (E) 18,112

Section 2.25

Sample Exam Problem Solutions

1.

We solve this problem by putting things in terms of 4-year periods. For each 4-year period the effective interest rate *j* is given in terms of the annual rate *i* by the equation $(1+i)^4 = (1+j)$.

We can think of 40 years as ten 4-year periods. Thus the accumulated value at the end of 40 years is $FV_{40} = 100 \cdot \ddot{s}_{\overline{10}|j} = X$.

Similarly, the accumulated value at the end of 20 years is $FV_{20} = 100 \cdot \ddot{s}_{\overline{5}|j} = \dfrac{X}{5}$.

We can calculate *X* if we know what *j* is. To find *j* we use the fact that the accumulated value at time 40 is 5 times the accumulated value at time 20:

$$\ddot{s}_{\overline{10}|j} = 5\ddot{s}_{\overline{5}|j}$$

Now we use the definition of $\ddot{s}_{\overline{n}|}$ to create an equation that can be solved for *j*:

$$\left[\frac{(1+j)^{10}-1}{d}\right] = 5\left[\frac{(1+j)^{5}-1}{d}\right]$$

$$(1+j)^{10} - 1 = 5\left[(1+j)^5 - 1\right]$$

The standard trick here is to make the substitution $x = (1+j)^5$. Then we have: $x^2 - 1 = 5 \cdot (x-1)$. Dividing by $(x-1)$ gives $x+1=5$, so $x=4$.

Thus we have: $x = (1+j)^5 = 4 \;\; \rightarrow \;\; j = 0.31951$.

Using a 4-year effective rate of 31.951% to find $\ddot{s}_{\overline{10}|j}$, we have:

$$X = 100 \cdot \ddot{s}_{\overline{10}|j} = 100 \cdot \frac{1.31951^{10} - 1}{0.31951 / 1.31951} = 6{,}194.72$$

We didn't need to find *i*, but its value is: $1.31951^{1/4} - 1 = 0.07177$.

Answer E

2.
We will start by looking at a timeline for the perpetuity.

payment			1	2	3	⋯	n	n	n
time t	0	1	2	3	4	⋯	n+1	n+2	n+3

This perpetuity can be viewed as the sum of an n-year arithmetic increasing annuity-immediate deferred 1 year, and a perpetuity-immediate of n deferred (n+1) years (first payment at time (n+2)). Thus its present value (its cost) is:

$$\underbrace{{}_{1|}(Ia)_{\overline{n}|}}_{\substack{\text{1-year-deferred}\\ n\text{-year increasing}\\ \text{annuity}}} + \underbrace{n \cdot {}_{(n+1)|}a_{\overline{\infty}|}}_{\substack{(n+1)\text{-year–deferred}\\ \text{perpetuity of amount } n}} = v\left[(Ia)_{\overline{n}|}\right] + v^{n+1}\frac{n}{i} = v\left[\frac{\ddot{a}_{\overline{n}|} - n\cdot v^{n}}{i}\right] + v^{n+1}\cdot\frac{n}{i} = \frac{v\cdot\ddot{a}_{\overline{n}|}}{i} = \frac{a_{\overline{n}|}}{i}$$

We are given that the cost (PV) is 77.1 and $i = 0.105$.

Thus: $\dfrac{a_{\overline{n}|0.105}}{0.105} = 77.1 \rightarrow a_{\overline{n}|0.105} = 8.0955$.

We can solve for n using the TVM worksheet of the BA II Plus calculator:
I/Y = 10.5, PV = −8.0955, and PMT = 1. The computed solution is N = 19.

Alternatively, we can recognize that the annuity is equivalent to a 1-year-deferred increasing perpetuity-immediate, minus an (n+1)-year-deferred increasing perpetuity-immediate. This leads to the following:

$${}_{1|}(Ia)_{\overline{\infty}|} - {}_{(n+1)|}(Ia)_{\overline{\infty}|} = 77.1 \quad \rightarrow \quad v\cdot\frac{1}{i\cdot d} - v^{(n+1)}\cdot\frac{1}{i\cdot d} = \frac{1}{i^2} - \frac{v^n}{i^2} = 77.1$$

We are given that i = 0.105. Substituting that value into the previous equation and simplifying, we have:

$$\frac{1-1.105^{-n}}{(0.105)^2} = 77.1 \quad \rightarrow \quad 1.105^n = \left[1 - 77.1\cdot(0.105)^2\right]^{-1} = 6.6679$$

Solving for n: $n = \dfrac{\ln 6.6679}{\ln 1.105} = 19.00$

Answer C

3.
Each year all of the interest earned in Fund X, plus 100 of principal, is transferred from Fund X to Fund Y. Here is a diagram of the deposits to Fund Y:

Time	0	1	2	• • •	9	10
Principal from Fund X	--	100	100	• • •	100	100
Interest from Fund X	--	.06(1000)=60	.06(900)=54	• • •	.06(200)=12	.06(100)=6
Total Deposit to Fund Y	--	160	154	• • •	112	106

The *withdrawals of principal* from Fund X represent a 10-year level annuity-immediate of 100 per year, and the *interest payments* from Fund X form a decreasing annuity-immediate with payments of 60, 54, 48, …, 6. Both are deposited into Fund Y, where they accumulate at a 9% annual effective rate. The accumulated value at time 10 is $100 \cdot s_{\overline{10}|} + 6 \cdot (Ds)_{\overline{10}|}$.

By formula or using the financial calculator, $s_{\overline{10}|} = 15.1929$.

And $(Ds)_{\overline{10}|} = 1.09^{10}(Da)_{\overline{10}|} = 1.09^{10}\left[\dfrac{10 - a_{\overline{10}|}}{0.09}\right] = 94.23$.

The total accumulated value is $100 \cdot (15.1929) + 6 \cdot (94.23) = 2{,}084.67$.

Alternatively, we can apply the *PQ* formula with $P = 160$ and $Q = -6$:

$$FV = 160 \cdot s_{\overline{10}|} - 6 \cdot \frac{s_{\overline{10}|} - 10}{i} = 2{,}430.87 - 346.20 = 2{,}084.67$$

Answer C

4.
The first thing we should notice is that the exchange from the perpetuity could have occurred on *any* annual payment date, since the value of a perpetuity-immediate is the same after each payment. (It is the present value of an infinite number of future payments, where the next payment is one period in the future.) That value is $100 \cdot a_{\overline{\infty}|0.08} = 1,250$.

Next we notice that the interest rate (i) and the payment increase rate (g) are both 8%. This means that the formula for the value of a geometric annuity can't be applied, because the denominator $(i-g)$ equals 0. Also, since these two values are equal, each of the 25 payments has the same present value. (The amount of the payment increases each year by a factor of 1.08, but the present value factor decreases by a factor of $\frac{1}{1.08}$, so the present values of all 25 payments are identical.) This means that each of the payments has a present value equal to 1/25 of the value of the perpetuity that was exchanged. So each payment has a present value of $\frac{1250}{25} = 50$.

The first payment has an amount of X and it occurs one year after the exchange from the perpetuity. Its present value on the date of the exchange is 50, so we can write $\frac{X}{1.08} = 50$ and solve for X: $X = 1.08 \cdot (50) = 54$.
Answer A

5.
Divide the perpetuity into two parts: the first 4 payments, and the 5th and later payments. The first 4 payments are a level annuity of 10. The 5th and later payments are a geometric perpetuity. Because the complex part of this series of payments is the geometric perpetuity, we will choose a valuation date of $t = 4$. As of $t = 4$, its value is can be calculated using the formula for the present value of a geometric perpetuity-immediate: $a^{g}_{\overline{\infty}|i} = \frac{1}{i-g}$, where i is 0.092 and g is $K\% = \frac{K}{100}$.

The equation of value is:

$$167.5 \cdot 1.092^4 = 10 \cdot s_{\overline{4}|0.092} + 10 \cdot \frac{1}{0.092 - \frac{K}{100}}$$

Solving for K yields $K = 4$.

Answer A

6.
We can look at each payment of 196 in the last $2n$ years as the sum of 2 payments of 98 each. Then the total annuity is the sum of two level-payment annuities, the first starting at time 1 and continuing to time $3n$, and the second starting at time n+1 and continuing to time $3n$.

n-yr def'd $2n$-yr ann			...		98	...	98
$3n$-yr ann		98	...	98	98	...	98
Total Payments		98		98	196		196
time	$t=0$	1	...	n	$n+1$	...	$3n$

Then the accumulated value of 8,000 equals $98s_{\overline{3n}|}+98s_{\overline{2n}|}$.

Thus: $98\left[\dfrac{(1+i)^{3n}-1}{i}\right]+98\left[\dfrac{(1+i)^{2n}-1}{i}\right]=8000$.

Now we can use the given value of $(1+i)^n=2$:

$$98\left[\frac{(2)^3-1}{i}\right]+98\left[\frac{(2)^2-1}{i}\right]=8000$$

$$98\cdot\left[\frac{7}{i}+\frac{3}{i}\right]=\frac{980}{i}=8000 \qquad \rightarrow \qquad i=0.1225$$

We didn't need to find n, but its value is: $n=\ln 2/\ln 1.01225=6.00$ years.

Answer C

7.
Olga has a monthly increasing annuity extending over 60 months. The nominal interest rate convertible quarterly is 9%, giving an effective rate of 2.25% per quarter. We first solve for the monthly effective rate, which we will call i.

Exponent is 3 because there are three months in a quarter-year. A common mistake is to use 4 instead.

$$(1+i)^3=1.0225 \qquad 1+i=1.007444$$

Thus the monthly effective rate is 0.7444%.

The present value of the increasing annuity is:

$$2(Ia)_{\overline{60}|0.007444}=2\left[\frac{\ddot{a}_{\overline{60}|}-60v^{60}}{0.007444}\right]=2\left[\frac{48.61-38.45}{0.007444}\right]=2{,}729.71$$

Answer B

8.

We will write an expression for the accumulation to time 10 of the amount deposited in a small interval of length dt at time t. Then we will integrate that expression from t=0 to t=10 to sum the accumulations of all amounts contributed from time 0 to time 10.

Payment rate at time t: $\rho(t) = 8k + tk = k(8+t)$

The accumulated value at time 10 is:

$$\int_{t=0}^{10} \rho(t) \cdot e^{\int_{u=t}^{10} \delta_u du} dt = \int_{t=0}^{10} (8k+tk) \cdot e^{\int_t^{10} \left(\frac{1}{8+u}\right) du} dt = \int_{t=0}^{10} (8k+tk) \cdot e^{\ln(8+u)\big|_t^{10}} dt$$

$$= \int_{t=0}^{10} (8k+tk) \cdot e^{\ln(18)-\ln(8+t)} dt = \int_{t=0}^{10} \frac{18}{8+t} \cdot (8k+tk) dt$$

$$= \int_{t=0}^{10} 18k \cdot dt = 18kt\Big|_{t=0}^{10} = 180k$$

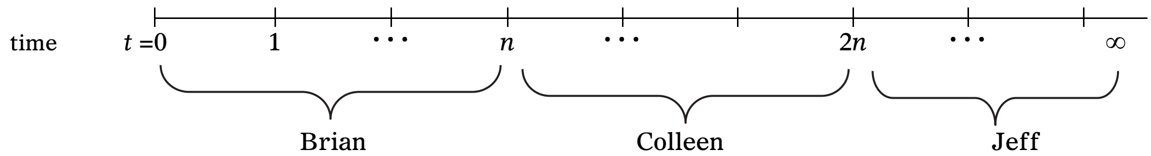
Since we are given that the account is worth 20,000 in 10 years, we have:

$180k = 20{,}000 \qquad k = 111.11$

Note: This problem is somewhat challenging to set up (expressing the <u>*accumulated*</u> *value as an integral), but the values are such that it is relatively simple to evaluate. This is typical for exam problems where the rate of payment and the force of interest both vary continuously.*

Answer A

9.



Brian receives an annuity-immediate of X for n periods. The present value of his annuity is $B = X \cdot a_{\overline{n}|}$. The total perpetuity has a present value of $\frac{X}{i}$. Thus Brian's share of the total is $\frac{Xa_{\overline{n}|}}{\left(\frac{X}{i}\right)} = ia_{\overline{n}|} = i\left(\frac{1-v^n}{i}\right) = 1 - v^n$.

Since Brian's share is 40%, we have $0.4 = 1 - v^n$. Therefore, $v^n = 0.6$.

Colleen has an immediate annuity of X for n periods, but it is deferred for n periods.
Her present value is $C = v^n X a_{\overline{n}|} = 0.6 X a_{\overline{n}|} = 0.6B$.

Since Brian's share (B) is 40% of the total, Colleen's share is $0.6(40\%) = 24\%$.
Thus Jeff's share is $100\% - (40\% + 24\%) = 36\%$.

More simply, we can observe that at time 0 the perpetuity is worth $a_{\overline{\infty}|}$, the payments after the first n years have a present value (at time 0) of $v^n \cdot a_{\overline{\infty}|}$, and the payments after the first $2n$ years have a present value (at time 0) of $v^{2n} \cdot a_{\overline{\infty}|}$. Since Brian's share is 40% of $a_{\overline{\infty}|}$, we know that the remainder ($0.60 \cdot a_{\overline{\infty}|}$) equals the present value (at time 0) of the payments after n years. So $v^n = 0.6$ and $v^{2n} = 0.36$. Therefore, Jeff's share is $0.36 \cdot a_{\overline{\infty}|}$, which is 36% of the perpetuity.

Answer D

10.

Let j denote the effective rate per 3-year period. For the perpetuity that pays 10 at the end of each 3-year period, we can apply the basic formula for the present value of a perpetuity-immediate:

$$\text{Present value} = 32 = \frac{10}{j} \qquad j = \frac{10}{32} = 0.3125$$

A four-month period is one-ninth of a three-year period. Thus the effective rate for a four-month period is:

$$(1+j)^{1/9} - 1 = (1.3125)^{1/9} - 1 = 0.030676$$

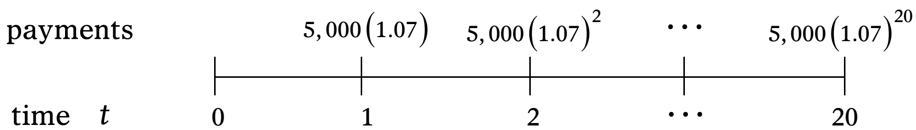
The present value of a unit perpetuity-immediate payable every 4th month is:

$$\frac{1}{0.030676} = 32.6$$

Answer B

11.
The present value is calculated here under the assumption that the claimant survives for exactly the expected 20 years. His payments will be:

$$5{,}000(1.07),\ 5{,}000(1.07)^2,\ldots,\ 5{,}000(1.07)^{20}$$



The present value of these payments at 5% is

$$5{,}000\left[\frac{1.07}{1.05}+\frac{1.07^2}{1.05^2}+\ldots+\frac{1.07^{20}}{1.05^{20}}\right]=\frac{5{,}000(1.07)}{1.05}\left[1+\frac{1.07}{1.05}+\ldots+\frac{1.07^{19}}{1.05^{19}}\right]$$

The final term in brackets is a geometric series with ratio $r=\dfrac{1.07}{1.05}$.

Thus the present value is: $\dfrac{5{,}000(1.07)}{1.05}\left[\dfrac{1-\left(\dfrac{1.07}{1.05}\right)^{20}}{1-\left(\dfrac{1.07}{1.05}\right)}\right]=122{,}634$

Alternatively, we can use the artificial interest rate method:

$$PV=1.07\cdot 5{,}000\cdot a_{\overline{20}|5\%}^{7\%}=\frac{5{,}350}{1.05}\cdot \ddot{a}_{\overline{20}|5\%}^{7\%}=\frac{5{,}350}{1.05}\cdot \ddot{a}_{\overline{20}|j}$$

where: $$1+j=\frac{1+i}{1+g}=\frac{1.05}{1.07}=0.9813084$$

Thus $j=-1.86916\%$. Note that this is a *negative* artificial interest rate. However, the formulas remain unchanged, and we find the present value as follows:

$$\frac{5{,}350}{1.05}\cdot \ddot{a}_{\overline{20}|j}=\frac{5{,}350}{1.05}\cdot\frac{1-(1+j)^{-20}}{j/(1+j)}=5{,}095.24\cdot\frac{1-0.9813084^{-20}}{(-0.0186916)/0.9813084}=122{,}634$$

Answer D

12.
Let k be the number of 1,000's of dollars in the fund at the man's 65th birthday. Each 1,000 in the fund will provide 9.65 of monthly income, so the total income from the fund will be $9.65k$. The desired monthly income is 3,000, so:

$$9.65k = 3{,}000 \rightarrow \qquad k = 310.88083$$

Thus the value of the fund on his 65th birthday must be $1{,}000k = 310{,}880.83$.

The man must accumulate 310,880.83 (FV) with beginning-of-month payments (PMT, which is the unknown) for $12(25) = 300$ monthly periods (N) at an effective rate per period (I/Y) of $8\% \div 12 = 0.6667\%$ per month. Using the BA II Plus set to BEGIN mode, we find that PMT = -324.725. Or leave the calculator in END mode, calculate an end-of-month PMT of -326.89, and divide by 1.006667: $\dfrac{326.89}{1.006667} = 324.725$.

Alternatively, $P = \dfrac{310{,}880.83}{\ddot{s}_{\overline{300}|0.006667}}$ =324.725

Answer A

13.
The equation of value must equate the value of 17 deposits to the value of 4 withdrawals. This eliminates choices B (16 deposits), C (18 deposits), and E (5 withdrawals), leaving choices A and D. Further, the values of the deposits and the withdrawals must be measured as of the same valuation date. Choice A has deposits that are valued as of age 0 and withdrawals valued as of age 17. Choice D has deposits that are valued as of age 18 and withdrawals that are also valued as of age 18. Therefore, D is a valid equation of value.

Answer D

14.
Choice A consists of an n-year present value factor $\left(v^n\right)$ times the accumulated value of an n-year annuity $\left(\dfrac{(1+i)^n-1}{i}=s_{\overline{n}|}\right)$. The value of this expression is $a_{\overline{n}|}$, so the answer is not A.
Choice B is the standard formula for $a_{\overline{n}|}$, so the answer is not B.
Choice C is the definition of $a_{\overline{n}|}$ (the sum of the present values of payments at times 1 through n), so the answer is not C.
Choice D consists of a 1-year present value factor (v) times an expression that is equivalent to $\ddot{a}_{\overline{n}|}$ (since $\ddot{a}_{\overline{n}|}=\dfrac{1-v^n}{d}=\dfrac{1-v^n}{1-v}$). Discounting an annuity-due $\left(\ddot{a}_{\overline{n}|}\right)$ by one year produces an annuity-immediate $\left(a_{\overline{n}|}\right)$, so this is a valid expression for $a_{\overline{n}|}$. The answer is not D.
Choice E is *not* a correct expression for $a_{\overline{n}|}$. (If the denominator were $(1+i)^n$ it would be correct, since $a_{\overline{n}|}$ equals $s_{\overline{n}|}$ discounted n years.) So E is the answer.

Answer E

15.
Since Seth gets the first n payments, he has an n-period annuity-immediate of X. His present value is $X\cdot a_{\overline{n}|}$.

Since Lori gets the next m payments after the first n years, she has an n-year-deferred, m-year annuity of X. Her present value is $v^n\cdot X\cdot a_{\overline{m}|}$.

The difference is $X\cdot a_{\overline{n}|}-v^n\cdot X\cdot a_{\overline{m}|}=X\cdot\left[a_{\overline{n}|}-v^n a_{\overline{m}|}\right]$

Answer A

16.
We can think of this annuity as the sum of a 50-payment level annuity of 99 and a 50-payment increasing annuity which starts at 1 and increases by 1 each year:

$$PV=99a_{\overline{50}|}+(Ia)_{\overline{50}|}=99(10.962)+125.287=1{,}210.525$$

Note: This problem could also have been solved using the PQ formula (P=100 and Q=1).

Answer D

17.

I. This is the definition of a *level* perpetuity, But not all perpetuities are level, so the statement is false.
II. True for all perpetuities.
III. False in general, although true for *level* perpetuities-*immediate*, because:

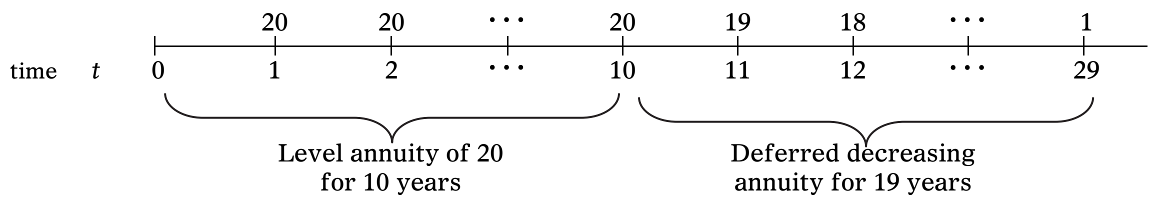
$$principal = \frac{payment}{i} \rightarrow payment = (principal) \cdot i$$

Again, not all perpetuities are level or immediate.

Answer B

18.

Here we have a level annuity of 20 for 10 years. Then payments drop to 19 in year 11 and by 1 each year thereafter –so that we have a deferred decreasing annuity for 19 years.



The equation of value is:

$$X = 20 \cdot a_{\overline{10|}} + v^{10} \cdot (Da)_{\overline{19|}} = 147.20 + 0.5584(130.6981) = 220.18$$

Alternatively, the payments can be separated into groups in other ways.
For example:

29 level payments of 20, offset by a series of 19 increasing payments that begin after the first 10 years:

$$X = 20 \cdot a_{\overline{29|}} - v^{10} \cdot (Ia)_{\overline{19|}} = 220.18$$

or:

9 level payments of 20, followed by 20 decreasing payments (beginning with 20 at time 10, decreasing to 1 at time 29):

$$X = 20 \cdot a_{\overline{9|}} + v^{9} \cdot (Da)_{\overline{20|}} = 220.18$$

Answer E

19.
In this problem we will use the fact that the present value of an increasing perpetuity-immediate with first payment P and periodic increase Q is:

$$\frac{P}{i}+\frac{Q}{i^2}$$

P=200 and Q=50, but the interest rate is unknown. The present value of the perpetuity is 46,530 so the equation of value is:

$$\frac{200}{i}+\frac{50}{i^2}=46{,}530$$

This gives us the quadratic equation $46{,}530i^2-200i-50=0$.
Applying the quadratic formula gives us the roots: $i=.035$ and $i=-.0307$.
The correct answer is the positive root, $i=0.035$.
(At a negative interest rate, a level perpetuity's present value would be infinite, and the formula $a_{\overline{\infty}|}=\frac{1}{i}$ would not be valid.)

Answer B

20.
We will first create a table to illustrate the pattern of payments.

Time	0	1	2	···	9	10
Payment invested	D	D	D	···	D	
Total payments to date	D	$2D$	$3D$	···	$10D$	$10D$
Interest on payments at 12%		$0.12D$	$0.12(2D)$	···	$0.12(9D)$	$0.12(10D)$

Note that the numbers in the last row are the deposits to the 8% reinvestment account, and these form an arithmetic increasing annuity. At the end of 10 years, the reinvestment account plus total deposits equals 10,000, so the equation of value is:

$$10{,}000=\underbrace{10D}_{\substack{\text{Total}\\ \text{deposits}}}+\underbrace{0.12D\cdot(Is)_{\overline{10}|.08}}_{\substack{\text{Increasing annuity of}\\ \text{0.12D for 10 years}}}$$

$$=10D+0.12D(70.5686)$$
$$=18.4682D$$

$$D=541.47$$

Answer A

21.
For each dollar of cost, the consumer makes 12 monthly payments of 0.10 starting at time 0. This is an annuity-due with an unknown interest rate. The monthly effective rate can be found directly on the BA II Plus in BGN mode.

Set N= 12, PV = 1, PMT = −0.10 and CPT I/Y = 3.5032. In actuarial notation, we have found:

$$\frac{i^{(12)}}{12} = 0.035032$$

Note: To avoid using BGN mode, you could treat this as a loan of ***0.9*** *at time 0 (after subtracting the first payment of 0.10) that is repaid in* ***11*** *installments.*

In this question we are asked for the annual effective rate i, which is given by:

$$(1+i) = \left(1+\frac{i^{(12)}}{12}\right)^{12} = (1.035032)^{12} = 1.5116 \rightarrow i = 0.5116$$

Answer D

22.
Here both n and i are unknown, so we cannot find the answer directly with the calculator. The annuity in this question is a unit annuity. We are given the accumulated value at time n+1 of an n-period annuity-immediate. That accumulated value is $s_{\overline{n}|} \cdot (1+i) = \ddot{s}_{\overline{n}|}$=13.776.

Using the fact that $(1+i)^n = 2.476$, and applying algebra, we can find i:

$$13.776 = \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \frac{2.476-1}{d}$$

$$13.776 \cdot d = 1.476 \rightarrow d = 0.107143 \rightarrow i = \frac{d}{1-d} = 0.12$$

Now we can find n using the given value for $(1+i)^n$:

$$(1.12)^n = 2.476 \quad \rightarrow \quad n \cdot \ln(1.12) = \ln(2.476) \quad \rightarrow \quad n = \frac{\ln(2.476)}{\ln(1.12)} = 8$$

Answer E

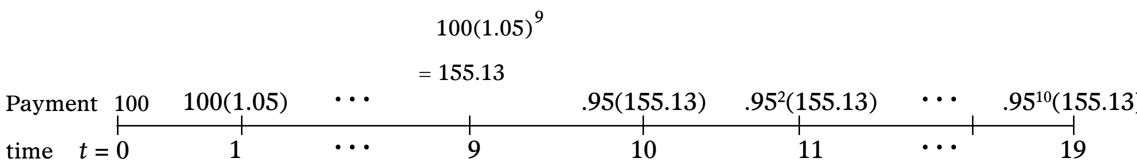
23.

Each separate block of 24 identical payments of amount P (where P is 50, 100, or 150) accumulates to $P \cdot \ddot{s}_{\overline{24}|j}$ at the end of its 24 months. These series of payments end at times 2 years, 4 years, and 6 years, so to find their values at the end of the 7th year, as the problem requires, these $P \cdot \ddot{s}_{\overline{24}|j}$ values must be accumulated for 5, 3, and 1 years, respectively. The total accumulation at time 7 is 10,000, so the equation of value is:

$$50\ddot{s}_{\overline{24}|j}(1+i)^5 + 100\ddot{s}_{\overline{24}|j}(1+i)^3 + 150\ddot{s}_{\overline{24}|j}(1+i)^1 = 10{,}000$$
$$\ddot{s}_{\overline{24}|j}(1+i)^5 + 2\ddot{s}_{\overline{24}|j}(1+i)^3 + 3\ddot{s}_{\overline{24}|j}(1+i)^1 = 200$$
$$(1+i)\ddot{s}_{\overline{24}|j}\left[(1+i)^4 + 2(1+i)^2 + 3\right] = 200$$

Answer C

24.



The present value at 7% is

$$100\left[1+\frac{1.05}{1.07}+\ldots+\left(\frac{1.05}{1.07}\right)^9\right]+\frac{155.13(0.95)}{1.07^{10}}\left[1+\frac{0.95}{1.07}+\ldots+\left(\frac{0.95}{1.07}\right)^9\right]$$

$$=100\left[\frac{1-\left(\frac{1.05}{1.07}\right)^{10}}{1-\left(\frac{1.05}{1.07}\right)}\right]+74.92\left[\frac{1-\left(\frac{0.95}{1.07}\right)^{10}}{1-\left(\frac{0.95}{1.07}\right)}\right]=919.95+464.70=1{,}384.65$$

Alternatively, using the geometric annuity formula, the present value is:

$$100\cdot\ddot{a}^{5\%}_{\overline{10}|7\%}+v^9\cdot 0.95\cdot\left(100\cdot 1.05^9\right)a^{-5\%}_{\overline{10}|7\%}=100\cdot\frac{1-\left(\frac{1.05}{1.07}\right)^{10}}{\frac{0.07}{1.07}-\frac{0.05}{1.07}}+1.07^{-9}\cdot 147.3762\cdot\frac{1-\left(\frac{0.95}{1.07}\right)^{10}}{0.07-(-0.05)}$$

$$=919.95+464.70=1{,}384.65$$

Answer B

25.
The pattern of payments is illustrated in the following table:

Time	0	1	2	3	...
Company deposits	1000	150	150	150	...
Company receives		100	$100(1.05)$	$100(1.05)^2$	...

The yield rate is the interest rate at which the company's deposits have the same present value as the payments it receives. Call this unknown rate i.

$$\text{PV (deposits)} = 1{,}000 + \frac{150}{i} \qquad \text{PV (receipts)} = 100 \cdot a_{\overline{\infty}|}^{0.05} = \frac{100}{i - 0.05}$$

$$1{,}000 + \frac{150}{i} = \frac{100}{i - 0.05}$$

We can solve for i as follows:

$$1000\left(i^2 - 0.05i\right) + 150\left(i - 0.05\right) = 100i$$

$$1000i^2 = 7.5$$

$$i = \sqrt{0.0075} = 0.0866$$

Answer E

26.
First we will find the unknown interest rate i .The value of Megan's perpetuity-immediate of 130 is 3,250, so we have: $\frac{130}{i} = 3{,}250 \quad \rightarrow \quad i = \frac{130}{3{,}250} = 0.04$

Let P be the unknown initial payment for Chris. The annuity that is purchased by Chris has payments P, P+15, P+2(15),..., P+19(15).

The present value of these year-end payments at 4% is

$$Pa_{\overline{20}|} + \frac{15}{1.04}(Ia)_{\overline{19}|} = 13.59P + \left(\frac{15}{1.04}\right)116.03 = 13.59P + 1{,}673.47$$

Thus: $3{,}250 = 13.59P + 1{,}673.47 \qquad P = 116.00$

Note: This problem can also be solved using the PQ formula, starting with:

$$Pa_{\overline{20}|} + 15 \cdot \frac{a_{\overline{20}|} - 20v^{20}}{i} = 3{,}250$$

Answer B

27.
This is a simple financial calculator problem. The quarterly effective yield can be found directly using N = 40 (quarters), PV = −10,000 (price), and PMT = 400. The computed quarterly yield is I/Y = 2.524%. This converts to a monthly yield of $1.02524^{1/3} - 1 = 0.00834$.

The nominal annual yield convertible monthly is $0.00834(12) = 0.1001$.

Answer A

28.
The table below shows the pattern of payments and interest earned at 8%

Time	0	1	2	...	18	19	20
Payment	X	X	X	...	X	X	
Cumulative payments	X	$2X$	$3X$	...	$19X$	$20X$	$20X$
Interest earned		$0.08X$	$0.08(2X)$ $=0.16X$	...	$0.08(18X)$ $=1.44X$	$0.08(19X)$ $=.1.52X$	$0.08(20X)$ $=1.60X$

The total value of 5,600 at time 20 is equal to:

$$20X + 0.08X(Is)_{\overline{20}|0.06} = 20X + 0.08X(316.5454) = 45.3236X = 5{,}600$$

Solving for X, we have: $X = 123.56$

Answer B

29.
This is a straightforward decreasing annuity problem.

$$X = 100(Da)_{\overline{25}|} = 100\left(\frac{25 - a_{\overline{25}|}}{0.10}\right) = 15{,}922.96$$

Answer C

Section 2.26

Supplemental Exercises

1. A man has a loan of 15,000 for 5 years with level quarterly (end-of-quarter) payments at a nominal interest rate of 6.8% convertible quarterly. What is the amount of his quarterly payment?

2. A man plans to work for 25 years, during which time he will create a retirement fund by making end-of-month payments of 100 per month. The fund earns interest at a nominal rate of 6% convertible monthly. He will use the accumulated funds to purchase a 20-year retirement annuity with end-of-month payments. Assuming the annuity is priced based on an interest rate of 6% convertible monthly, what will be the amount of his monthly payment?

3. A woman purchases an annuity that makes annual payments at the beginning of each year for 20 years. The first 10 payments are 1,000 and the last 10 are 1,500. The annuity is priced based on a 6.5% annual effective rate. Find the cost of the annuity.

4. A 20-year annual-payment annuity-immediate has a first payment of 500, and each subsequent payment is 4% larger than the previous one. Find the present value of this annuity at an annual effective rate of 6.2%.

5. A man borrows 25,000. He repays the loan by making quarterly end-of-quarter payments of 650 for 15 years. What is his loan's nominal rate of interest convertible quarterly?

6. A perpetuity-immediate makes quarterly payments. The first 20 payments are 10. Starting with the 21^{st} payment, the remaining payments under this perpetuity are 15. At an interest rate of 6% convertible quarterly, what is the perpetuity's present value?

7. A perpetuity-immediate makes annual payments. The first payment is 100, and each subsequent payment is 3% larger than the previous one. Find the present value of this perpetuity at a 5% annual effective rate.

8. You set up a retirement fund by making annual payments at the end of each year for 30 years. The first payment is 1,000 and each subsequent payment is 100 larger than the previous one. The fund earns 5.8% annually. How much has accumulated in the fund at the end of the 30 years?

9. You invest 1,000 at the beginning of each year for 20 years. The investment makes interest payments to you at the end of each year at a 7% annual effective rate. You invest the interest payments in a fund that pays a 6% annual effective rate. What is your total accumulation at the end of the 20 years?

10. An investor makes level end-of-year payments into a fund that earns interest at an annual effective rate *r*. The accumulation at the end of 20 years is 3 times the accumulation at the end of 10 years. Find *r*.

Section 2.27

Supplemental Exercise Solutions

1. The number of payments is 20 and the effective interest rate per period is 1.7%. Using the BA II Plus calculator, set N = 20, I/Y =1.7, PV = 15,000, and FV = 0. Then CPT PMT = -891.01. The payment amount is 891.01.

2. There are 300 monthly payments and the monthly effective rate is 0.5%. The accumulation is $100\,s_{\overline{300}|0.005} = 69{,}299.40$.
   To calculate the monthly benefit payment, set N = 240, I/Y = 0.5, PV = -69.299.40 and FV = 0.
   Then CPT PMT = 496.48.

3. The cost of the annuity is:
   $1{,}000\,\ddot{a}_{\overline{20}|0.065} + 500\,v^{10}\,\ddot{a}_{\overline{10}|0.065} = 1{,}000(11.73471) + 500(0.53273)(7.65610) = 13{,}774.03$

   If you have set your calculator to BGN mode to solve this problem, remember to reset it to END mode.

4. The present value of the annuity is:
   $500/1.062 + 500(1.04)/1.062^2 + \ldots + 500(1.04^{19})/1.062^{20}$
   $= (500/1.062)[1 + (1.04/1.062) + \ldots + (1.04/1.062)^{19}]$
   $= (500/1.062)[1 - (1.04/1.062)^{20}]/(1 - 1.04/1.062) = 7{,}774.43$

   Alternatively, using the geometric annuity formula:
   $$500 \cdot a^{4\%}_{\overline{20}|6.2\%} = 500 \cdot \frac{1-\left(\dfrac{1.04}{1.062}\right)^{20}}{0.062-0.04} = 7{,}774.43$$

5. Using the BA II Plus, set N = 60, PV = 25,000, PMT = -650, and FV = 0. Then CPT I/Y = 1.592. The quarterly effective rate is 1.592%, so the nominal annual rate convertible quarterly is $4 \times 1.592\% = 6.37\%$.

6. This can be viewed as a perpetuity-immediate with quarterly payments of 10, plus a 20-period-deferred perpetuity-immediate with quarterly payments of 5. The quarterly effective rate is 1.5%.
   The present value is $\dfrac{10}{0.015} + 1.015^{-20}\left(\dfrac{5}{0.015}\right) = 914.16$.

7. The present value of the perpetuity is:
   $$100 \cdot a^{3\%}_{\overline{\infty}|5\%} = \frac{100}{0.05-0.03} = 5{,}000$$

8. The total accumulation A is found by the PQ formula as follows:

$$A = 1000 \cdot s_{\overline{30}|} + 100 \cdot \frac{(s_{\overline{30}|} - 30)}{i}$$

$i = 5.8\%$ and $s_{\overline{30}|} = 76.3298$

$$A = 76,329.80 + \frac{100(46.3298)}{0.058} = 156,208.77$$

9. The year-by-year interest-earning balances of the original investment are 1,000; 2,000; 3,000; ….; 20,000.
The interest amounts earned on the original investment and deposited into the second fund are 70; 140; 210; ….; 1,400. The total accumulation in this fund is:

$$70(Is)_{\overline{20}|0.06} = 70(316.5454) = 22,158.18.$$

The total accumulation in the two accounts at the end of 20 years is:

$$20,000 + 22,158.18 = 42,158.18$$

10. The accumulation at the end of 10 years is $P \cdot \dfrac{(1 + r)^{10} - 1}{r}$.

The accumulation at the end of 20 years is $P \cdot \dfrac{(1 + r)^{20} - 1}{r}$.

Then $P \cdot \dfrac{(1 + r)^{20} - 1}{r} = 3 \cdot P \cdot \dfrac{(1 + r)^{10} - 1}{r}$

$$(1+r)^{20} - 1 = 3 \cdot \left[(1+r)^{10} - 1\right]$$

$$\frac{(1+r)^{20} - 1}{(1+r)^{10} - 1} = \left[(1+r)^{10} + 1\right] = 3$$

(Recall that $x^2 - 1 = (x+1)\cdot(x-1)$, and set $x = (1+r)^{10}$.)

$$(1+r)^{10} = 2$$

$$r = 0.072$$

Module 3

Loan Repayment

In Module 2 we developed a formula to calculate the level payment amount to repay a loan. This module will go into more detail on how loans can be structured and repaid, including non-level payments and sinking-fund loans. We will also develop methods for determining the outstanding balance of a loan as of any date, and for calculating the amount of interest paid and the amount of principal repaid in each loan payment.

Section 3.1

The Amortization Method of Loan Repayment

Consider a loan for 30,000 with level payments to be made at the end of each year for 5 years at an annual effective interest rate of 8%. We already know how to find the annual payment. The equation of value is $\text{Loan} = \text{Pmt} \cdot a_{\overline{n}|i}$, and the loan payment is $\text{Pmt} = \dfrac{30,000}{a_{\overline{5}|8\%}} = 7,513.69$. Or, using the TVM worksheet, set N=5, I/Y=8, PV=30,000 and CPT PMT = -7,513.69.

Once we have found the loan payment amount, we can track the loan balance year by year, based on the fact that the balance increases each year with interest, and decreases at the end of each year when a payment is made. Here are the calculations for the first two years of this loan:

<u>Year 1:</u>	Beginning Bal.	$Bal_0 = 30,000$
	Interest due	$Int_1 = Bal_0 \cdot i = 30,000(0.08) = 2,400$
	Payment	$Pmt = 7,513.69$
	Principal paid	$PRin_1 = Pmt - Int_1 = 7,513.69 - 2,400 = 5,113.69$
	Ending Balance	$Bal_1 = Bal_0 - PRin_1 = 30,000 - 5,113.69 = 24,886.31$
<u>Year 2:</u>	Beginning Bal.	$Bal_1 = 24,886.31$
	Interest due	$Int_2 = Bal_1 \cdot i = 24,886.31 \cdot (0.08) = 1,990.90$
	Payment	$Pmt = 7,513.69$
	Principal paid	$PRin_2 = Pmt - Int_2 = 7,513.69 - 1,990.90 = 5,522.79$
	Ending Balance	$Bal_2 = Bal_1 - PRin_2 = 24,886.31 - 5,522.79 = 19,363.52$

We can generalize this process to develop a **successive value formula** for the loan balance:

(3.1)

$$Bal_t \cdot (1+i) - Pmt_{t+1} = Bal_{t+1}$$

The successive value formula shows that the beginning-of-year balance, plus the interest on that balance during the year, minus the end-of-year loan payment equals the end-of-year loan balance.

Amortization is the gradual reduction of an amount over time. In this case, we are talking about amortizing a debt (a loan) by means of periodic loan payments. The following **amortization table** shows how the entire loan balance of 30,000 is amortized over 5 years by the level annual payments of 7,513.69.

Table (3.2)

Time	Payment	Interest Paid	Principal Paid	Balance
0				30,000.00
1	7,513.69	2,400.00	5,113.69	24,886.31
2	7,513.69	1,990.90	5,522.79	19,363.52
3	7,513.69	1,549.08	5,964.61	13,398.90
4	7,513.69	1,071.91	6,441.78	6,957.12
5	7,513.69	556.57	6,957.12	0.00

You should check the calculations on a few lines of the table to verify your understanding of the amortization procedure and the successive value formula. The key points are:

a) Each loan payment is applied to pay the interest that is due for that year; any remaining part of the payment is applied to repay principal.

b) The final balance is 0. The 5 level payments pay off the loan as intended.

c) As the balance declines over time, the amount of interest due in each period decreases.

d) As the amount of interest due decreases each year, the amount of principal paid in each period increases. It will be shown later that the amount of principal repaid increases each year by a factor of $(1+i)$.

   You can verify this using the Principal column of Table **(3.2)**.

Note: It is important to add a comment on rounding here. The amortization table was generated in EXCEL with amounts calculated to 10 decimal places. The payment was actually 7,513.6936370051, but it is displayed rounded to 2 decimal places. If you recalculate values using the <u>*rounded*</u> *amounts shown, you will find apparent discrepancies of a few pennies in some of the table's values.*

The ability to do amortization calculations easily is of great practical value. For example, you need to know the amount of interest paid on your mortgage to fill out your tax return. Or, if you wish to pay off a loan early, the outstanding balance is the amount you have to pay.

Section 3.2

Calculating the Loan Balance

In calculating the amount of the loan payment for the loan in Table **(3.2)**, we used $t = 0$ as the valuation date. We could have used $t = 5$ as the valuation date and set the accumulated value of the loan amount as of time 5 equal to the accumulated value of the repayments as of time 5. This results in the following equation of value:

$$30{,}000 \cdot (1.08)^5 = Pmt \cdot s_{\overline{5}|8\%}.$$

By rearranging this equation, we can calculate the amount of the loan payment:

$$Pmt = \frac{30{,}000 \cdot 1.08^5}{s_{\overline{5}|8\%}} = 7{,}513.69$$

This value matches the loan payment we calculated using $t = 0$ as the valuation date. Now suppose that we choose $t = 3$ as the valuation date. Setting the value of the loan amount as of time 3 equal to the value of the 5 loan payments as of that date, we have:

$$30{,}000 \cdot (1.08)^3 = Pmt \cdot s_{\overline{3}|8\%} + Pmt \cdot a_{\overline{2}|8\%}$$

Solving for the amount of the loan payment produces the same value as before:

$$Pmt = \frac{30{,}000 \cdot 1.08^3}{s_{\overline{3}|8\%} + a_{\overline{2}|8\%}} = 7{,}513.69$$

We can now rearrange this equation to calculate the outstanding balance of the loan at time 3:

$$Bal_3 = 30{,}000 \cdot 1.08^3 - Pmt \cdot s_{\overline{3}|} = Pmt \cdot a_{\overline{2}|}$$

The expression $30{,}000 \cdot 1.08^3 - Pmt \cdot s_{\overline{3}|}$ is the outstanding loan balance at time 3 calculated by the retrospective method (i.e., the "backward-looking" or historical method). It shows that the loan balance at time 3 equals the net value at time 3 of the cash flows between times 0 and 3. These consist of the accumulated value of the amount the borrower received (at time 0), less the accumulated value of the first 3 loan payments (at times 1, 2, and 3). The calculated balance at time 3 by the retrospective method is:

$$Bal_3 = 30{,}000 \cdot 1.08^3 - Pmt \cdot s_{\overline{3}|} = 37{,}791.36 - 7{,}513.69 \cdot \frac{1.08^3 - 1}{0.08} = 13{,}398.90$$

This calculated value matches the Year 3 ending balance in the amortization table (Table **(3.2)**).

The retrospective formula for the outstanding balance of a level-payment loan is:

(3.3)

$$Bal_t = Loan \cdot (1+i)^t - Pmt \cdot s_{\overline{t|}}$$

This equation assumes that the entire loan amount (*Loan*) is received at time 0 and is repaid by *n* level payments (*Pmt*) to be made at times 1, , 2, …, *n*.

More generally, the **retrospective valuation method** states that the balance of a loan as of time *t* equals the accumulated value of all loan amounts received as of time *t*, minus the accumulated value of all repayments made as of time *t*.

Exercise (3.4)

Use the retrospective method to find the balance after the second loan payment for the loan in Table **(3.2)**

Solution.
The loan was for 30,000 at *i* = 8% with 5 payments. The annual payment was 7,513.69.

$$Bal_2 = 30{,}000(1.08)^2 - 7{,}513.69 \cdot s_{\overline{2|}} = 34{,}992.00 - 15{,}628.48 = 19{,}363.52$$

The balance at time 2 equals the accumulated value of the amount borrowed (34,992), minus the accumulated value of the 2 loan payments that have been made (15,628.48).
This matches the amount shown in Table **(3.2)**.

Exercise (3.5)

A loan for 40,000 at a 7% annual effective rate has an annual payment of 9,755.63. Use the retrospective method to find the balance after the fourth payment.

Answer : 9,117.40

Another method for determining the outstanding balance of a loan is the prospective ("forward-looking") method. In the earlier equation for the outstanding loan balance at time 3 ($Bal_3 = 30{,}000 \cdot 1.08^3 - Pmt \cdot s_{\overline{3|}} = Pmt \cdot a_{\overline{2|}}$), the expression $Pmt \cdot a_{\overline{2|}}$ is the outstanding balance at time 3 calculated by the prospective method. It shows that the loan balance at time 3 equals the value of the two remaining loan payments. This value is:

$$Bal_3 = Pmt \cdot a_{\overline{2|}} = 7{,}513.69 \cdot \frac{1-1.08^{-2}}{0.08} = 13{,}398.90$$

This matches the value calculated by the retrospective method, as well as the amount shown in Table **(3.2)**.

The prospective formula for the outstanding balance of a loan at time t is:

(3.6)

$$Bal_t = Pmt \cdot a_{\overline{n-t}|}$$

This equation assumes that the loan is being repaid by n level payments at times 1, 2, …, n.

More generally, the **prospective valuation method** states that the balance of a loan as of time t equals the present value of all repayments to be made after time t. This is reasonable. At any time during the term of a loan, the borrower has an obligation to make the remaining payments. The outstanding loan balance is the present value of those remaining payments.

Exercise (3.7)

Use the prospective method to find the balance after the first loan payment for the loan in Table **(3.2)**

Solution.
The loan has 5 payments of 7,513.69.
The balance at time 1 equals the present value of the 4 remaining payments:

$$Bal_1 = 7{,}513.69 \cdot a_{\overline{4}|} = 24{,}886.31$$

This matches the amount shown in Table **(3.2)**.

Exercise (3.8)

A loan made at an annual effective rate of 6.5% has 7 remaining payments of 950.
What is the loan balance?

Answer : 5,210.29

Calculator Note

The BA II Plus calculator has amortization features that can be helpful with amortization calculations for level-payment loans. To demonstrate these features, we will look again at the example loan for 30,000 at 8% for 5 years. (That loan is probably still in your calculator. If not, compute the payment again to follow the discussion below.) There are two ways to calculate a loan balance using the calculator's features:

a) The FV key will give you the loan balance after N periods. (Initially, FV=0, which is the loan balance after N=5 periods.) To see the loan balance after 3 periods, set N=3 and CPT FV = -13,398.90. This is the amount that would have to be paid (thus the negative sign) at time 3 in order to pay off the loan after 3 annual loan payments. The calculator is showing that 3 payments of 7,513.69 (PMT) and 1 payment (at time 3) of 13,398.90 (FV) have the same total value as the amount borrowed (at time 0) of 30,000 (PV).
Note: This calculation is applying the retrospective method. (FV equals the accumulated value of PV as of time 3, less 3 PMT's accumulated to time 3).

b) Above the PV key you will see AMORT for the amortization worksheet. To open the amortization worksheet use the keystrokes [2nd] AMORT.
This worksheet allows you to find the principal and interest paid over a time span starting at a first period (P1) and ending at a second period (P2). If you wish to focus on just one period, set both P1 and P2 equal to the value of that period. Thus to see what happens in the first 3 periods of the loan, set P1 = 1 and P2 = 3.The first display asks you to enter the value of P1. Press 1 [ENTER] and then scroll down and press 3 [ENTER] for P2. As you scroll down three more times, you will see the following three values displayed:
BAL = 13,398.90
PRN = −16,601.10
INT = −5,939.99
These represent the balance (BAL) after the selected payments (Bal_3), the amount of principal (PRN) repaid by those payments, and the amount of interest (INT) included in those payments. These functions can be very useful (provided you are confident that you understand how to use them).

Exercise (3.9)

Use the AMORT worksheet to check balance, principal and interest for period 2 for the loan in table **(3.2)**.

Answer: 19,363.52, −5,522.79, −1,990.90

More Challenging Problems

On Exam FM, problems are made tougher by introducing loans whose payments are arithmetic or geometric series. The prospective and retrospective methods can be used to find loan balances for these loans, as the next examples indicate.

Example (3.10)

A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 2% each year. Find the loan balance after the 4th payment.

Solution.

The payments are $100,\ 100(1.02),\ 100(1.02)^2,\ldots,\ 100(1.02)^9$. After the 4th payment the remaining payments are $100(1.02)^4,\ldots,\ 100(1.02)^9$. Using the prospective method, the balance immediately after the fourth payment is the present value of those remaining payments.

$$\begin{aligned} Bal_4 &= \frac{100\left(1.02^4\right)}{1.10} + \ldots . + \frac{100\left(1.02^9\right)}{1.10^6} \\ &= \frac{100\left(1.02^4\right)}{1.10}\left[1 + \frac{1.02}{1.10} + \left(\frac{1.02}{1.10}\right)^2 + \ldots + \left(\frac{1.02}{1.10}\right)^5\right] \\ &= \frac{100\left(1.02^4\right)}{1.10}\left[\frac{1-\left(\frac{1.02}{1.10}\right)^6}{1-\left(\frac{1.02}{1.10}\right)}\right] = 492.93 \end{aligned}$$

Alternatively, using the geometric annuity formula and applying the prospective method, we have:

$$Bal_4 = 100\left(1.02^4\right)\cdot a_{\overline{6}|0.10}^{0.02} = 100\left(1.02^4\right)\cdot \frac{1-\left(\frac{1.02}{1.10}\right)^6}{0.10-0.02} = 492.93$$

Exercise (3.11)

A loan at an 8% annual effective rate has an initial payment of 1,000, and 9 further payments. The payment amount decreases by 2% each year. Find the loan balance immediately after the 3rd payment.

Answer : 4,644.38

Example (3.12)

A loan at a 10% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount increases by 10 each year. Find the loan balance after the 4th payment.

Solution.
Payments are $100, 110, 120, \ldots, 190$. Immediately after the 4th payment the remaining payments are $140, 150, \ldots, 190$. Using the prospective method, the balance immediately after the 4th payment is the present value of those remaining payments. Using $P = 140$ and $Q = 10$, we have:

$$Bal_4 = Pa_{\overline{n|}} + Q\left(\frac{a_{\overline{n|}} - nv^n}{i}\right) = 140a_{\overline{6|}} + 10\left(\frac{a_{\overline{6|}} - 6v^6}{0.10}\right)$$

$$= 140(4.355) + 10\left(\frac{4.355 - 6(0.5645)}{0.10}\right) = 706.57$$

Exercise (3.13)

A loan at an 8% annual effective interest rate has an initial payment of 100, and 9 further payments. The payment amount decreases by 5 each year. Calculate the loan balance immediately after the 6th payment.

Answer : 208.60

Loan Balance Between Payment Dates

All of the preceding analysis involved calculating loan balances as of a payment date. What happens if you need to know the loan balance (the amount needed to pay off a loan) on a date other than a payment date? Between payment dates, the balance simply increases at the loan interest rate. So the balance is equal to the balance immediately after the preceding loan payment, increased by interest for the time since that payment. On the next payment date, of course, the loan balance is reduced by the amount of the payment.

Exercise (3.14)

What is the balance for the loan in Table **(3.2)** 3 months after the 2nd loan payment?

Solution.
The balance after the 2nd payment is 19,363.82. Increasing that balance for 3 months of interest results in a balance of
$19{,}363.52 \cdot 1.08^{0.25} = 19{,}739.68$.

Exercise (3.15)

What is the balance for the loan in Table **(3.2)** eight months after the 3rd loan payment?

Answer : 14,104.31

Section 3.3

Loans With Varying Payments

Many loans have payments that vary over time in patterns other than arithmetic or geometric progressions. In the next example we will demonstrate how to set up a particular kind of loan with non-level payments.

Example (3.16)

A borrower wants to borrow 30,000 at 8% for 5 years, but would like to pay only 5,000 for the first 2 years and then catch up with larger payments in the final 3 years of the loan. What is the payment amount for the final 3 years?

Solution.
First we will use the calculator to find the outstanding loan balance after 2 years if the payment each year is 5,000:

Set N=2, I/Y=8, PV=30,000, and PMT=−5,000.
Then CPT FV=−24,592.00.
This is the amount needed to repay the loan at time 2.

The borrower can pay off the 24,592 balance in 3 years by making larger payments.

Set N=3, I/Y=8, PV=24,592, FV=0, and CPT PMT =−9,542.52.
The payment amount for the final 3 years is 9,542.52.

Below we show the amortization table for the loan of Example **(3.16)**. As before, you should calculate the values for a few rows of the table to make sure that you understand the process.

Table (3.17): Variable-payment loan

Year	Payment	Interest Paid	Principal Paid	Balance
0				30,000.00
1	5,000.00	2,400.00	2,600.00	27,400.00
2	5,000.00	2,192.00	2,808.00	24,592.00
3	9,542.52	1,967.36	7,575.16	17,016.84
4	9,542.52	1,361.35	8,181.17	8,835.67
5	9,542.52	706.85	8,835.67	0.00

Notice that the amount of Principal in each payment is larger than the Principal in the previous payment by a factor of 1.08 $(=1+i)$, provided that the amount of the payment is the same as in the prior year. This is *not* the case for the 3rd loan payment, because its amount is different from the 2nd payment.

Exercise (3.18)

What would the payment for the final 3 years be if the borrower in the above loan paid only 4,000 in each of the first 2 years?

Answer: 10,349.63

Some Notation

We can summarize amortization calculations using the following notation. For a loan with periodic interest rate i:

Loan Payment at time k	Pmt_k
Loan Balance after Pmt_k is made:	Bal_k
Principal paid in Pmt_k	$PRin_k$
Interest paid in Pmt_k	Int_k

Bal_0 is the original loan amount. For $k \geq 1$, values for an amortizing loan can be found by the recursive relations:

Interest paid in Pmt_{k+1} $\quad Int_{k+1} = i(Bal_k)$

Principal paid in Pmt_{k+1} $\quad PRin_{k+1} = Pmt_{k+1} - i(Bal_k)$

Balance after Pmt_{k+1}

$$\begin{aligned} Bal_{k+1} &= Bal_k - PRin_{k+1} \\ &= Bal_k - \left[Pmt_{k+1} - i \cdot (Bal_k)\right] \\ &= (1+i) \cdot Bal_k - Pmt_{k+1} \end{aligned}$$

Section 3.4

Formulas for Level-Payment Loan Amortization

Consider a loan with n level payments of amount Pmt. We will analyze the activity for this loan during year t. The outstanding balance at the beginning of year t (after $t-1$ payments), using the prospective valuation method, is:

$$Bal_{t-1} = Pmt \cdot a_{\overline{n-(t-1)}|} = Pmt \cdot \frac{1-v^{n-t+1}}{i}$$

The interest on this balance during year t is:

$$Int_t = i \cdot Bal_{t-1} = Pmt \cdot \left(1-v^{n-t+1}\right).$$

The payment at time t includes this interest plus an amount of principal equal to:

$$PRin_t = Pmt - Int_t = Pmt - Pmt \cdot \left(1-v^{n-t+1}\right) = Pmt \cdot v^{n-t+1}.$$

(3.19)

For a level-payment loan:

Interest paid in Pmt_t: $Int_t = Pmt\left(1-v^{n-t+1}\right)$

Principal paid in Pmt_t: $PRin_t = Pmt \cdot v^{n-t+1}$

Example (3.20)

For the loan in table **(3.2)** we had $n=5$, $i=0.08$ and $Pmt=7{,}513.69$.

For the payment at $t = 4$, we have $n-t+1=2$.

The principal in the 4th payment is:

$$PRin_4 = \frac{7{,}513.69}{1.08^2} = 6{,}441.78$$

This matches the value given in table **(3.2)**.

Exercise (3.21)

A 10-year annual-payment loan has an annual effective interest rate of 6% and a level payment of 1,000. Find the amount of principal and interest in the 6th payment.

Answer : Principal 747.26 Interest 252.74

We can develop a simple relationship between the amounts of principal repaid on two different payment dates. For example, if the two payment dates are time t and time $t+k$, we have:

$$PRin_{t+k} = Pmt \cdot v^{n-(t+k)+1} = v^{-k} \cdot Pmt \cdot v^{n-t+1} = (1+i)^k \cdot PRin_t$$

The only requirement is that Pmt_t, Pmt_{t+k}, and all of the intervening payments be for equal amounts. For a level-payment loan, this will always be the case. So we have the following relation for a level-payment loan:

(3.22)

$$\boxed{PRin_{t+k} = (1+i)^k \cdot PRin_t}$$

Note that k in this formula can be positive or negative. That is, payment $t+k$ can occur either before or after payment t.

Formulas **(3.19)** and **(3.22)** can often be used to solve exam problems very efficiently, as we shall see in the next example. **You should know these formulas.**

Example (3.23)

For a level-payment loan with an 8% annual effective rate, the amount of principal in the 2nd payment is 5,522.79. Find the amount of principal in the 4th payment.

Solution.
We are not given the total payment amount (Pmt) or the term of the loan, but that information is not needed. Using **(3.19)** we have:

$$PRin_2 = v^{(n-2+1)} Pmt = 5{,}522.79$$
$$PRin_4 = v^{(n-4+1)} Pmt = (1+i)^2 PRin_2 = 1.08^2 (5{,}522.79) = 6{,}441.78$$

Or we can apply **(3.22)** to solve the problem more efficiently:

$$PRin_4 = (1+i)^2 \cdot PRin_2 = 5{,}522.79 \cdot 1.08^2 = 6{,}441.78$$

This problem is based on the loan in table **(3.2)**, and you can check the answer of 6,441.78 there.

Exercise (3.24)

For a level-payment loan at an annual effective rate of 6%, the amount of principal in the first payment is 5,321.89. Find the amount of principal in the 4th payment.

Answer : 6,338.46

Section 3.5

Monthly-Payment Loans

Exams can have questions that involve things such as nominal rates convertible every 4 years applied to a loan with semi-annual payments. But what you will usually encounter in the real world are monthly-payment loans quoted with a nominal rate convertible monthly. In this section, we will review what we have done so far in the context of the familiar monthly-payment loan.

Example (3.25)

A thirty-year monthly-payment mortgage loan for 250,000 is offered at a nominal rate of 6% convertible monthly.

Find:
a) the monthly payment
b) the total principal and interest that will be paid on the loan over 30 years
c) the balance after 5 years
d) the principal and interest paid over the first 5 years.

Solution.

a) Find the monthly payment using the BA II Plus calculator.
The loan has a monthly effective rate of $6\% \div 12 = 0.5\%$.
Set N=360, I/Y=0.5, PV = 250,000, and CPT PMT = −1,498.88.

b) The total principal paid is just the amount of the loan, i.e., 250,000.
The total interest can be found as:
Total payments − Principal Paid = 1,498.88(360) – 250,000 = 289,596.80

You could also do this with the AMORT worksheet (P1=1; P2=360). The answer will be slightly different because we used the *rounded* payment amount in the above calculation.

c) To find the balance after 5 years (60 months), set N=60 and CPT FV =-232,635.89

d) Finding the principal and interest paid during the first 5 years can be done in two ways (as was true for part b).
The amount of principal paid in 5 years is just the original amount of the loan less the balance in 5 years: 250,000 – 232,635.89=17,364.11.
The total interest paid in 5 years can be found as:
Total payments - Principal Paid = 1,498.88(60) – 17,364.11 = 72,568.69
The AMORT worksheet (P1=1; P2=60) will again give a slightly different answer.

Exercise (3.26)

A 15-year monthly-payment mortgage loan for 250,000 is offered at a nominal rate of 6% convertible monthly.

Find:

a) the monthly payment
b) the total principal and interest paid on the loan over 15 years
c) the balance after 5 years, and
d) the principal and interest paid over the first 5 years.

Answers:
a) 2,109.64
b) Principal 250,000 Interest 129,735.57
c) 190,022.75
d) Principal 59,977.25 Interest 66,601.27

Example (3.27)

A 30-year monthly-payment mortgage loan for 250,000 has a nominal interest rate of 6% convertible monthly. The borrower arranges to make graduated payments where the first year's monthly payment is *P*, the second year's monthly payment is *P*+100 and all subsequent monthly payments are *P*+200.

a) Find the initial payment *P*
b) Find the balance at the end of one year.

Solution.

a) The equation of value is:

$$250{,}000 = Pa_{\overline{360}|0.005} + v^{12}(100)a_{\overline{348}|0.005} + v^{24}(100)a_{\overline{336}|0.005}$$
$$= P\cdot 166.7916 + 0.9419(100)(164.7434) + 0.8872(100)(162.5688)$$
$$= P\cdot 166.7916 + 29{,}940.28$$

This gives *P* = 1,319.37

b) Using the prospective method:

$$Bal_{12} = (P+100)a_{\overline{348}|0.005} + v^{12}(100)a_{\overline{336}|0.005}$$
$$= 1{,}419.37(164.7434) + 0.9419(100)(162.5688) = 249{,}144.19$$

By the retrospective method:

$$Bal_{12} = 250{,}000\cdot 1.005^{12} - P\cdot s_{\overline{12}|0.005}$$
$$= 250{,}000\cdot 1.005^{12} - 1319.37(12.3356) = 249{,}144.28$$

The answers differ slightly due to rounding.

Exercise (3.28)

A 15-year monthly-payment mortgage loan for 250,000 is offered at a nominal rate of 6% convertible monthly. The borrower will make graduated payments where the first year's monthly payment is P, the second year's monthly payment is $P+100$, and all subsequent monthly payments are $P+200$.

Find:

a) the initial payment P
b) the balance at the end of one year

Answers: a) 1,938.49 b) 241,507.13

Lenders sometimes charge percentage fees called "points" on a loan that raise their yield. The following example illustrates the effect of these points.

Example (3.29)

A 30-year monthly-payment mortgage loan for 300,000 is offered at a nominal rate of 7.2% convertible monthly. Thus the monthly effective interest rate is 0.6% and the calculated monthly payment is 2,036.36. (Calculate the payment on your calculator and leave it there for the moment.)

When the loan closes, the lender includes a fee of 3 "points" for which no service is performed. It is taking 3% of the loan amount (9,000) as a fee that raises its yield. In effect, the borrower is receiving a loan of only 291,000. This raises the borrower's interest rate. To see this, modify the loan in your calculator by setting PV = 291,000 and CPT I/Y. The result is a monthly rate of 0.63%. Multiply this by 12 to find the nominal rate: 7.51%. That is the borrower's actual nominal annual rate.

The actual nominal annual rate is called the **annual percentage rate** or APR in the United States. Lenders are required by law to reveal this rate to the borrower. The intent of the law is to provide information that would prevent lenders from quoting lower rates and then charging a very large fee in the form of "points." The **annual percentage yield** or APY for a loan is the annual *effective* rate being charged.

Exercise (3.30)

A 15-year monthly-payment mortgage loan for 200,000 is offered at a nominal rate of 7.2% convertible monthly. The lender charges a fee of 2% for which no services are provided. Find the APR.

Answer : 7.53%

Variable Interest Rate Loans

It has become common for mortgage loans to be written with variable interest rates. The concept of variable interest rates will be discussed in more detail in Module 9, but the following example demonstrates calculations for a variable-rate loan.

Example (3.31)

A homebuyer takes out a 30-year monthly-payment variable-rate mortgage loan for 250,000. The initial interest rate for the mortgage is 6.0% convertible monthly. This rate is level for the first 5 years, after which the rate can increase or decrease annually, based on the level of an interest rate index. (Interest rate indexes will be discussed in Module 9.)

To find the initial monthly payment, set N=360, I/Y=0.5 and PV=250,000. Then CPT PMT=-1,498.88. The initial monthly payment for this mortgage is 1,498.88.

Suppose that at the end of 5 years the interest rate adjusts upward to 6.6% convertible monthly. To find the new monthly payment amount, set N=300 and CPT PV=232,635.89. This is the outstanding balance at time 5, calculated by the prospective method (the present value of the 300 remaining mortgage payments).

To calculate the new monthly payment for years 6 and later, set I/Y=0.55 and CPT PMT=-1,585.34. The new payment in year 6 is 1,585.34, and payments will continue at this level until the next interest rate adjustment.

Each time the interest rate is adjusted, it is necessary to determine the outstanding balance (the present value of future payments based on the number of payments remaining, the old payment amount, and the old interest rate). Then the new monthly payment is calculated based on the outstanding balance, the number of payments remaining, and the new interest rate.

Exercise (3.32)

For the mortgage described in Exercise **(3.31)**, suppose that at time 7, the interest rate changes again. This time it is reduced from 6.6% to 5.7%. What is the outstanding balance at time 7, and what is the new payment amount?

Answers: 224,812.08 1,463.60

Section 3.6

An Example With Level Payment of Principal

Loan payments can have any pattern that the borrower and lender agree on. In the following example, the borrower pays a level amount of principal each year, plus the interest that is due at that time.

Example (3.33)

You borrow 30,000 and will repay it over 5 years at an 8% annual effective interest rate. You agree to pay off the principal in installments of 6,000 per year, and to pay interest on the outstanding balance each year. The amortization table is shown below.

Year	0	1	2	3	4	5
Interest		2,400	1,920	1,440	960	480
Principal		6,000	6,000	6,000	6,000	6,000
Total Payment		8,400	7,920	7,440	6,960	6,480
Balance	30,000	24,000	18,000	12,000	6,000	0

Here we first find the interest due and add it to the level principal payment to get the total payment. For example, at time 2 we apply 8% interest to the previous outstanding balance of 24,000:

$$0.08(24,000) = 1,920 .$$

Then we add that amount to the principal payment of 6,000 to find the total payment amount: 7,920.
Note that the principal payments for this loan are level, but the interest due and the total payment amount decrease.

It is easy to find the interest due for a period without constructing the table. Suppose that you wanted to find the interest due in the 4th payment. Note that

$$Bal_3 = 30,000 - 3(6,000) = 12,000$$

$$Int_4 = i \cdot Bal_3 = (0.08)12,000 = 960$$

The interest due in the 4th payment is 960 and the total payment is 6,960.

Exercise (3.34)

You have a 30-year loan for 30,000 at an 8% annual effective rate. You agree to pay the principal in installments of 1,000 per year, and to pay interest on the outstanding balance each year.

Find: a) the interest due in the 11th payment
b) the total amount of the 11th payment.

Answers : a) 1,600 b) 2,600

Section 3.7

Sinking Fund Repayment of a Loan

When a borrower uses a sinking fund, each period the borrower pays the lender only the interest due on the loan at the loan interest rate, i. In addition, the borrower makes level deposits to an account called a "**sinking fund**" that earns interest at a rate j.

The amount of the sinking fund deposits is determined so that the fund will accumulate to the amount of the loan at the end of the loan term. Thus the borrower can pay off the loan when it is due using the final sinking fund balance.

Example (3.35)

A loan of 100,000 is made for a term of 10 years at 10% interest. The lender will receive annual payments of only the interest due until the end of year 10 when the 100,000 principal must be repaid. The borrower will make level annual year-end payments to a sinking fund earning 8%.

Find the amount of the annual sinking fund deposit, and the balance in the sinking fund at times 3 and 4.

Solution.
The first task is to find the required sinking fund deposit. This is easily done using the BA II Plus calculator. Set N=10, I/Y=8, FV=100,000, and CPT PMT= -6,902.95. The annual sinking fund deposit is 6,902.95.
(Note that each year the borrower will also pay $100,000(0.10)=10,000$ of interest to the lender, resulting in total annual payments of 16,902.95.)

Next we will look at the balance in the sinking fund. The balance at time k is the future value of k payments of 6,902.95 to the fund. To get the balance at time 3, set N=3 and CPT FV = 22,409.73. To get the balance at time 4, set N=4 and CPT FV = 31,105.46.

Exercise (3.36)

A loan of 70,000 is made for a term of 10 years at 8% interest. The lender wants only payments of interest until the end of year 10 when the 70,000 principal must be repaid. The borrower will make level annual year-end deposits to a sinking fund earning 6%.

Find the amount of the sinking fund deposits and the balance in the sinking fund at times 5 and 6.

Answers : *SFD*=5,310.76; Balances: 29,937.23, 37,044.22

The sinking fund balance is important, and we will refer to the balance at time k as $SFBal_k$. The amount owed to the lender does not change from period to period, since the borrower is paying only the interest due. However, we can view the borrower's **net indebtedness** at time k as being the amount of the loan minus the balance in the sinking fund: $Loan - SFBal_k$. This net indebtedness decreases over time as the sinking fund balance increases.

In Example **(3.35)** we looked at the sinking fund balances at two successive time periods. The difference between the balances at times $k-1$ and k is the amount by which the borrower's net indebtedness has decreased during year k. We will refer to this difference as the amount of *principal* in the k^{th} payment. In Example **(3.35)**, we have:

$$\text{Principal in } 4^{\text{th}} \text{ Payment } = SFBal_4 - SFBal_3 = 31{,}105.46 - 22{,}409.73 = 8{,}695.73$$

The sinking fund is where the borrower accumulates the funds to repay the loan principal of 100,000. Thus the change in the sinking fund from time 3 to time 4 is considered to be principal, even though it is not applied to repay loan principal at time 4. Note that this "principal" consists of the sinking fund deposit at time 4, *plus* interest earned in the sinking fund from time 3 to time 4. Using the notation *SFD* for the sinking fund deposit, we have:

$$\text{Principal in } k^{\text{th}} \text{ payment} = SFBal_k - SFBal_{k-1} = SFBal_{k-1} \cdot j + SFD$$

Once you know the amount of principal in a payment, you can find the interest in that payment too.

$$\text{Interest in } 4^{\text{th}} \text{ Pmt} = \text{Total Pmt} - \text{Principal Paid} = 16{,}902.95 - 8{,}695.73 = 8{,}207.22$$

There is an alternative way to find the interest in a sinking fund payment. In each period the borrower pays interest to the lender, but also earns interest on the sinking fund balance. The difference between the interest paid and the interest earned is called the **net interest**.

At time 3, the balance in the sinking fund was 22,409.73, so the amount of interest earned during the 4^{th} year was $0.08 \cdot (22{,}409.73) = 1{,}792.78$. The interest paid to the lender at time 4 was 10,000. So we have:

$$\text{Net Interest in } 4^{\text{th}} \text{ Payment } = 10{,}000 - 1{,}792.78 = 8{,}207.22$$

Exercise (3.37)

For the sinking fund loan in Exercise **(3.36)** find the amount of principal in the 6^{th} payment and the amount of net interest during the 6^{th} year.

Answers: Principal=7106.99; Interest=3803.77

We have proceeded intuitively and relied on the calculator so far. Now we will introduce some formulas that are commonly found in actuarial texts. We will denote the loan amount by L and the term of the loan by n. Recall that the loan interest rate is i and the sinking fund interest rate is j.The sinking fund deposit satisfies the equation $SFD\left(s_{\overline{n|}}\right)=L$. Thus:

(3.38)
$$SFD=\frac{L}{s_{\overline{n|}j}}$$

The interest payable to the lender each period is Li, so the total loan payment each period is given by:

(3.39)
$$\text{Total sinking fund loan payment} = \frac{L}{s_{\overline{n|}j}}+Li=L\left(\frac{1}{s_{\overline{n|}j}}+i\right)$$

The balance in the sinking fund at time k is given by:

(3.40)
$$SFBal_k=SFD\cdot s_{\overline{k|}j}=L\cdot\frac{s_{\overline{k|}j}}{s_{\overline{n|}j}}$$

The principal paid in payment k is:

$$\begin{aligned}SFBal_k-SFBal_{k-1}&=SFD\cdot s_{\overline{k|}j}-SFD\cdot s_{\overline{k-1|}j}\\&=SFD\cdot\left(s_{\overline{k|}j}-s_{\overline{k-1|}j}\right)\\&=SFD\cdot(1+j)^{k-1}\end{aligned}$$

This last expression is useful for exam problems:

(3.41)
$$\text{Principal paid in sinking fund payment } k = SFD\cdot(1+j)^{k-1}$$

The net interest paid in payment k can be given in two ways:

(3.42)
$$\begin{aligned}&\text{Net interest = Total Payment – Principal Paid}\\&=(SFD+Li)-SFD(1+j)^{k-1}\end{aligned}$$

(3.43)
$$\begin{aligned}&\text{Net Int = Int Paid to Lender – Int Earned on Sinking Fund}\\&=L\cdot i-SFBal_{k-1}\cdot j\end{aligned}$$

This is a large assortment of formulas. A crucial one for exam problems is **(3.41)**. In the next examples we will give a simple application and then an example of a tougher problem that is easy to solve if you know **(3.41)**.

Example (3.44)

For the sinking fund loan in **(3.35)** we found that *SFD* = 6,902.95 and the principal paid in the 4th payment was 8,695.73. We can check the principal paid amount using **(3.41)**.

$$SFD \cdot (1+j)^{k-1} = 6{,}902.95 \cdot (1.08)^3 = 8{,}695.73$$

Exercise (3.45)

Use **(3.41)** to verify the principal paid amount in Exercise **(3.37)**.

Example (3.46)

For a sinking fund loan *SFD* = 5,310.76 and the amount of principal in the third payment is 5,967.17. What is the interest rate?

Solution.

$$\begin{aligned}\text{Principal Paid} &= SFD \cdot (1+j)^{k-1} \\ &= 5{,}310.76(1+j)^2 \\ &= 5{,}967.17\end{aligned}$$

$$1+j = \sqrt{\frac{5967.17}{5310.76}} = 1.06 \rightarrow j = 0.06$$

Exercise (3.47)

For a sinking fund loan *SFD* = 5,066.43 and the amount of principal in the fourth payment is 6,206.59. What is the sinking fund interest rate?

Answer : 7%

A note about "sinking fund loans" and "installment loans":
For the sinking fund loans described in this section, none of the principal was repaid until the end of the loan term. These are **"interest-only loans,"** which is the most common type to involve sinking funds (and the type you are most likely to see on Exam FM). But it is *the existence of a sinking fund* that makes it a "sinking fund loan." The details of the loan and its repayments are separate from the sinking fund, and the sinking fund does not affect the lender.

In the previous section, Example (3.33) referred to a loan with level payments of principal and called these payments "installments." You may encounter this use of the word "installments" on Exam FM (see Sample Exam Question #3 at the end of this module). But you should know that the phrase **"installment loan"** generally refers to any loan where a specific amount is borrowed and is repaid by a series of periodic payments.

Section 3.8

Capitalization of Interest and Negative Amortization

Capitalization of interest and negative amortization occur when a loan payment is less than the interest due on the loan. Let's look at an example:

Example (3.48)

A borrower would like to borrow 30,000 at 8% for 5 years, but would like to pay only 2,000 for the first two years and then catch up with a higher payment for the final three years. What is the payment for the final 3 years?

Solution.
First we will use the calculator to find the loan balance in 2 years if the payment each year is 2,000.

Set N=2, I/Y=8, PV = 30,000, PMT=−2,000, and CPT FV= −30,832.00.
The balance at time 2 is 30,832.

After the second payment of 2,000, the borrower owes 30,832, which can be paid off in 3 years by making larger payments. To find the new payment amount, set N=3, I/Y=8, PV=30,832, and FV=0. Then CPT PMT=−11,963.85. The payment for the final 3 years is 11,963.85.

Below we show the amortization table for the loan.

Year	0	1	2	3	4	5
Payment		2,000.00	2,000.00	11,963.85	11,963.85	11,963.85
Interest		2,400.00	2,432.00	2,466.56	1,706.78	886.21
Principal		-400.00	-432.00	9,497.29	10,257.07	11,077.64
Balance	30,000.00	30,400.00	30,832.00	21,334.71	11,077.64	0.00

Notice what happened in years 1 and 2. The total payment was less than the interest required, so the "principal paid" amount was negative. When we subtract the negative "principal paid" from the prior balance, the effect is to add the shortfall to the balance of the loan. The unpaid portion of the interest is "capitalized." In period 1 there is a shortfall of 400, and this means that the borrower owes 30,000+400 = 30,400 at the end of the first year.

Such an increase in the loan balance is called **negative amortization** because the amount of principal amortized is negative. It is also referred to as **capitalization of interest**, since the unpaid interest becomes capital (i.e., part of the principal) when it is added to the loan balance.

Negative amortization has been present in many United States mortgage loans which were structured to have low initial payments. Low initial monthly payments facilitated the sale of homes, increasing the demand for houses and causing house prices to rise. But it also affected the soundness of the loans. The "housing bubble" burst when many borrowers could not make their mortgage payments, in many cases because the payment had adjusted upward from the artificially low initial level. Defaults on mortgages caused many large financial institutions to fail and contributed to the financial crisis of 2007-2008.

Exercise (3.49)

Find the interest required, principal paid, and balance at time 1 if the initial payment in Example 3.47 is 1,500 instead of 2,000.

Answers: Interest=2,400; Principal=–900; Balance=30,900

Section 3.9

Formula Sheet

- Loan amount: Bal_0
- Loan Payment at time k: Pmt_k
- Loan Balance after Pmt_k is made: Bal_k
- Interest paid in Pmt_k: $Int_k = i\left(Bal_{k-1}\right)$
- Principal paid in Pmt_k: $PRin_k = Pmt_k - i \cdot \left(Bal_{k-1}\right)$
- Balance after Pmt_k:

$$\begin{aligned} Bal_k &= Bal_{k-1} - PRin_k \\ &= Bal_{k-1} - \left[Pmt_k - i \cdot Bal_{k-1}\right] \\ &= (1+i) \cdot Bal_{k-1} - Pmt_k \end{aligned}$$

- For a level-payment loan with n payments of Pmt:
  - Interest paid in Pmt_t: $Int_t = Pmt \cdot \left(1 - v^{n-t+1}\right)$
  - Principal paid in Pmt_t: $PRin_t = Pmt_t - Int_t = Pmt \cdot v^{n-t+1}$
  - Principal paid in Pmt_{t+k}: $PRin_{t+k} = (1+i)^k \, PRin_t$

- Prospective Method: The loan balance equals the present value of the remaining payments.

  For a level-payment loan: $Bal_k = Pmt \cdot a_{\overline{n-k|}}$

- Retrospective Method: The loan balance equals the accumulated value of the amount(s) borrowed, minus the accumulated value of the payments that have been made.

  For a level-payment loan: $Bal_k = Bal_0 \cdot (1+i)^k - Pmt \cdot s_{\overline{k|}}$

Sinking Fund Loan

- $SFD = \dfrac{L}{s_{\overline{n|}j}}$ (i=int. rate on loan j=int. rate on sinking fund)
- Total sinking fund loan payment = $SFD + Li = \dfrac{L}{s_{\overline{n|}j}} + Li = L\left(\dfrac{1}{s_{\overline{n|}j}} + i\right)$
- $SFBal_k = SFD \cdot s_{\overline{k|}j} = L \cdot \dfrac{s_{\overline{k|}j}}{s_{\overline{n|}j}}$
- Principal Paid:
  $PRin_k = SFBal_k - SFBal_{k-1} = SFD \cdot \left(s_{\overline{k|}j} - s_{\overline{k-1|}j}\right) = SFD \cdot (1+j)^{k-1}$
- Net Int = Int Paid – Int Earned
  $= L \cdot i - SFBal_{k-1} \cdot j = (SFD + Li) - SFD \cdot (1+j)^{k-1}$

Section 3.10

Basic Review Problems

1. A loan for 50,000 has level payments to be made at the end of each year for 10 years at an annual effective rate of 9%. Find a) the balance at the end of 3 years and b) the principal and interest paid in the third payment.

2. A borrower would like to borrow 50,000 at 7.5% for 5 years, but wants to pay only 6,000 in each of the first 2 years and then catch up with a higher payment amount for the final 3 years. What is the annual payment amount for the final 3 years?

3. You have a 20,000 loan at 7.2% annually for 8 years. You agree to pay the principal in level installments of 2,500 per year, and to pay interest on the outstanding balance each year. Find the interest due and the total payment at time 6.

4. For a level-payment loan at an effective interest rate of 6.3% per payment period, the amount of principal in the third payment is 845.28. Find the amount of principal in the 7th payment.

5. A loan made at an annual effective rate of 6% has 10 remaining annual payments of 1,000. What is the loan balance?

6. A loan at a 7% annual effective rate has an initial payment of 250 and 9 further payments. The payment amount increases by 3% each year. Find the loan balance immediately after the 7th payment.

7. A loan at a 6.5% annual effective rate has an initial payment of 300 and 9 further payments. The payment amount increases by 20 each year. Find the loan balance immediately after the 6th payment.

8. A 30-year monthly-payment mortgage for 500,000 is offered at a nominal annual rate of 8.4% convertible monthly. Find a) the monthly payment, b) the total principal and interest that will be paid over 30 years, c) the balance after 5 years, and d) the amount of principal and amount of interest paid over the first 5 years.

9. A 30-year monthly-payment mortgage loan for 325,000 is offered at a rate of 6.6% convertible monthly. The borrower would like to have graduated payments where the first year's monthly payment is P and all subsequent monthly payments are P+500. Find a) the initial payment P, and b) the balance at the end of one year.

10. A 30-year monthly-payment mortgage loan for 250,000 is offered at a nominal rate of 6.3% convertible monthly. The lender charges a fee of 2.5% for which no services are provided. Find the APR.

11. A 65,000 annual-payment loan is made for a term of 10 years at 7.3% interest. The lender wants only payments of interest each year until the end of year 10 when the 65,000 must be repaid. The borrower will make level annual year-end deposits to a sinking fund earning 4.8%. Find the level sinking fund deposit and the balance in the sinking fund at time 5.

12. For the loan in problem 11, find the amount of principal and the net interest in the 6th payment.

Section 3.11

Basic Review Problem Solutions

1. Use the AMORT worksheet. First input the TVM data: Set N=10, I/Y=9, PV=50,000, and CPT PMT=-7,791.00. Then press 2ND AMORT and enter 3 for both P1 and P2. The remaining balance after the 3rd payment is 39,211.77. The principal paid in the 3rd payment is 3,910.04 and the interest is 3,880.96. Alternatively:

$$Pmt = \frac{50,000}{a_{\overline{10}|}} = 7,791.00$$

$$PRin_3 = 7,791 \cdot v^{10-3+1} = 3,910.04$$

$$Int_3 = Pmt - PRin_3 = 3,880.96$$

2. First we will use the calculator to find the loan balance after 2 years if the payment each year is 6,000.
   Set N=2, I/Y = 7.5, PV = 50,000, PMT=−6,000 , and CPT FV=−45,331.25. The balance is 45,331.25.

   After the second payment of 6,000, the borrower owes 45,331.25. He can pay this off in 3 years by making larger payments. To find the amount of these payments, set N=3, I/Y = 7.5, PV=45,331.25, FV=0, and CPT PMT=−17,431.57. The payment for the final 3 years is 17,431.57.

3. After 5 years, the balance is $Bal_5 = 20,000 - 5 \cdot (2,500) = 7,500$.
   The interest due on the 6th payment date is $7,500 \cdot (0.08) = 540$.
   The total payment is $2,500 + 540 = 3,040$.

4. We will use the formula $PRin_{n+k} = (1+i)^k \cdot PRin_n$.
   $PRin_7 = (1+i)^4 \cdot PRin_3 = (1.063)^4 \cdot (845.28) = 1,079.28$

5. Applying the prospective method: $Bal = 1,000 \cdot a_{\overline{10}|6\%} = 7,360.09$

6. The payments are: $250,\ 250(1.03),\ 250(1.03)^2, \ldots,\ 250(1.03)^9$.
   Immediately after the 7th payment the remaining payments are:
   $250(1.03)^7,\ 250(1.03)^8,\ 250(1.03)^9$
   Using the prospective method, the balance immediately after the 7th payment is the present value of those remaining payments:

$$Bal_7 = \frac{250(1.03^7)}{1.07} + \frac{250(1.03^8)}{1.07^2} + \frac{250(1.03^9)}{1.07^3} = 830.24$$

7. First we will show the (much faster) calculator solution using the Cash Flow worksheet, then the traditional increasing annuity method (in case you want to practice it).
   Calculator method:
   In the CF worksheet, clear the registers by pressing 2ND CLR WORK, then set CF_0=0, C01=420, C02=440, C03=460, and C04=480. Next press NPV, set I=6.5, CPT NPV=1,536.22.

   Traditional method:
   The payments are $300,\ 320,\ 340,\ldots,\ 480$. Immediately after the 6th payment the remaining payments are 420, 440, 460, 480. Using the prospective method, the balance immediately after the 6th payment is the present value of those remaining payments.
   Using P = 420 and Q = 20, we have:
$$Bal_6 = Pa_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right) = 420a_{\overline{4}|} + 20\left(\frac{a_{\overline{4}|} - 4v^4}{0.065}\right)$$
$$= 420(3.4258) + 20\left(\frac{3.4258 - 4(0.7773)}{0.065}\right) = 1,536.22$$

8. 8.4% convertible monthly $\Rightarrow$ 0.7% monthly effective rate.
   The number of payment periods (months) is 360 (= 30 yrs × 12 months/year).
   a) Set N=360, I/Y=0.7, PV = 500,000, and CPT PMT = -3,809.19.
   b) The total principal paid is just the amount of the loan: 500,000.
      The total interest can be found as:
      Total pmts – Principal Paid = 3,809.19(360) – 500,000 = 871,307.79
   c) *Retrospective method:*
      Set N=60 (for 5 years) and CPT FV = −477,043.37.
      Prospective method:
      Set N=300, FV=0, and CPT PV = 477,043.37.
   d) The AMORT worksheet with P1=1 and P2=60 gives principal of 22,956.75 and interest of 205,594.65.

   Note: To use the AMORT worksheet , the values of PV, I/Y, and PMT must reflect the original loan amount. So if you used the Prospective method in part c)*, you will need to re-enter 500,000 for PV.*

9. a) The monthly effective rate is 0.55%. The equation of value is:

$$325{,}000 = Pa_{\overline{360}|0.0055} + v^{12}(500)a_{\overline{348}|0.0055}$$
$$= P \cdot 156.5781 + .9363(500)(154.861)$$
$$= P \cdot 156.5781 + 72{,}498.19$$

Solving for P: $P = \dfrac{325{,}000 - 72{,}498.19}{156.5781} = 1{,}612.63$

b) We can apply the prospective method (because payments are level after the first year at 2,112.63):
Set N=348, I/Y=0.55, PMT=−2,112.63 and CPT PV=327,163.90.
Alternatively, we can apply the retrospective method (because payments are level during the first year at 1,612.63):
Set N=12, I/Y=0.55, PV=325,000, PMT=−1,612.63,
and CPT FV=−327,163.90.
Note: The balance at time 1 year is greater than the original loan. This is an example of negative amortization (because the monthly payments during the first year are less than the monthly interest).

10. First find the monthly payment. The monthly effective rate is 0.525%.
Set N=360, I/Y=0.525, PV=250,000, and CPT PMT=−1,547.43.
The 2.5% fee is 6,250, so the actual loan amount is 250,000−6,250 = 243,750.

Set PV = 243,750 and CPT I/Y=0.5452%. Multiply by 12 to find the nominal rate (the APR): 6.5422%.

11. Sinking fund deposit: Set N=10, I/Y=4.8, FV=65,000, and CPT PMT=−5,216.23.

Balance at time 5: Set N=5 and CPT FV=−28,708.06.

12. The annual interest payment is 65,000(0.073) = 4,745.
Thus, the total payment is 4,745 + 5,216.23 = 9,961.23
The principal in the 6th payment is:

$$PRin_6 = (1+j)^{6-1} \cdot SFD = 1.048^5(5{,}216.23) = 6{,}594.22.$$

The net interest equals:

$$Total\ Pmt - PRin_6 = 9{,}961.23 - 6{,}594.22 = 3{,}367.01.$$

Alternatively, net interest can be calculated as the interest paid on the loan, minus interest earned on the sinking fund:

$$Int\ pd = L \cdot i = 65{,}000 \cdot 0.073 = 4{,}745$$
$$Int\ earned = j \cdot SFBal_5 = 0.048 \cdot 5{,}216.23 \cdot s_{\overline{5}|0.048} = 1{,}377.99$$
$$Net\ interest = 4{,}745.00 - 1{,}377.99 = 3{,}367.01$$

Section 3.12

Sample Exam Problems

1. **(2005 Exam FM Sample Questions #4)**
John borrows 10,000 for 10 years at an annual effective interest rate of 10%. He can repay this loan using the amortization method with payments of 1,627.45 at the end of each year. Instead, John repays the 10,000 using a sinking fund that pays an annual effective interest rate of 14%. The deposits to the sinking fund are equal to 1,627.45 minus the interest on the loan and are made at the end of each year for 10 years.

   Determine the balance in the sinking fund immediately after repayment of the loan.

   (A) 2,130 (B) 2,180 (C) 2,230 (D) 2,300 (E) 2,370

2. **(2005 Exam FM Sample Questions #9)**
A 20-year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%. Calculate X.

   (A) 32 (B) 57 (C) 70 (D) 97 (E) 117

3. **(2005 Exam FM Sample Questions #15)**
A 10-year loan of 2000 is to be repaid with payments at the end of each year. It can be repaid under the following two options:
(i) Equal annual payments at an annual effective rate of 8.07%.
(ii) Installments of 200 each year plus interest on the unpaid balance at an annual effective rate of i.
The sum of the payments under option (i) equals the sum of the payments under option (ii).

   Determine i.

   (A) 8.75% (B) 9.00% (C) 9.25% (D) 9.50% (E) 9.75%

4. **(2005 Exam FM Sample Questions #16)**
A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment.

   Calculate the outstanding loan balance immediately after the 40th payment is made.

   (A) 6751 (B) 6889 (C) 6941 (D) 7030 (E) 7344

5. (2005 Exam FM Sample Questions #24)
A 20-year loan of 20,000 may be repaid under the following two methods:
i) amortization method with equal annual payments at an annual effective rate of 6.5%
ii) sinking fund method in which the lender receives an annual effective rate of 8% and the sinking fund earns an annual effective rate of j
Both methods require a payment of X to be made at the end of each year for 20 years.

Calculate j.

(A) $j \le 6.5\%$ (B) $6.5\% < j \le 8.0\%$ (C) $8.0\% < j \le 10.0\%$
(D) $10.0\% < j \le 12.0\%$ (E) $j > 12.0\%$

6. (2005 Exam FM Sample Questions #26)
Seth, Janice, and Lori each borrow 5000 for five years at a nominal interest rate of 12%, compounded semi-annually.

- Seth has interest accumulated over the five years and pays all the interest and principal in a lump sum at the end of five years.
- Janice pays interest at the end of every six-month period as it accrues and pays the principal at the end of five years.
- Lori repays her loan with 10 level payments at the end of every six-month period.

Calculate the total amount of interest paid on all three loans.

(A) 8718 (B) 8728 (C) 8738 (D) 8748 (E) 8758

7. (2005 Exam FM Sample Questions #28)
Ron is repaying a loan with payments of 1 at the end of each year for n years. The amount of interest paid in period t plus the amount of principal repaid in period $t + 1$ equals X.

Calculate X.

(A) $1+\dfrac{v^{n-1}}{i}$ (B) $1+\dfrac{v^{n-1}}{d}$ (C) $1+v^{n-t}i$ (D) $1+v^{n-t}d$ (E) $1+v^{n-t}$

8. (2005 Exam FM Sample Questions #46)
Seth borrows X for four years at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding loan balance at the end of the third year is 559.12.

Calculate the principal repaid in the first payment.

(A) 444 (B) 454 (C) 464 (D) 474 (E) 484

9. (May 05 #8)
A loan is being repaid with 25 annual payments of 300 each. With the 10^{th} payment, the borrower pays an extra 1000, and then repays the balance over 10 years with a revised annual payment.
The annual effective interest rate is 8%.

Calculate the amount of the revised annual payment.

(A) 157 (B) 183 (C) 234 (D) 257 (E) 383

10. (May 05 #2)
Lori borrows 10,000 for 10 years at an annual effective interest rate of 9%. At the end of each year, she pays the interest on the loan and deposits the level amount necessary to repay the principal to a sinking fund earning an annual effective interest rate of 8%.

The total payments made by Lori over the 10-year period equal X.

Calculate X.

(A) 15,803 (B) 15,853 (C) 15,903 (D) 15,953 (E) 16,003

11. (May 05 #25)
A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter.

Calculate the amount of principal in the fourth payment.
(A) 0 (B) 0.90 (C) 2.70 (D) 5.20
(E) There is not enough information to calculate the amount of principal.

12. (Nov 05 #18)
A loan is repaid with level annual payments based on an annual effective interest rate of 7%. The 8th payment consists of 789 of interest and 211 of principal.

Calculate the amount of interest paid in the 18th payment.

(A) 415 (B) 444 (C) 556 (D) 585 (E) 612

Section 3.13

Sample Exam Problem Solutions

1.

John owes interest of $10{,}000(0.10)=1{,}000$ at the end of each year. He puts the remaining 627.45 into the sinking fund at 14%.

We can find the accumulated value of the sinking fund in 10 years using the financial calculator with N=10, I/Y = 14, and PMT = 627.45. Computing FV gives the accumulated value of 12,133.19.

John must use this accumulated value to pay the loan amount of 10,000. The balance in his account after repaying the loan is 12,133.19−10,000.00=2,133.19. In this situation, the amount deposited into the sinking fund each year was not based on the amount needed to repay the loan. It was greater than the amount needed to accumulate 10,000.

Answer A

2.

The loan bears interest at 10%, and each of the first 10 payments is 150% of the interest due. Therefore, each payment pays the 10% accrued interest plus 5% of the remaining principal. This reduces the principal by 5% each year (and also reduces the annual payment by 5% each year).

After 10 payments, the loan balance has been reduced by 5% ten times, so the loan balance at time 10 is $1{,}000\cdot(1-5\%)^{10}=598.74$

(We could have found this value by the retrospective method by accumulating the loan 10 years at 10% and subtracting the accumulated value of the loan payments, which form a geometric annuity. But the above method is simpler.)

After 10 years, we have a level-payment loan with payments of X for 10 more periods at a rate of 10%. Using the financial calculator with N=10, I/Y=10, and PV=-598.74, the computed PMT is 97.44 = X.

Alternatively, $X=\dfrac{598.74}{a_{\overline{10}|}}=97.44$.

Answer D

3.
We will first find the total payments for each of the two options.
Option (i):
We can calculate the payment for option (i) using the calculator with N=10, I/Y=8.07, and PV=2,000. The payment is 299.00, so the total payments over 10 years are 2,990.
Option (ii):
Since the full amount of interest on the unpaid balance is being paid each year, the additional payment of 200 reduces the principal. So the loan balance at the beginning of each of the 10 years is: 2000, 1800, 1600, … 200. We do not know the unknown rate i, but we can use this pattern of principal amounts to derive an expression for the total interest payments in terms of i:

Total int. paid $= i[2{,}000 + 1{,}800 + \ldots + 200] = 200i[10 + 9 + \ldots 1] = 200i[55] = 11{,}000i$

The amount of the loan is 2,000, so we have:

Total Principal Paid = 2,000

Total Payments $= 2{,}000 + 11{,}000 \cdot i$

Since total payments under both options are the same, we can calculate i:

$$2{,}990 = 2{,}000 + 11{,}000 \cdot i \qquad i = \frac{990}{11{,}000} = 0.09$$

Answer B

4.
With a nominal rate of 9%, the monthly effective rate is 0.75%. The sequence of 60 payments is a geometric annuity with payments of:

$$1{,}000;\ 1000(0.98);\ 1{,}000(0.98)^2;\ \ldots;\ 1{,}000(0.98)^{59}$$

We were not told the amount of the loan. We could calculate it (by finding the present value of the above geometric annuity) and then find the outstanding balance after the 40th payment by the retrospective method. But it will be simpler to apply the *prospective* method by calculating the present value of the remaining 20 payments at time 40:

$$1{,}000\left[\frac{0.98^{40}}{1.0075} + \frac{0.98^{41}}{1.0075^2} + \ldots + \frac{0.98^{59}}{1.0075^{20}}\right]$$

$$= 1{,}000\left(\frac{0.98^{40}}{1.0075}\right)\left[1 + \left(\frac{0.98}{1.0075}\right) + \left(\frac{0.98}{1.0075}\right)^2 \ldots + \left(\frac{0.98}{1.0075}\right)^{19}\right]$$

$$= 1{,}000\left(\frac{0.98^{40}}{1.0075}\right)\left[\frac{1 - \left(\frac{0.98}{1.0075}\right)^{20}}{1 - \left(\frac{0.98}{1.0075}\right)}\right] = 6{,}889.11$$

Alternatively, by the geometric annuity formula:

$$1{,}000 \cdot \left(0.98^{40}\right) \cdot a_{\overline{20}|0.75\%}^{-2\%} = 445.70 \cdot \frac{1 - \left(\frac{0.98}{1.0075}\right)^{20}}{0.0075 - (-0.02)} = 6{,}889.11$$

Answer B

5.
For payment method (i), set N=20, I/Y = 6.5, and PV = 20,000, and CPT PMT = −1,815.13. Therefore, the X referred to in the problem is 1,815.13.

For payment method (ii), the total payment is also X = 1,815.13. The 8% interest paid each year is $0.08(20{,}000) = 1{,}600$. This leaves 215.13 to be deposited to the sinking fund. The sinking fund accumulates to FV = 20,000 over N = 20 years with an annual payment of PMT = −215.13. To find the yield, CPT I/Y = 14.18.

Answer E

6.
Total Interest Paid = Total Payments – Loan Amount
The effective interest rate per semi-annual period is 12% ÷ 2 = 6%.

Seth makes a single payment at the end of five years. He pays:
$5{,}000(1.06)^{10} = 8{,}954.24$. His total interest is 8,954.24 – 5,000 = 3,954.24.

Janice pays 6% interest at the end of each semi-annual period and repays the principal at the end. Thus she pays 0.06×5,000 = 300 of interest 10 times. Her total interest is $10(300) = 3{,}000$.

Lori has a level-payment loan with N = 10, I/Y = 6, and PV = 5,000. Use the financial calculator to CPT PMT = -679.34. Her total interest is:

$$10(679.34) - 5{,}000 = 1{,}793.40$$

Total Interest Paid on all loans is the sum: $3{,}954.24 + 3{,}000 + 1{,}793.40 = 8{,}747.64$

Answer D

7.
The key identities for amortization of a level-payment loan, applied to a loan with a payment of 1 are:

Period t: $PRin_t = Pmt \cdot v^{n-t+1} = v^{n-t+1}$ $\qquad Int_t = Pmt_t \cdot \left(1 - v^{n-t+1}\right) = 1 - v^{n-t+1}$

Thus for Ron's loan:

Interest pd in period t + Principal pd in period t+1 $= (1 - v^{n-t+1}) + v^{n-(t+1)+1}$

$$= 1 - v^{n-t+1} + v^{n-t}$$

$$= 1 + v^{n-t}\left(1 - v\right) = 1 + v^{n-t} \cdot d$$

Answer D

8.
We know that the loan balance after 3 years is 559.12. This is the amount of principal that will be repaid by the last payment (at time 4). Therefore, the principal paid in the first payment (at time 1) is:

$$\frac{559.12}{1.08^{(4-1)}} = 443.85$$

Answer A

Note: The problem provides enough information to calculate the annual payment and loan amount, from which the first-year principal payment can be calculated. But it is easier to solve the problem using the fact that the principal repaid increases each year by a factor of (1+*i*).

9.
We will first find the amount of the original loan. Using the BA II Plus, set N=25, I/Y = 8, and PMT=−300, and CPT PV = 3,202.43. That is the amount of the original loan.

Next we look at what happens at the time of the 10th payment. To find the balance after the regular payment of 300 is made at time 10, set N=10 and CPT FV=-2,567.84. The extra payment of 1,000 reduces the balance to 1,567.84.

The loan is now revised to pay off the remaining balance over 10 years. To find the annual payment on that revised loan, set N=10, I/Y=8, PV=1567.84, FV=0, and CPT PMT = −233.65.

Answer C

10.
Lori pays interest of 900 (9% of 10,000) each year. She also makes a sinking fund deposit to accumulate 10,000 in an 8% account in 10 years. To find *SFD*, set N=10, I/Y=8, FV=10,000, and CPT PMT= −690.29. Thus each year she pays a total of 900 + 690.29 =1,590.29. In 10 years she pays a total of 15,902.90.

Answer C

11.
The nominal rate of 16% convertible quarterly translates to an effective rate of 4% per quarter. Note that the interest on 500 at 4% is 20. Thus the customer's payments of 20 are covering interest only, and no principal is paid in any payment. (Incidentally, this means the term of the loan is infinite.)

Answer A

12.
We know that $PRin_{18} = PRin_8 \cdot (1+i)^{10} = 211 \cdot (1.07)^{10} = 415.07$

Since payments are level, we know that $Pmt_{18} = Pmt_8 = 789 + 211 = 1{,}000$.

Therefore, $Int_{18} = 1{,}000 - 415.07 = 584.93$.

Answer D

Section 3.14

Supplemental Exercises

1. A man has a 30-year home loan for 200,000 at a nominal annual rate of 7.5% convertible monthly. Find:
   a) his monthly payment
   b) his balance after 12 years
   c) the amount of interest paid in the 40^{th} payment

2. Suppose that at the end of 10 years the man in Problem 1 is able to refinance the balance of his loan at 6% convertible monthly. What is his new monthly payment?

3. A woman has a 10-year loan of 55,000 at an annual effective rate of 6.8%. She makes annual end-of-year payments consisting of the interest on the unpaid balance plus repayment of part of the principal. Her principal payments start at 1,000 at the end of year 1, and increase by 1,000 each year. What is her total payment at the end of year 6?

4. A loan at a 5.8% annual effective rate has level annual end-of-year payments. The principal repaid in the 8^{th} payment is 1,234.08. What is the amount of principal repaid in the 15^{th} payment?

5. A fixed-rate loan has level annual payments. The principal repaid in the 5^{th} payment is 1,489.40. The principal repaid in the 15^{th} payment is 2,795.81. What is the annual interest rate on the loan?

6. A man has a loan of 50,000 for 10 years at a 6.5% annual effective rate. He repays the loan with annual payments of 4,500 for the first 5 years and X for the next 5 years. Find X.

7. A company has a loan of 80,000 to be paid in 20 level annual payments. The interest and the principal repayment in the 13^{th} payment are equal. Find the amount of principal repaid in the 6^{th} payment.

8. On a 20-year loan of 50,000 at a 6.2% annual effective rate, the lender wants the interest paid annually and the principal repaid at the end of the 20 years. The borrower makes level annual payments into a sinking fund to accumulate 50,000. The sinking fund earns a 5.8% annual effective rate. What are the borrower's total annual payments?

9. A man borrows 250,000 for 30 years at a 6.8% annual effective rate. His annual payments are 15,000 for the first 10 years, 15,000 + P for the next 10 years and 15,000 + $2P$ for the last 10 years. Find P.

10. For Problem 9, find the loan balance at the end of year 20.

Section 3.15

Supplemental Exercise Solutions

1. a) Using the BA II Plus calculator set N = 360, I/Y = 0.625, PV = 200,000, and FV = 0. Then CPT PMT = -1,398.43. The payment is 1,398.43.

   b) To get the balance after 12 years (144 months) set N = 216 (the number of months remaining) and CPT PV = 165,499.78.
   That solution uses the "prospective method."

   Or, by the retrospective method (with PV still set at 200,000), set N=144 and CPT FV = -165.499.78.

   c) The interest in the 40th payment is:
   $$\text{PMT}\cdot\left(1-v^{(360-40+1)}\right)=1{,}398.43\cdot\left(1-1.00625^{-321}\right)=1{,}209.17$$

2. To get the balance at the end of 10 years (240 months left), set N= 240, I/Y = 0.625, PMT = - 1,389.43, and FV = 0.
   Then CPT PV = 173,589.97.

   To get the new payment, reset I/Y = 0.5.
   Then CPT PMT = - 1,243.65. The new payment is 1,243.65.

3. At the end of year 5 the woman has paid 15,000 toward the principal (=1,000+2,000+…+5,000), so her balance is 40,000. The interest due at the end of year 6 is 40,000(0.068) = 2,720.
   Total payment for year 6 is 6,000 + 2,720 = 8,720.

4. If $PRin_n$ is the principal repaid in the n^{th} payment,
   then $PRin_{n+k} = (1 + i)^k\, PRin_n$. Hence $PRin_{15} = (1.058^7)PRin_8$.
   $PRin_{15} = 1.4839(1{,}234.08) = 1{,}831.23$.

5. $PRin_{15} = (1 + i)^{10}\, PRin_5 \Rightarrow 2{,}795.81 = (1 + i)^{10}(1{,}489.40)$
   Hence $i = (2{,}795.81/1{,}489.40)^{1/10} - 1 = 0.065$

6. We will first find the remaining balance at the end of the first 5 years, and then calculate the annual payment required to repay that amount in the last 5 years.

   Set N = 5, I/Y = 6.5, PV = 50,000 and PMT = -4,500.
   Then CPT FV = -42,882.95, the outstanding balance after 5 years.

   To find X, reset PV = 42,882.95 and FV = 0.
   Then CPT PMT = -10,319.12. $X = 10{,}319.12$

   Alternatively: $50{,}000 = 4{,}500\cdot a_{\overline{5}|} + X\cdot v^5\cdot a_{\overline{5}|} \quad X = 10{,}319.12$

7. $PRin_{13}$ = PMT$v^{20-13+1}$ = PMT/2 $\Rightarrow v^8 = 0.5 \Rightarrow i = 0.0905$.
To calculate the payment, set N= 20, I/Y = 9.05, PV = 80,000 and FV = 0.
Then CPT PMT = -8,794.97.

The principal repaid in the 6th period is:
$$PMT \cdot v^{(20-6+1)} = 8,794.97 \cdot 1.0905^{-15} = 2,398.00$$

8. The annual interest on the loan is 50,000(0.062) = 3,100.
To find the deposits to the sinking fund, set N = 20, I/Y = 5.8,
PV = 0 and FV = 50,000.
Then CPT PMT = -1,388.72.
The total annual payments are 1,388.72 + 3,100 = 4,488.72.

9. The present value of the man's payments is:
$$15,000a_{\overline{30}|0.068} + P\,v^{10}\,a_{\overline{20}|0.068} + P\,v^{20}\,a_{\overline{10}|0.068} = 250,000$$
$$a_{\overline{30}|0.068} = 12.6625, \quad a_{\overline{20}|0.068} = 10.7607, \quad a_{\overline{10}|0.068} = 7.0890$$
$$v^{10} = 0.51795 \qquad v^{20} = 0.26827$$
15,000(12.6625) + P[0.51795(10.7607) + 0.26827(7.0890)] = 250,000
P = 8,034.83

10. For the last 10 years the payments are:
15,000 + 2P = 15,000 + 2(8,034,83) = 31,069.66.
The balance due at time 20 is:
$$31,069.66 \cdot a_{\overline{10}|0.068} = 31,069.66 \cdot (7.0890) = 220,252.82$$
Note: For the first 10 years there was negative amortization, since the annual payments of 15,000 were less than the interest due on the loan.

"Midterm" #1

Questions

1. An amount of 1,000 is invested today at an interest rate of 5% per annum. How much more interest will it earn during the 4th year if it earns 5% compound interest than if it earns 5% simple interest?

A) 0 B) 4.3 C) 7.9 D) 10.8 E) 12.2

2. A loan of 2,000 is taken out at an annual effective interest rate of 6%. The borrower makes level annual interest-only payments at the end of each year for 10 years, and repays the principal at the end of 10 years. In addition, the borrower makes level deposits at the end of each year to a sinking fund that earns interest at an annual rate of 5%. At the end of 10 years, the sinking fund accumulates to the amount of the loan principal.

What is the total amount the borrower pays each year (to the lender *and* the sinking fund)?

A) 120 B) 159 C) 215 D) 279 E) 298

3. An annuity-due pays an initial benefit of 2 per year, with the benefit increasing by 5.25% every four years. The annuity consists of 40 annual payments.

Using an annual effective rate of 3%, calculate the value of this annuity as of the date of the last payment.

A) 58 B) 76 C) 125 D) 183 E) 197

4. A 300,000 home loan is amortized by equal monthly payments for 25 years, starting one month from the time of the loan at a nominal rate of 7% convertible monthly. Which of the following is closest to the total interest paid during the last 10 years of the loan?

A) 71,820 B) 71,910 C) 72,530 D) 72,660 E) 77,050

5. Given $d^{(4)} = 0.05$ and $A(1.5) = 100$, find $A(0)$, the present value, at time 0.

A) 92.7 B) 93.0 C) 93.1 D) 93.8 E) 94.0

6. An annuity-immediate has 32 initial quarterly payments of 20, followed by a perpetuity of quarterly payments of 25 starting in the 9^{th} year. Find the present value at a nominal rate of 16% convertible quarterly.

A) 982 B) 877 C) 715 D) 610 E) 536

7. Hank purchases a home and borrows 200,000. Mortgage payments are to be made monthly for 30 years with the first payment to be made one month from loan origination. The annual effective rate of interest is 5%.

Starting with the 100th payment, $400 is added to each payment in order to repay the mortgage earlier. What will be the amount of the last payment?

A) 565 B) 567 C) 1020 D) 1060 E) 1460

8. If $i^{(4)} = 0.05$, find $d^{(2)}$.

A) 4.6% B) 4.7% C) 4.8% D) 4.9% E) 5.0%

9. An annuity-immediate pays 10 annually for 20 years. After 20 years, the payment amount decreases by 1 each year until it reaches an amount of 1. The payments of 1 continue forever. The annual effective rate of interest is 6%. Calculate the present value of this annuity.

A) 99 B) 105 C) 129 D) 136 E) 140

10. Suppose you can earn interest at a 10% annual effective rate for the next 10 years and 6% interest convertible semi-annually for the 10 years after that. What would be the required investment today to accumulate to 1,000,000 in 20 years?

A) 215,285 B) 213,466 C) 213,270 D) 212,115 E) 211,980

11. Annual deposits of 500 are made at the beginning of each year for 30 years to an account earning an annual effective rate of 7%. The interest earned each year is reinvested in another fund that earns a 4.5% annual effective rate. At the end of the 30 years, what is the total accumulated value of the 30 payments and the reinvested interest?

A) 15,000 B) 26,250 C) 31,500 D) 41,250 E) 42,575

Solutions

1. The fourth year occurs between time 3 and time 4. The interest earned during the fourth year is the accumulated value at time 4 minus the accumulated value at time 3.

Using compound interest: $1{,}000 \cdot 1.05^4 - 1{,}000 \cdot 1.05^3 = 57.8813$.

Using simple interest: $1{,}000(1+4(0.05)) - 1{,}000(1+3(0.05)) = 50$

For simple interest, we could just note that the amount of interest earned each year is equal to 5% of the unchanging principal of 1,000.
The difference is $57.8813 - 50 = 7.8813$.

***Answer:* C**

2. Total payment in a sinking fund situation is given by the formula:

$$P = Li + \frac{L}{s_{\overline{n}|j}}$$

Thus, the total payment is: $2{,}000(0.06) + \dfrac{2{,}000}{s_{\overline{10}|0.05}} = 279.01$

***Answer:* D**

3. You could set up the present value calculation with separate four-payment annuities-due.

$$PV = 2\ddot{a}_{\overline{4}|0.03} + 2(1.0525)\ddot{a}_{\overline{4}|0.03}v^4 + 2(1.0525)^2\ddot{a}_{\overline{4}|0.03}v^8 + \ldots + 2(1.0525)^9\ddot{a}_{\overline{4}|0.03}v^{36}$$

$$= 2\ddot{a}_{\overline{4}|0.03}\left[1 + 1.0525v^4 + 1.0525^2v^8 + \ldots + 1.0525^9v^{36}\right]$$

$$= 2\ddot{a}_{\overline{4}|0.03}\left[\frac{1-\left(1.0525v^4\right)^{10}}{1-1.0525v^4}\right] = 7.6572(7.5328) = 57.6803$$

This is the present value. The future value immediately after the last payment *at time 39* (since the first payment was at time 0) would then be:

$$FV = PV(1.03)^{39} = 57.6803(1.03)^{39} = 182.6751$$

More efficiently, you can recognize that this is equivalent to the future value of a geometric annuity-*immediate* (since we want the value on the date of the last payment, not one period *after* the last payment), where the annuity has quarterly payments (beginning at 2) and annual increases of 5.25%. We want to calculate the annuity's future value (as of the last payment date) at a nominal rate of 12% convertible quarterly.

$$FV = 8\left(s^{5.25\%}_{\overline{10}|i}\right)^{(4)} = 8\cdot\frac{1.0525^{10}-(1+i)^{10}}{i-0.0525}\cdot\frac{i}{i^{(4)}}$$

$i^{(4)}$ is 12%, and i is $1.03^4 - 1 = 0.1255$.

$$FV = 8\left(s^{5.25\%}_{\overline{10}|}\right)^{(4)} = 8\cdot\frac{1.1255^{10}-1.0525^{10}}{0.1255-0.0525}\cdot\frac{0,1255}{0.12} = 182.68$$

Answer: D

4. First, find the outstanding balance when 10 years of payments remain.

The payment amount is: $P = \dfrac{300,000}{a_{\overline{300}|\frac{0.07}{12}}} = 2,120.34$

(Or N=300, I/Y=7/12=0.5833%, PV=300,000, FV=0. CPT PMT = 2,120.34)

Then the outstanding balance after 180 payments, with 120 payments (10 years) remaining, by the prospective method, is:

$$Bal_{180} = 2,120.34\cdot a_{\overline{120}|\frac{0.07}{12}} = 182,616.94$$

If 120 payments of 2,120.34 are made during the last ten years, these payments total 254,440.80 to pay off 182,616.95 of principal.

The total amount of interest in these payments is the difference:

$$254,440.80 - 182,616.95 = 71,823.85$$

Answer: A

5. From the given value of $d^{(4)}$, we can find the quarterly effective interest rate, which we will call j:

$$(1+j)^{-1} = 1 - \frac{d^{(4)}}{4}$$

$$j = \left(1 - \frac{d^{(4)}}{4}\right)^{-1} - 1 = \left(1 - \frac{0.05}{4}\right)^{-1} - 1 = 0.012658$$

Since 1.5 years equals 6 quarters, we can write:

$$100 = A(1.5) = A(0)\cdot(1+j)^6 = A(0)\cdot 1.012658^6 = 1.078394 \cdot A(0)$$

$$A(0) = \frac{100}{1.078394} = 92.73$$

Answer: A

6. The quarterly effective interest rate is 0.04. Since payments are made quarterly, this is the only interest rate we need. We can analyze this as a 32-payment annuity-immediate plus a deferred perpetuity:

$$PV = 20a_{\overline{32}|0.04} + v^{32}\cdot 25a_{\overline{\infty}|0.04} = 20(17.8736) + 1.04^{-32}\cdot 25\left(\frac{1}{0.04}\right) = 535.63$$

Alternatively, we can analyze it as a perpetuity-immediate with annual payments of 25, *less* a 32-period annuity with annual payments of 5:

$$PV = 25a_{\overline{\infty}|0.04} - 5a_{\overline{32}|0.04} = 25\left(\frac{1}{0.04}\right) - 5(17.8736) = 535.63$$

Answer: E

7. First, solve for the payment amount:
Let i be the monthly effective rate: $i = (1.05)^{1/12} - 1 = 0.004074$

$$200{,}000 = P \cdot a_{\overline{360}|i} = P \cdot \frac{1 - 1.05^{-30}}{0.004074}$$

$$P = 1{,}060.11$$

This is the amount of the first 99 payments. The outstanding balance immediately after the 99th payment is:

$$Bal_{99} = 1060.11 a_{\overline{261}|i} = 170{,}162.81$$

Starting with the 100th payment, payments are 1,060.11 + 400 = 1,460.11. The number of additional payments that will be required is n such that:

$$170{,}162.81 = 1{,}460.11 \cdot a_{\overline{n}|i} = 1{,}460.11 \cdot \frac{1 - 1.004074^{-n}}{0.004074}$$

$$1.004074^{-n} = 1 - 170{,}162.81 \cdot \frac{0.004074}{1{,}460.11} = 0.525197$$

$$n = -\frac{\ln 0.525197}{\ln 1.004074} = 158.3880$$

There will be a total of 159 + 99 payments = 258 payments.

The last payment, the 258th payment, will be a smaller amount. To determine that amount, we need to figure out the outstanding balance after 257 payments (99 at the original amount and 158 at 1,460.11). We'll use the "refinanced" loan and the retrospective method to find that balance:

$$Bal_{158} = 170{,}162.8103(1+i)^{158} - 1{,}460.11 s_{\overline{158}|i} = 564.89$$

The last payment will be 564.98 plus one month's interest:

$$564.89(1+i) = 567.19$$

The problem can also be solved using the BA II Plus's TVM functions:
Find the annual payment:
N=360, I/Y=$\left(1.05^{1/12} - 1\right) \cdot 100 = 0.4074$, PV=200,000, FV=0
CPT PMT=1,060.11
Find the balance after 99 payments:
N=360 – 99 = 261 CPT PV = 170,162.81
Increase the payments by 400; solve for N (number of required payments):
RCL PMT (-1,060.11 will display) Enter -400 = and press PMT
CPT N=158.388
Calculate the (negative of the) loan balance if 159 full payments are made:
N=159, CPT FV=892.92 This is the amount by which we have OVERpaid the outstanding balance by making a payment of 1,460.11. The outstanding balance *before* this payment must therefore be: $1{,}460.11 - 892.92 = 567.19$, and that is the amount of the actual final payment.

***Answer:* B**

8.

$$\left(1+\frac{i^{(m)}}{m}\right)^m=\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

$$\left(1+\frac{0.05}{4}\right)^4=\left(1-\frac{d^{(2)}}{2}\right)^{-2}$$

$$\left(1+\frac{0.05}{4}\right)^{-2}=1-\frac{d^{(2)}}{2}$$

$$d^{(2)}=2\cdot\left(1-\left(1+\frac{0.05}{4}\right)^{-2}\right)=0.049078$$

Answer:* *D

9. The annuity in this problem consists of a 20-payment annuity-immediate, followed by a 9-year arithmetic decreasing annuity, then a perpetuity. Don't forget to defer both the decreasing annuity and the perpetuity.

$$\begin{aligned}PV&=10a_{\overline{20}|0.06}+v^{20}(Da)_{\overline{9}|0.06}+v^{29}a_{\overline{\infty}|0.06}\\&=10a_{\overline{20}|0.06}+v^{20}\left(\frac{9-a_{\overline{9}|0.06}}{0.06}\right)+v^{29}\frac{1}{0.06}\\&=114.6992+11.4240+3.0759=129.1991\end{aligned}$$

Answer:* *C

10. The future value is 1,000,000, and we are solving for present value (PV).

$$PV\cdot(1+0.10)^{10}\cdot\left(1+\frac{0.06}{2}\right)^{20}=1{,}000{,}000$$

$$PV=\frac{1{,}000{,}000}{(1.1)^{10}(1.03)^{20}}=213{,}465.97$$

Answer:* *B

11. The first fund has a future value equal to the sum of the 30 payments of 500 = 15,000.
The second fund receives payments of 35, 70, 105, ... , 1050 (at the end of each year for 30 years).

The total accumulated value is:

$$\begin{aligned}15{,}000+35(Is)_{\overline{30}|0.045}&=15{,}000+35\left(\frac{\ddot{s}_{\overline{30}|0.045}-30}{0.045}\right)\\&=15{,}000+26{,}251.8572=41{,}251.86\end{aligned}$$

Answer:* *D

Module 4

Bonds

Section 4.1

Introduction to Bonds

Corporations need to obtain money in order to start projects. One way to secure funding is to sell stock in the company, but that expands the ownership of the company, since every stockholder is an owner of a proportional share of the company. Another way to get money is to borrow it. Corporations can and do borrow money from banks and other lenders, but a more widely used method for large-scale borrowing is to issue bonds. Bonds are also issued by national governments, municipalities, school districts and various other entities. We will illustrate how a corporation can use bonds to borrow with a simple example.

Example (4.1)

Company A needs $100,000,000 to build new manufacturing facilities. The company creates a bond issue consisting of 100,000 bonds in denominations of $1,000. This amount is called the **face value** or **par value** of the bond.

The bonds will pay a nominal interest rate of 10% convertible semi-annually for 10 years and then pay the face value of $1,000 along with the final interest payment. The 10% rate is called the **coupon rate** of the bond. The $1,000 paid at the end of 10 years is called the **redemption value** or **maturity value.** Most bonds are issued so that the redemption value equals the face value, although it is possible to have a redemption value that does not equal the face value.

Individual investors who wish to earn interest can buy Company A's bonds and thus make a loan to the company. If Company A can sell all 100,000 bonds at $1,000 each, they will have the $100,000,000 they need.

In the above example, Company A's bonds pay 10% coupons. However, interest rates change daily. If current market interest rates are such that an investor can earn more than 10% on a similar investment, investors will not want to pay full price for Company A's bond. Conversely, if rates on similar investments are *less* than 10%, investors will be willing to pay *more* than the $1,000 face value. This is demonstrated in the following example.

Example (4.2)

Company A puts its bonds up for sale on a day when investors in the marketplace are demanding a yield rate of 10.2% convertible semi-annually (a 5.1% semi-annual effective rate). They want to buy each bond at a price that gives that yield. Over 10 years each bond will make 20 semi-annual payments of 50 and a final payment of 1,000 at the same time as the last payment of 50. The bond's price is the present value of this series of payments at $i^{(2)} = 10.2\%$. On the BA II Plus:

Set N=20, I/Y=5.1, PMT=50, FV=1,000, and CPT PV=-987.64.

Investors will pay only \$987.64 for each bond; so Company A will collect a total of \$98,764,000, which is 1,236,000 short of the target amount.

Exercise (4.3)

Suppose Company A's bonds were sold on a day when the market interest rate was 9.8% convertible semi-annually. What is the price of an individual bond with a 1,000 face amount?

Answer: 1,012.57

Note that when the interest rate demanded by investors went up, the price of the bond decreased to 987.64. In this case the bond is said to have sold at a **discount** of 12.36 ($= 1{,}000 - 987.64$). When the interest rate demanded by investors went down, the price of the bond increased to 1,012.57. In this case the bond is said to have sold at a **premium** of 12.57.

Remember: The *price* of the bond moves in the *opposite* direction from *interest rates*: a higher price means a lower yield; a lower price means a higher yield. Note also that, although coupon payments seem similar to interest payments, a bond's coupon rate is *not* the interest rate (yield) that the investor earns.

The examples above are simplified to illustrate the basics. In practice, there are also expense and risk issues associated with issuing bonds. We will not review these issues here, since they are covered in finance courses. Instead, we will focus on the mathematics of bonds.

Note that in the above examples we stated the coupon rate and yield as nominal rates convertible semi-annually. This is the most common practice in Canada and the United States. However an issuer (or an exam question) can use any conversion period. Annual rates have been used in Europe and are used in some Exam FM questions.

Also in the above examples we assumed that the borrower would pay back the face value at the maturity of the bond. It is possible for the bond to specify a different amount for the redemption value.

Example (4.4)

The company issuing the bonds of Example **(4.1)** decides to make the bonds more attractive by paying 1,100 at the time of the final coupon payment of 50. If the bonds are purchased to yield 10.2%, we can find the price of this new bond on the BA II Plus. Set N=20, I/Y=5.1, PMT=50, FV=1,100, and CPT PV=−1,024.62.

At a price of 1,024.62, the bond yields 10.2% convertible semi-annually.

Exercise (4.5)

What is the price of the bond in **(4.4)** if the required yield is 9.8% convertible semi-annually?

Answer: 1050.98

In Example **(4.4)**, the redemption value (1,100) is different from the face value (1,000). **If an exam problem does *not* specify a separate redemption value, you can assume that the bonds will be redeemed at par.**

Note that raising the redemption value to 1,100 resulted in a price above 1,000. It is natural to ask what redemption value would give a price of 1,000.

Example (4.6)

Suppose that the company issuing the bonds of Example **(4.1)** wants to increase the redemption value to keep the price at 1,000 for a required nominal yield of 10.2% We can find the required redemption value on the BA II Plus.

Set N=20, I/Y=5.1, PMT=50, PV =−1,000, and CPT FV= 1,033.42.
The required redemption value is 1,033.42.

Exercise (4.7)

What redemption value would assure a price of 1,000 for the above bonds if the required nominal yield were 10.1%?

Answer: 1016.62

An investor who wishes to buy a bond is usually quoted a market price for the bond. In that case, the investor would like to know the bond's yield at that price.

Example (4.8)

A 1,000 par value bond with a term of 10 years and a coupon rate of 10% payable semi-annually is offered at a price of 990. We can find the yield per semi-annual period on the BA II Plus.

Set N=20, PV = −990, PMT=50, FV=1,000, and CPT I/Y=5.08.
This gives a nominal yield convertible semi-annually of
$5.08\% \times 2 = 10.16\%$.

Exercise (4.9)

A 1,000 par value bond with a term of 10 years and a coupon rate of 10% payable semi-annually is offered at a price of 1,020. Find the implied yield (convertible semi-annually).

Answer: 9.68%

You may have observed by now that we have solved all the problems to this point using the financial calculator and have not introduced any mathematical notation or formulas. It is important to recognize when exam problems can be solved easily and directly like this. As always, there are formulas and notation to learn, and some problems will require their use. The key variables are:

F = par value (or face value)
r = coupon rate *per coupon period* (e.g., per semi-annual period)
Fr = coupon amount
C = redemption (maturity) value (= F in most cases)
n = number of coupon periods to redemption
P = price
i = yield *per coupon period*
$v_i = \dfrac{1}{1+i}$

The most basic formula for the price P is:

(4.10)

$$P = PV(coupons) + PV(redemption\ payment)$$
$$-or-$$
$$P = (Fr)a_{\overline{n}|i} + Cv_i^n$$

If the bond is redeemed at par, then $F = C$, and we can write $P = (Fr)a_{\overline{n}|i} + Fv_i^n$.

Since $Fv_i^n = F - Fi\dfrac{1-v_i^n}{i} = F - Fi \cdot a_{\overline{n}|i}$, we can derive the following formula for a bond's price in terms of its premium or discount:

(4.11)

When $F = C$:

$$\underbrace{P}_{\text{price}} = \underbrace{F}_{\text{face value}} + \underbrace{F(r-i)a_{\overline{n}|i}}_{\text{premium or discount}}$$

We illustrate this with another example.

Example (4.12)

Consider again a 1,000 par value bond redeemable at par in 10 years with a nominal coupon rate of 10% payable semi-annually. This bond has F=1,000, r=0.05 and n=20. We have already shown in Example **(4.2)** that the price of this bond at a yield of 10.2% is 987.64. We can check this using the above formula with $i = 0.051$.

$$P = F + F(r-i)a_{\overline{n}|i} = 1{,}000 + 1{,}000(0.05 - 0.051)a_{\overline{20}|0.051}$$
$$= 1{,}000 + (-1)a_{\overline{20}|0.051} = 1{,}000 - 12.36 = 987.64$$

Note that the term $F(r-i)a_{\overline{n}|i}$ gave us the discount of 12.36 for the bond. The discount is the present value of the difference between Fr (the actual coupon amount) and Fi (what the coupon would be if the coupon rate *were equal to* i). Formula **(4.11)** is referred to as the **premium-discount formula.**

Exercise (4.13)

Use **(4.11)** to verify the price of the premium bond in Exercise **(4.3)**

There are a number of other possible formulas for the price of a bond, but we have found that most problems for which these formulas were used historically can now be solved directly using the BA II Plus.

Mathematics of Investment and Credit has another formula worth mentioning, **Makeham's Formula**:

(4.14)

When $F=C$, if we let $K=Fv_i^n$,

then: $P=K+\frac{r}{i}(F-K)$

Example (4.15)

Consider again the 1,000 bond redeemable at par in 10 years with a nominal coupon rate of 10% payable semi-annually that is purchased at a nominal yield of 10.2% convertible semi-annually. For this bond:

$$K=\frac{1,000}{1.051^{20}}=369.78$$

$$P=K+\frac{r}{i}(F-K)=369.78+\frac{0.05}{0.051}(1,000-369.78)=987.64$$

The specialized formulas in **(4.11)** and **(4.14)** are valid only if the bond is redeemable at par and F=C.

If $F \neq C$, you must analyze the problem from first principles. (*See #12 in the Sample Exam Problems at the end of this module for an example of a problem of this type.)*

The amortization formulas in the next section also have this same disclaimer.

Section 4.2

Amortization of Premium or Discount

Given the variability of interest rates, bonds are usually sold at either a premium or a discount. The premium or discount must then be amortized for accounting and valuation purposes. The method is similar to the method used in the last module for amortizing loans. We will illustrate this with an example of **amortization of premium**.

Example (4.16)

A 3-year 1,000 par value bond pays semi-annual coupons at a 6% annual rate. It is sold at a yield of 5% convertible semi-annually. We can see immediately that this will be a premium bond (since the coupon rate is higher than the yield). We find the price using the BA II Plus.

Set N=6, I/Y = 2.5, PMT = 30, FV = 1,000, and CPT PV=−1,027.54.
The price is 1,027.54 and the premium is 27.54.

Thus, the buyer of the bond has an investment of 1,027.54 that pays interest at the yield rate of 2.5% (per half-year). Now we will break down the first payment using the amortization method.

First payment: 30

Interest Paid: $1,027.54(0.025) = 25.69$

Principal Paid: $30 - 25.69 = 4.31$

The table below shows the result of continuing this process over the life of the bond.

Period	0	1	2	3	4	5	6
Coupon		30	30	30	30	30	30
Redemption Value							1,000
Interest Paid		25.69	25.58	25.47	25.36	25.24	25.12
Principal Paid		4.31	4.42	4.53	4.64	4.76	4.88
Balance	1,027.54	1,023.23	1,018.81	1,014.28	1,009.64	1,004.88	1,000.00
Remaining Premium	27.54	23.23	18.81	14.28	9.64	4.88	0
Premium Amortized		4.31	4.42	4.53	4.64	4.76	4.88

In calculating the amortization amounts for this premium bond, we see that each coupon is partly interest and partly a repayment of principal. As the principal amount is partly repaid each period, the remaining premium decreases by the amount of principal paid. Thus the amount of principal paid is referred to as the amount of amortization of premium**.** When the final coupon is paid, the balance owed is equal to the face amount of 1,000, which is paid off by the redemption payment.

For accounting purposes, this decreasing balance is referred to as the bond's **book value**. Depending on the accounting method being used, this may be the asset value that appears on the company's balance sheet.

The table in the preceding example illustrates how the method works, but **for exam questions you do not want to build the entire table**. Fortunately, the amortization method and the premium-discount formula can be used to derive a simple formula that is useful for exam problems.

(4.17)

> For a bond that is redeemable at face value,
> the **amortization of premium in period k** is:
> $$F(r-i)v^{n-k+1}$$

Formula **(4.17)** is easy to remember, since it uses the same power of v as the loan amortization formula in Module 3. To understand why this is so, consider that the bond's coupon, Fr, can be broken down into 2 parts: Fi and $F(r-i)$. The payments of Fi on each coupon date, plus the payment of F at maturity are just sufficient to pay off a loan of F with interest at rate i. But the investor receives an additional amount equal to $F(r-i)$ each coupon period. The present value of these additional amounts is $F(r-i)\cdot a_{\overline{n}|i}$, which equals the *premium* the investor paid to purchase the bond, as we can see in Formula **(4.11)**. This is an additional amount (over and above the face amount, F) that the investor has lent to the bond issuer. This additional loan amount (the "premium") is being repaid by the payments of $F(r-i)$. So the amount of principal repaid in period k is $F(r-i)v^{n-k+1}$, based on the principal repayment formula developed for loans in Module 3. As was the case with repayment of loan principal, **the amount of premium that is repaid (amortized) increases each period by a factor of (1+*i*)**.

Example (4.18)

> For the bond in, **(4.16)**, the amortization of premium in period 5 is found using Formula **(4.17)** with $F=1{,}000$, $(r-i)=0.03-0.025=0.005$, and $(n-k+1)=6-5+1=2$.
>
> The result is identical with the value in the table of Example **(4.16)**:
> $$F(r-i)v^{n-k+1}=1{,}000(0.005)\left(\frac{1}{1.025}\right)^2=4.76$$

Exercise (4.19)

> Use Formula **(4.17)** to verify the amount of premium amortized in period 3 in the table of Example **(4.16)**.

In the next example we will look at **amortization of discount**.

Example (4.20)

A 3-year 1,000 par value bond pays semi-annual coupons at a 6% annual coupon rate. It is sold at a yield of 7% convertible semi-annually. We can see immediately that this will be a discount bond (because the coupon rate is less than the yield). We find the price using the BA II Plus.

Set N=6, I/Y = 3.5, PMT = 30, FV = 1,000, and CPT PV=-973.36.
The price is 973.36 and the discount is 26.64.

Thus the buyer of the bond has an investment of 973.36 which pays interest at the yield rate of 3.5% per semi-annual period. Now we will break down the first payment using the amortization method.

First payment: 30

Interest Paid: $973.64(0.035) = 34.07$

Principal Paid: $30 - 34.07 = -4.07$

The table below shows this process continuing over the bond's life:

Period	0	1	2	3	4	5	6
Coupon		30	30	30	30	30	30
Redemption Value							1,000
Interest Paid		34.07	34.21	34.36	34.51	34.67	34.83
Principal Paid		−4.07	−4.21	−4.36	−4.51	−4.67	−4.83
Balance	973.36	977.43	981.63	985.99	990.50	995.17	1,000
Remaining Discount	26.64	22.57	18.37	14.01	9.50	4.83	0
Discount Amortized		4.07	4.21	4.36	4.51	4.67	4.83

Under the amortization method for this discount bond, the actual coupon payment is *less* than the interest due. As a result, some of the interest due is capitalized by being added to the principal. This is called **negative amortization**. The amount of principal added is sometimes referred to as the **accumulation of discount** or **accrual of discount**. The loan balance (or "book value" of the bond) increases as the amount of discount decreases. When the final coupon is paid, the balance owed is equal to the face amount of 1,000, which is paid off by the redemption payment.

Formula **(4.17)** works for discount bonds as well as premium bonds, but we must recognize that it produces a *negative* value for discount bonds. This reflects the fact that the book value of the bond (the amount of the loan) is *increasing* as the discount is amortized. We restate Formula **(4.17)** here in a form that is specific to a discount bond.

(4.21) For a bond that is redeemable at face value,
the ***negative*** of the **amortization of discount in period *k*** is:

$$F(r-i)v^{n-k+1}$$

Example (4.22)

For the bond in **(4.20)** the amortization of discount in period 5 is found using **(4.21)** with $F=1{,}000$, $(r-i)=0.03-0.035=-0.005$, and $(n-k+1)=6-5+1=2$.

The result is identical with the value in the table of Example **(4.20)**:

$$F(r-i)v^{n-k+1}=1{,}000(-0.005)\left(\frac{1}{1.035}\right)^2=-4.67$$

Exercise (4.23)

Verify the amortization of discount in period 3 as shown in the table of Example **(4.20)** using Formula **(4.21)**.

As we have seen for loan principal in Module 3, the amount of premium or discount amortized each period increases geometrically.

Amount amortized in period *k* : $F(r-i)v^{n-k+1}$

Amount amortized in period *k+m* : $F(r-i)v^{n-(k+m)+1}=(1+i)^m F(r-i)v^{n-k+1}$

This leads to amortization questions similar to those seen in Module 3.

Example (4.24)

A premium bond is purchased to yield 4% convertible semi-annually. The amount of premium amortized in the 2nd payment is 8.37. Find the amount of premium amortized in the 7th payment.

Solution.

$$8.37(1.02)^5=9.24$$

Exercise (4.25)

A discount bond is purchased to yield 4% convertible semi-annually. The amount of discount amortized in the third payment is 4.10. Find the amount of discount amortized in the 6th payment.

Answer: 4.35

You can also use the AMORT feature of the BAII Plus to find the numbers in the preceding examples. To apply AMORT to the premium bond in Example **(4.16)**, re-enter the bond information (set FV = 1,000, I/Y = 2.5, PMT = 30, N=6 and CPT PV=-1,027.54). Then press 2ND AMORT. To check the first entry, set P1=1 and P2=1. Scroll down and you will see PRIN=4.31, the same result given in the table. Check the final entry by entering P1=6 and P2=6. Scroll down and you will see PRIN=4.88, the same result given in the table.

You can also apply the AMORT worksheet to the discount bond in Example **(4.20)** to reproduce the values in the table. The AMORT worksheet can be a valuable time-saving tool.

Section 4.3

Callable Bonds

Sometimes it is advantageous to pay off a loan early. For example, when mortgage interest rates drop, homeowners often re-finance their home loans by taking out new loans at lower rates and using the proceeds to pay off the old higher-rate loan. Corporations have similar motivations when interest rates fall; they would like to pay off their bonds early and issue new bonds at lower interest rates. For this reason, some bonds are designed with **call provisions** that allow the issuer to pay off or "call" the bonds at some specified future date before maturity. Any bond that does not have a call provision must pay coupons until maturity.

A call provision typically states that the bond may be "called" on a specified coupon date or any coupon date thereafter. It also specifies the **call price** that the issuer will pay to redeem the bond on a call date. The call price is often higher than the bond's maturity value. For example, a 10-year 1,000 par value bond might have a call provision allowing the issuer to redeem the bond after 5 years by paying the bondholder 1,040 (plus the coupon due on that date). The additional redemption amount (40, in this case) is referred to as a **call premium**. When an issuer exercises its right to redeem a bond early, the bondholder is forced to reinvest the funds at a time of lower interest rates. The call premium provides the bondholder some compensation for the resulting lost interest.

When you buy a callable bond, the price is based on either the *earliest call date* or the *final maturity date*. The rule for which date to use is given below. It calls for using whichever date results in the *lower* price.

(4.26)

Redemption date to use in pricing a callable bond:	
Type of Bond	**Find N using:**
Premium Bond	Earliest Possible Redemption Date
Discount Bond	Latest Possible Redemption Date

Consider a 10-year bond with semi-annual coupons. Its maturity date is at the end of N=20 coupon periods. If the bond is also *callable* after 5 years, the earliest redemption date is at N=10. Since the latest possible maturity is N=20 and the earliest is N=10, the investor will price the bond using N=20 if it is trading at a discount, and N=10 if it is selling at a premium.

Example (4.27)

A 10-year 1,000-face bond has a 10% coupon rate payable semi-annually. It is callable in 6 years. An investor wishes to buy the bond to yield 8% convertible semi-annually. That means it is a premium bond.

Using **(4.26)**, the investor would price the bond using N=12. On the BA II Plus, set N=12, I/Y=4, PMT=50, FV=1,000 and CPT PV = −1,093.85.

To understand the reasoning in this example, consider what the price would be if the investor used the latest maturity date instead:

Set N=20 and CPT PV = −1,135.90.

That gives a higher price. If the investor pays 1,135.90 for the bond and it is called after 6 years (at N=12), the yield will be only 7.2% (set N=12 and CPT I/Y=3.6%). The investor's target (minimum) return of 8% is assured by choosing the redemption date that gives the lowest price.

Another way to think of this is to remember that a premium bond pays coupons at a rate *above* the desired yield. If these higher payments are cut off early, there is a loss of value to the investor. The investor must *anticipate* this adverse event (the bond's being called early), and *assume* that it will happen by calculating the price based on a call at the earliest possible call date.

Example (4.28)

A 10-year bond with 1,000 face amount has a 10% annual coupon rate, with coupons payable semi-annually. The bond is callable at any coupon payment date on or after 6 years. An investor wishes to buy the bond to yield 12% convertible semi-annually. Because the coupon rate is less than the yield, this is a discount bond.

Using the rules of **(4.26)**, the investor would price the bond based on N=20. Using the BA II Plus, set N=20, I/Y=6, PMT=50, and FV=1,000. Then CPT PV = −885.30.

To see the reasoning behind pricing the bond this way, take a look at what the price would be if the investor used the earliest maturity date. Set N=12 and again CPT PV = −916.16. That is a higher price than when we used N=20. If the investor pays that higher price and the bond is *not* called after 6 years, the yield will be *less* than the intended rate of 12%. (If the bond is never called, its yield to maturity is 11.4% (set N=20 and CPT I/Y). But even if it is called after only 6.5 years, the yield is 11.9% (set N=13 and CPT I/Y). To assure the desired 12% return, the price must be based on the assumption that the bond will not be called before its maturity date.

Another way to think of this is to remember that a discount bond earns a rate of return (yield) higher than the coupon rate by "recapturing" discount. That is, the bondholder gradually increases the book value of the bond on its balance sheet as the discount is amortized. If the bond is called before maturity, the bondholder will realize a sudden *increase* in the bond's value. That would be favorable for the bondholder, but it is not likely to occur on a discount bond. So the investor should not pay a price based on the assumption that a discount bond will be called.

Exercise (4.29)

A 5-year 1,000 bond has a 6% coupon rate and pays coupons semi-annually. It is callable in 2 years. At what price should the investor buy the bond to assure a yield of at least 6.2% convertible semi-annually?

Answer: 991.51

Example (4.30)

A 10-year 1,000-face bond has a 10% coupon rate payable semi-annually. It is callable on or after its 6^{th} anniversary at a call price of 1,100. What is the maximum price an investor can pay for this bond and be assured of earning at least 8% convertible semi-annually?

Solution.
Because the target yield (8%) is less than the coupon rate (10%), this is a premium bond. The bond's call price is 1,100, so it has a call premium of 100. When a premium bond has a call premium, as in this case, it is necessary to calculate its price based on the earliest call date and also based on the maturity date, and use the lower of the two prices.

Based on the maturity date, the bond's price is found by setting N=20, I/Y=4, PMT=50, and FV=1,000. Then CPT PV=1,135.90.
Based on earliest call, set N=12 and FV=1,100. Then CPT PV=1,156.31.

We choose the lower price of 1,135.90, based on the assumption the bond will *not* be called. In this case, the call premium is large enough that the investor would realize a *higher* yield if this premium bond were called before maturity, so we should not assume that this will happen.

Example **(4.30)** demonstrates that sometimes a premium bond's price should be calculated assuming *no call*. To understand why that is the case, consider what happens if we purchase the bond for 1,135.90 and amortize the premium of 135.90 over the bond's 10-year term. At the end of 6 years, the unamortized premium is 67.32 (the bond's book value is 1,067.32). If the bond is called at that time, the yield will be *increased* above 8%, since the call price (1,100) is greater than the bond's book value based on an 8% yield. That means that the call would be a *favorable* event for the bondholder. If we had priced the bond *assuming* that favorable event, then there would be a risk that the bond might *not* be called and the 8% target yield would not be realized.

This demonstrates that a premium bond should sometimes be priced based on not being called *if there is a call premium*. When a premium bond has a call premium, you should calculate its price with a call at the earliest call date, and again with no call. Then choose the lower of the two prices to assure that the target return will be realized.

Exercise (4.31)

A 10-year 1,000-face bond pays semi-annual coupons at a 10% (annual) coupon rate. It is callable on or after its 3^{rd} anniversary at a call price of 1,100. What is the maximum price an investor can pay for this bond and be assured of earning at least 8% convertible semi-annually? Is this price based on the maturity date or the earliest call date?

Answers: 1,131.45 earliest call date

In Exercise **(4.31)**, unlike Example **(4.30)**, the price is based on the earliest possible call date. If the buyer paid a price that was calculated based on the maturity date (1,135.90), the amortized value of the bond after 3 years would be 1,105.63. The bondholder would then have to receive 1,105.63 in order to have an 8% yield. But the call price is only 1,100, so the buyer can't pay 1,135.90 for the bond and be assured of an 8% yield. (Note that a price of 1,135.90 based on the maturity date is 4.45 too high (1,135.90 - 1,131.45 = 4.45). The accumulated value of that amount after 3 years is $4.45 \cdot 1.04^6 = 5.63$, which is the amount by which the call price (1,100) is smaller than the bond's book value after 3 years.)

Some bonds have a call premium that varies depending on *when* the bond is called. For example, a 10-year bond might be callable after 5 years with a call premium equal to 5% of the face amount, but if it is called after 8 years, the call premium is only 2% of face. In that case, you should calculate the bond's price three ways (call at 5 years, call at 8 years, and continue to the maturity date) and choose the smallest of the three values.

Section 4.4

Pricing Bonds Between Payment Dates

Every problem we have solved so far used an integer for N, implying that every calculation was either on the bond's origination date or on a coupon date (immediately after the coupon was paid). However bonds are bought and sold daily, and we need to discuss how to price a bond *between* coupon payment dates. If a bond is priced between payment dates, the number of coupon periods remaining is not an integer. The number we will need in order to calculate the bond's price is the *fraction of a period that has elapsed* since the last coupon payment (or since issue, if the bond is in its first coupon period). This fraction is defined by:

$$k = \frac{\text{number of days from previous coupon date to settlement date}}{\text{number of days in the current coupon period}}$$

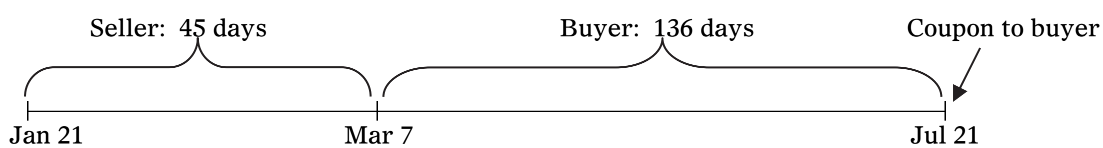
The following example illustrates this.

Example (4.32)

In this problem, we will give you the number of days between dates, as an exam problem might. Later, we will show how to use the BA II Plus to find the number of days between dates.

A bond with a par value of 1,000 has coupon payment dates of January 21 and July 21. The nominal coupon rate payable semi-annually is 6%. The bond matures on January 21, 2009. On January 21, 2007 a coupon payment of 30 was made. The bond is sold 45 days later with a **settlement date** of March 7, 2007, to yield 8% convertible semi-annually. There are 181 days between the coupon payment dates of January 21, 2007 and July 21, 2007. The seller of the bond owned it for a fraction of the coupon period equal to $\frac{45}{181} = 0.24862$.

A timeline is helpful for visualization:



On January 21, immediately after the coupon payment was made, there were 4 coupon payments of 30 remaining. At that point, the price of the bond based on an 8% yield would have been 963.70. (Using the BA II Plus, set N=4, I/Y=4, PMT=30, FV=1,000, and CPT PV = -963.70.)

The seller of the bond is entitled to interest for 45 days (0.24862 semi-annual periods) at 4% per period. The total sale price due to the seller is:

$$963.70(1.04)^{\frac{45}{181}} = 973.14$$

The price that is quoted to a potential buyer is *not* this total price of 973.14. Instead, the total price is broken down into two parts: the **market price** and the **accrued interest**. The "accrued interest" is actually a fraction of the next coupon payment, based on the fraction of a coupon period that has elapsed. In this case, the accrued interest is equal to $\frac{45}{181}\cdot 30 = 7.46$. Thus the market price of the bond is 965.68 ($= 973.14 - 7.46$ = total price – accrued interest). In other words, a bond buyer would be quoted a "market price" of 965.68, but would be required to pay the market price plus the "accrued interest" (accrued coupon), for a total price of 973.14.

The *total sale price* of a bond is referred to by various terms, including **price-plus-accrued**, **flat price**, and **dirty price.**
Similarly, the *market price* is also referred to by various terms, including **quoted price**, **true price**, **clean price**, or simply the **price**.

Consider a bond that has paid t coupons, and suppose that we want to calculate its price at a time during the t+1st coupon period. Let k be the fraction of the t+1st coupon period that has elapsed. We will use P_t to denote the price immediately after the t^{th} coupon payment and P_{t+k} to denote the price on the date of sale (settlement date). If i is the required yield per period for the buyer, then:

(4.33)

$$\text{Price-plus-accrued} = P_t\left(1+i\right)^k$$

If F is the face value of the bond and r is the coupon rate, then

(4.34)

$$\text{Accrued interest} = k\left(Fr\right)$$

(4.35)

$$\begin{aligned}\text{Price} &= (\text{Price-plus-accrued}) - (\text{Accrued interest})\\ &= P_t\left(1+i\right)^k - k\left(Fr\right)\end{aligned}$$

Notice that we have used compound interest in Formula **(4.33)** to find the price-plus-accrued and simple interest (or linear interpolation) in Formula **(4.34)** to find the accrued interest (accrued coupon). This is the convention that is used in section 4.1.3 of the official Exam FM reference text *Mathematics of Investment and Credit*. However problems need to be read carefully to assure that some other convention is not being specified, such as using simple interest ($\left(1+ki\right)$ instead of $\left(1+i\right)^k$) to calculate price-plus-accrued.

Exercise (4.36)

Find the market price and accrued interest of the bond in Example **(4.32)** if it is purchased on March 11, 2007 to yield 7%.

Answers: Price=982.70 Accrued interest=8.12

The BA II Plus has a bond worksheet that will calculate a bond's price as of any date (or calculate its yield, given the price). To illustrate this, we will use the worksheet to solve the problem in Example **(4.32)**. There are a number of entries in the process, but they are natural and easy to learn. The bond worksheet can be a time-saver.

Example (4.37)

Example **(4.32)** involved a bond with face amount 1,000 that pays semi-annual coupons at a 6% (annual) rate. The bond, which will mature on January 21, 2009, is sold on March 7, 2007, at a price such that its yield to maturity is 8%. We will use the bond worksheet to find the sale price.

The BOND legend appears above the [9] key. Enter the worksheet by pressing 2ND BOND. You will see the display SDT=. This is where you enter the settlement date (the date on which the purchase takes effect). To enter March 7, 2007, key in 3.0707 and press ENTER.

Note: the method for entering a date is to enter a number which has the month before the decimal point and then two digits for the day and two digits for the year after the decimal point. Years that are entered as values from 00 to 49 are interpreted as 21st century years; numbers from 50 to 99 are treated as 20th century years.

Now scroll down and you will see CPN. Enter 60 (the *annual* coupon amount, which is 6% of 1,000). Scroll down again and you will see RDT=. This is where you enter the redemption date. Enter 1.2109 for Jan. 21, 2009. Next scroll down to RV=. Here you enter a redemption value of 1,000.

Scroll down again and you will see either ACT or 360. Bonds can be priced using either actual days (ACT) or the convention that a year is composed of 12 months of 30 days each (a 360-day year). Using 2ND SET to toggle between ACT and 360, select ACT.

Scroll down again and you will see either 2/Y for semi-annual coupons or 1/Y for annual coupons. Use 2ND SET to select 2/Y. Scroll down again and you will see YLD=. Enter the required yield as 8 (meaning 8%). Scroll down again and you will see PRI= . Press CPT and you will see the price: 965.69. This is the market price, not the total sales price.

Scroll down again and you will see AI=. This is the amount of accrued interest: 7.46. The calculator does not show the total sales price, but we can find it by adding the market price and accrued interest:

$$965.6855 + 7.4586 = 973.1441$$

Note that these numbers match the answers in Example **(4.32)**.

Note: The bond worksheet can also be used to calculate a bond's yield to maturity as of any date. To calculate the yield, make entries as described in Example **(4.37)**, *but enter a Price (market price), scroll UP to "YLD=" and press CPT. The calculator also automatically calculates the accrued interest (AI).*

Exercise (4.38)

Suppose the bond of Example **(4.37)** was sold on March 7, 2007 at a total sales price of 980 and we want to find the buyer's yield to maturity. We can use all the entries from **(4.37)** except we need to change the Price to $980 - 7.46 = 972.54$ (subtracting accrued interest (AI) to find the market price). Then scroll up to YLD= and press CPT. Result: YLD= 7.59%.

Note: If we had not already calculated the AI in **(4.37)**, *we could enter any value for Price and CPT YLD in order to find AI (which is unaffected by Price and Yield). Then we can find the market price of 972.54 to enter.*

Calculator Note

The BOND worksheet automatically finds the days between the dates involved, although it does not display them. There is also a DATE worksheet which will calculate the number of days between two dates. The DATE legend appears above the [1] key.

As an example, we will demonstrate how to find the number of days from January 21, 2007 to July 21 2007:

Enter the worksheet by keying 2ND DATE. You will see the display DT1=. This is where you enter the first date. To enter January 21, 2007, key in the number 1.2107 and press the ENTER key. Scroll down and you will see the display DT2=. Key in the number 7.2107 and press the ENTER key. Scroll down again and you will see DBD =. This is where you can compute the days between dates, but you first scroll down again to see the display where you choose between ACT and 360. Make sure that you have chosen ACT and then scroll back to DBD and CPT DBD = 181. Note that if you were in 360 mode the answer would be 180.

Exercise (4.39)

Calculate the number of days between today and January 1 of next year.

You could also calculate the number of days that you have been alive if you were born in 1950 or later. This author is out of luck, since I was born in 1948.

Section 4.5

Formula Sheet

F = par value (or face value)
r = coupon rate *per coupon period* (e.g., per semi-annual period)
Fr = coupon amount
C = redemption (maturity) value (= F in most cases)
n = number of coupon periods to redemption date
P = price
i = yield *per period*
$v_i = \frac{1}{1+i}$

P = PV(coupons) + PV(redemption value) $= (Fr)a_{\overline{n}|i} + Cv_i^n$

If $F = C$:

Basic Formula: $P = (Fr)a_{\overline{n}|i} + Fv_i^n$

Premium-Discount Formula: $P = F + F(r-i)a_{\overline{n}|i}$

Makeham's Formula: $P = K + \frac{r}{i}(F-K)$, where $K = Fv_i^n$

Amortization of Premium and Discount

Amount amortized in period $k = F(r-i)v^{n-k+1}$

Amount amortized in period $k+m = (1+i)^m \times$ Amount amortized in period k

Redemption date to use in pricing a callable bond:

Type of Bond	**Find N using:**
Premium Bond	Earliest Possible Redemption Date
Discount Bond	Latest Possible Redemption Date (= maturity date)

Note: If a premium bond has a call premium, calculate the price at both the earliest and latest redemption dates and use the smaller value.

Price Between Payment dates

$$k = \frac{\text{number of days from previous coupon date to settlement date}}{\text{number of days in the current coupon period}}$$

Price-plus-accrued = $P_t(1+i)^k$ where P_t = price on previous coupon date

Accrued interest $= k(Fr)$

Price (market price) = (Price-plus-accrued) – (Accrued interest)
$= P_t(1+i)^k - k(Fr)$

Section 4.6

Basic Review Problems

1. A 1,000 par value bond with a term of 5 years and a coupon rate of 6% payable semi-annually is purchased to yield 8% convertible semi-annually. Find the purchase price.

2. A 1,000 par value bond with a term of 5 years and a coupon rate of 6% payable semi-annually is offered at a price of 975. Find the yield (expressed as a nominal rate convertible semi-annually).

3. Use the premium-discount formula **(4.11)** to verify the price of the discount bond in problem #1.

4. Find the amount of discount amortized in period 4 for the bond in problem #1.

5. A premium bond is purchased to yield 5% convertible semi-annually. The amount of premium amortized in the 3rd payment is 4.10. Find the amount of premium amortized in the 8th payment.

6. A 5-year 1,000 par value bond has a 6% coupon rate payable semi-annually. It is callable after 2 years. An investor buys the bond to yield 5.5% convertible semi-annually. Find the purchase price of the bond.

7. A bond with par value 1,000 has payment dates of April 15 and October 15. The nominal coupon rate payable semi-annually is 7%. The bond matures on October 15, 2009. On April 15, 2007 a coupon payment of 35 was made. The bond is sold 80 days later with a settlement date of July 4, 2007, to yield 6% convertible semi-annually. There are 183 days between the coupon payment dates of April 15, 2007 and October 15, 2007. Find the price-plus-accrued, the accrued interest, and the price.

8. A 20-year bond has a par value of 1,000 and pays semi-annual coupons at a 6% (annual) rate. An investor purchases the bond at issue at a price such that its yield to maturity is 5.5% convertible semi-annually. The investor sells the bond immediately after the 11th coupon payment at a price such that its new owner's yield to maturity is 5.0% convertible semi-annually. What was the investor's yield (convertible semi-annually) on this investment over the 5.5-year period the bond was owned?

Section 4.7

Basic Review Problem Solutions

1. On the BA II Plus, set N=10, I/Y=4, PMT=30, FV=1,000, and CPT PV=−918.89. The price is 918.89.

2. On the BA II Plus, set N=10, PV= -975, PMT=30, FV=1,000, and CPT I/Y=3.30. The yield per semi-annual period is 3.30%. The nominal yield is 6.6%.

3. $P = F + F(r-i)a_{\overline{n}|\,i} = 1{,}000 + 1{,}000(0.03-0.04)a_{\overline{10}|\,.04}$
   $= 1{,}000 + (-10)(8.111) = 918.89$

4. $F = 1000$, $r - i = 0.03 - 0.04 = -0.01$, $n - k + 1 = 10 - 4 + 1 = 7$.
   $F(r-i)v^{n-k+1} = 1{,}000(-0.01)\left(\frac{1}{1.04}\right)^7 = -7.60$

5. $4.10(1.025)^5 = 4.64$

6. Use the earliest possible redemption date. On the BA II Plus, set N=4, I/Y=2.75, PMT=30, FV=1,000, and CPT PV= -1,009.35. The price is 1,009.35.

7. First find the price on the previous coupon date. On the BA II Plus, set N=5, I/Y=3, PMT=35, FV=1,000, and CPT PV= −1,022.90.
   $P_t = 1{,}022.90 \qquad k = \frac{80}{183}$
   Price-plus-accrued = $P_{t+k} = 1{,}022.90(1.03)^{\frac{80}{183}} = 1{,}036.20$
   Accrued interest = $35\left(\frac{80}{183}\right) = 15.30$
   Price = (Price-plus-accrued) – (Accrued interest) = 1,036.20 – 15.30 = 1,020.90

8. Initial price: Set N=40, I/Y=2.75, PMT=30, and FV=1,000. CPT PV=-1,060.20.
   Sale price: N=29 (periods remaining) and I/Y=2.5. CPT PV=-1,102.27.
   Yield: N=11 (periods the bond was owned), PV=-1,060.20 (price paid), and FV=1,102.27 (sale price). CPT I/Y=3.1373.
   Yield = 6.275% convertible semi-annually.

Section 4.8

Sample Exam Problems

1. **(2005 Exam FM Sample Questions #10)**
A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%.

Calculate the interest portion of the 7th coupon.
(A) 632 (B) 642 (C) 651 (D) 660 (E) 667

2. **(2005 Exam FM Sample Questions #2)**
You have decided to invest in Bond X, an n-year bond with semi-annual coupons and the following characteristics:
- Par value is 1000.
- The ratio of the semi-annual coupon rate to the desired semi-annual yield rate, $\frac{r}{i}$, is 1.03125.
- The present value of the redemption value is 381.50.

Given $v^n = 0.5889$, what is the price of bond X?

(A) 1019 (B) 1029 (C) 1050 (D) 1055 (E) 1072

3. **(2005 Exam FM Sample Questions #30)**
As of 12/31/03, an insurance company has a known obligation to pay \$1,000,000 on 12/31/2007. To fund this liability, the company immediately purchases 4-year 5% annual coupon bonds totaling \$822,703 of par value. The company anticipates reinvestment interest rates to remain constant at 5% through 12/31/07. The maturity value of the bond equals the par value.

Under the following reinvestment interest rate movement scenarios effective 1/1/2004, what best describes the insurance company's profit or (loss) as of 12/31/2007 after the liability is paid?

	Interest Rates Drop by ½%	Interest Rates Increase by ½%
(A)	+6,606	+11,147
(B)	(14,757)	+14,418
(C)	(18,911)	+19,185
(D)	(1,313)	+1,323
(E)	Breakeven	Breakeven

4. (**2005 Exam FM Sample Questions #47)**
Bill buys a 10-year 1000 par value 6% bond with semi-annual coupons. The price assumes a nominal yield of 6%, compounded semi-annually. As Bill receives each coupon payment, he immediately puts the money into an account earning interest at an annual effective rate of i.

At the end of 10 years, immediately after Bill receives the final coupon payment and the redemption value of the bond, Bill has earned an annual effective yield of 7% on his investment in the bond. Calculate i.

(A) 9.50% (B) 9.75% (C) 10.00% (D) 10.25% (E) 10.50%

5. (2005 Exam FM Sample Questions #50)
A 1000 bond with semi-annual coupons at $i^{(2)} = 6\%$ matures at par on October 15, 2020. The bond is purchased on June 28, 2005 to yield the investor $i^{(2)} = 7\%$. What is the purchase price?

Assume simple interest between bond coupon dates and note that:

Date	Day of the Year
April 15	105
June 28	179
October 15	288

(A) 906 (B) 907 (C) 908 (D) 919 (E) 925

6. (2005 Exam FM Sample Questions #54)
Matt purchased a 20-year par value bond with semi-annual coupons at a nominal annual rate of 8% convertible semi-annually at a price of 1722.25. The bond can be called at par value X on any coupon date starting at the end of year 15 after the coupon is paid. The price guarantees that Matt will receive a nominal annual rate of interest convertible semi-annually of at least 6%. Calculate X.

(A) 1400 (B) 1420 (C) 1440 (D) 1460 (E) 1480

7. (2005 Exam FM Sample Questions #55)
Toby purchased a 20-year par value bond with semi-annual coupons at a nominal annual rate of 8% convertible semi-annually at a price of 1722.25. The bond can be called at par value 1100 on any coupon date starting at the end of year 15.

What is the minimum yield that Toby could receive, expressed as a nominal annual rate of interest convertible semi-annually?

(A) 3.2% (B) 3.3% (C) 3.4% (D) 3.5% (E) 3.6%

8. (2005 Exam FM Sample Questions #56)
Sue purchased a 10-year par value bond with semi-annual coupons at a nominal annual rate of 4% convertible semi-annually at a price of 1021.50. The bond can be called at par value X on any coupon date starting at the end of year 5. The price guarantees that Sue will receive a nominal annual rate of interest convertible semi-annually of at least 6%.

Calculate X.

(A) 1120 (B) 1140 (C) 1160 (D) 1180 (E) 1200

9. (2005 Exam FM Sample Questions #57)
Mary purchased a 10-year par value bond with semi-annual coupons at a nominal annual rate of 4% convertible semi-annually at a price of 1021.50. The bond can be called at par value 1100 on any coupon date starting at the end of year 5.

What is the minimum yield that Mary could receive, expressed as a nominal annual rate of interest convertible semi-annually?

(A) 4.8% (B) 4.9% (C) 5.0% (D) 5.1% (E) 5.2%

10. (May 05 #5)
Susan can buy a zero coupon bond that will pay 1000 at the end of 12 years and is currently selling for 624.60. Instead, she purchases a 6% bond with coupons payable semi-annually that will pay 1000 at the end of 10 years. If she pays X she will earn the same annual effective interest rate as the zero coupon bond.

Calculate X.

(A) 1164 (B) 1167 (C) 1170 (D) 1173 (E) 1176

11. (May 05 #11)
A 1000 par value bond pays annual coupons of 80. The bond is redeemable at par in 30 years, but is callable any time from the end of the 10th year at 1050. Based on her desired yield rate, an investor calculates the following potential purchase prices, P:

- Assuming the bond is called at the end of the 10th year, $P = 957$
- Assuming the bond is held until maturity, $P = 897$

The investor buys the bond at the highest price that guarantees she will receive at least her desired yield rate regardless of when the bond is called. The investor holds the bond for 20 years, after which time the bond is called.

Calculate the annual yield rate the investor earns.

(A) 8.56% (B) 9.00% (C) 9.24% (D) 9.53% (E) 9.99%

12. (Nov 05 #4)
A ten-year 100 par value bond pays 8% coupons semi-annually. The bond is priced at 118.20 to yield an annual nominal rate of 6% convertible semi-annually.

Calculate the redemption value of the bond.

(A) 97 (B) 100 (C) 103 (D) 106 (E) 109

13. (Nov 05 #11)
An investor borrows an amount at an annual effective interest rate of 5% and will repay all interest and principal in a lump sum at the end of 10 years. She uses the amount borrowed to purchase a 1000 par value 10-year bond with 8% semi-annual coupons bought to yield 6% convertible semi-annually. All coupon payments are reinvested at a nominal rate of 4% convertible semi-annually.

Calculate the net gain to the investor at the end of 10 years after the loan is repaid.

(A) 96 (B) 101 (C) 106 (D) 111 (E) 116

14. (Nov 05 #16)
Dan purchases a 1000 par value 10-year bond with 9% semi-annual coupons for 925. He is able to reinvest his coupon payments at a nominal rate of 7% convertible semi-annually.

Calculate his nominal annual yield rate convertible semi-annually over the ten-year period.

(A) 7.6% (B) 8.1% (C) 9.2% (D) 9.4% (E) 10.2%

15. (Nov 05 #22)
A 1000 par value bond with coupons at 9% payable semi-annually was called for 1100 prior to maturity. The bond was bought for 918 immediately after a coupon payment and was held to call. The nominal yield rate convertible semi-annually was 10%.

Calculate the number of years the bond was held.

(A) 10 (B) 25 (C) 39 (D) 49 (E) 54

16. (Nov 05 #24)
A 30-year bond with a par value of 1000 and 12% coupons payable quarterly is selling at 850. Calculate the annual nominal yield rate convertible quarterly.

(A) 3.5% (B) 7.1% (C) 14.2% (D) 14.9% (E) 15.4%

Section 4.9

Sample Exam Problem Solutions

1.
The interest paid in the 7th coupon is simply 6% of the value of the bond at time 6. We can use the financial calculator to find the value at time 6, when there are only 4 coupon payments of 800 and the redemption value of 10,000 left to be paid. Set N =4, I/Y = 6, PMT = 800, FV = 10,000, and CPT PV=10,693.02. The interest portion of the 7th payment is: $0.06(10,693.02) = 641.58$.

Answer B

2.
This problem cannot be done directly using the financial keys on the calculator. You must set up equations and do some algebra. The price (i.e., present value) of the bond is given by:

Present value of coupons + Present value of redemption value

We are given the present value of the redemption value: 381.50.
The coupons equal 1,000*r*, so the present value of the coupons is:

$$1{,}000r\left(a_{\overline{2n}|i}\right) = 1{,}000r\left(\frac{1-v^{2n}}{i}\right) = 1{,}000\left(\frac{r}{i}\right)\left(1-v^{2n}\right)$$

$$= 1{,}000\left(1.03125\right)\left(1-0.5889^2\right) = 673.61$$

(Note that v^n is the PV factor for n coupon periods, and there are $2n$ periods.)

Thus the present value of the bond is

381.50 + 673.61 = 1055.11

Note: It might appear this problem can be solved using Makeham's Formula: $P = K + \frac{r}{i}(F-K)$. *We are given* K *and* $\frac{r}{i}$; *if we apply the formula we get:* $P = 381.50 + 1.03125(1{,}000 - 381.50) = 1{,}019.33$, *and 1019 is choice A. But Makeham's Formula requires that* $F = C$. *If C were 1,000, then the present value of the redemption value would be* $1{,}000(0.5889)^2 = 346.80$, *not 381.50. (It turns out that the bond's redemption value is 1,100.) Actually, we <u>can</u> apply Makeham's Formula to find what the bond's price would be if it had a 1,000 redemption value* $\left(346.80 + 1.03125(1{,}000 - 346.80) = 1{,}020.41\right)$, *and then adjust that price by the difference between 346.80 and 381.50 (the given PV of the redemption value) to get the correct answer:*

$$1{,}020.41 + \left(381.50 - 346.80\right) = 1{,}055.11$$

Answer D

3.
i) First we will see why the bonds will cover the obligation of 1,000,000 in four years if reinvestment rates remain at 5%. This illustrates how the company's strategy works. ***On an exam you would start directly at ii) to solve the problem.***

The bonds have "822,703 of par value," and the annual coupon rate is 5%, so the total coupon payment each year is $0.05(822,703) = 41,135.15$. We know that "the maturity value of the bond equals the par value," so they mature for 822,703.

We can calculate the accumulated value of the coupons on the financial calculator: N=4, I/Y=5, and PMT = -41,135.15. CPT FV = 177,297.64. This is the accumulated value of the reinvested coupons. The total amount available at time 4, assuming the coupons can be reinvested at 5%, is:

$$\text{Accumulated value of reinvested coupons + Maturity value}$$
$$= 177,297.64 + 822,703 = 1,000,000.64$$

We see that, based on reinvesting the coupons at 5%, the company will have the amount needed to make the required payment of 1,000,000 at time 4.

ii) Now we can look at what happens if reinvestment rates drop by $1/2\%$. In this case the reinvestment rate is 4.5%. We can calculate the accumulated value of the coupons (reinvested at 4.5%) on the financial calculator with N=4, I/Y=4.5, and PMT = -41,135.15,. The computed value of FV = 175,984 is the accumulated value of the reinvested coupons. The total available at time 4 from the bonds with reinvestment at 4.5% is $175,984 + 822,703 = 998,687$. The company has a shortfall of $998,687 - 1,000,000 = -1,313$.

This matches the loss of (1,313) in choice D, and no other answer has this value for the result of a ½% drop in interest rates. Thus D is the only possible answer. (The second part of choice D is also correct, as can be confirmed by setting I/Y=5.5 and computing FV=178,620. Then $178,620 + 822,703 = 1,001,323$, for an excess of 1,323.)

iii) This problem can also be solved using rough approximations. We see that at maturity the company will have the face amount of the bonds, plus 4 coupon payments, plus interest earned on the 4 coupons. The only thing affected by a change in interest rate is the interest on the coupons. Each coupon is a little over 40,000 (5% of 822,703). The 1st coupon earns interest for 3 years, the 2nd for 2 years, the 3rd for 1 year, and the 4th earns no interest. Ignoring compounding, the total amount of interest is equivalent to 6 years (=3+2+1) of interest at 5% on about 40,000, which is $6 \times 5\% \times 40,000 = 12,000$. If the rate changes by 0.5%, that causes a 10% (=0.5%/5%) change in the amount of interest earned, or about 1,200. Only Answer D has amounts that are close to that number.

Answer D

4.

Since the bond pays semi-annual coupons at a 6% annual rate and is priced at 6% convertible semi-annually, the purchase price of the bond (and the amount of Bill's initial investment) was 1,000. For Bill to earn an annual effective yield of 7% over 10 years on his investment of 1,000, at the end of 10 years he must have $1{,}000(1.07)^{10} = 1{,}967.15$.

At the end of 10 years Bill actually has:

a) the 1,000 maturity value of the bond, plus
b) the accumulated value of the reinvested coupons

In order for the total to be the required 1,967.15, the accumulated value of the coupons must be 967.15.

The amount of each coupon payment is 3% of 1,000, or 30, so Bill's semi-annual deposit to the reinvestment account was 30, and there were a total of 20 semi-annual periods during the term of the 10-year bond. To calculate the semi-annual yield of the reinvestment account on the BA II Plus, set N = 20, PMT = -30, and FV = 967.15. The calculated semi-annual yield is 4.7596%. The annual effective yield is $1.047596^2 - 1 = 0.097458$

Answer B

5.

Note: there is some potential for confusion here, since the problem does not specify whether purchase price means price-plus-accrued or market price. The authors of the posted answer key take the words purchase price to imply the full price-plus-accrued.

The bond has semi-annual payment dates of April 15 and October 15 with coupons of 30. It is purchased between coupon dates on June 28, 2005. There are 183 days between April 15 and October 15, and 74 days between April 15, 2005 and June 28, 2005. The fractional period from the previous coupon date to the date of purchase is $k = \frac{74}{183}$.

The price of the bond immediately after the preceding (April 15, 2005) coupon payment can be found using the financial calculator with N = 31, I/Y = 3.5, PMT = 30, and FV = 1,000. (N=31 reflects one payment in 2005 and 2 in each of the remaining 15 years from 2006 to 2020). Then CPT PV = -906.32. This is P_t (the price on the previous coupon date). The total amount paid on the purchase date is the price-plus-accrued, which equals 906.32 accumulated with interest for 74/183 of a coupon period. The problem states that we are to use simple interest between bond coupon dates, so the calculation is:

$$P_t(1+ki) = 906.32\left(1+\frac{74}{183}(.035)\right) = 919.15$$

Note: If you used compound interest between coupon dates, your answer should be 919.01, leading to the same answer choice.

Answer D

6.
Since the semi-annual coupon rate of 4% is greater than the semi-annual yield rate of 3%, this is a premium bond. The bond is callable in 15 years, so because it is a premium bond, it is priced as if it will be redeemed in 15 years. The problem does not directly give the par value *X* or the coupon amount 0.04*X*, so the financial calculator cannot be used directly. Instead we will set up an equation of value for the price of 1,722.25.

$$1{,}722.25 = (0.04X)a_{\overline{30}|.03} + Xv^{30} = 0.784X + 0.412X = 1.196X$$

$$X = \frac{1{,}722.25}{1.196} = 1{,}440$$

Note: This problem can also be solved using the calculator's TVM functions. Calculate what the bond's price would be if its face amount were 1,000: set N=30, I/Y=3, PMT=40, FV=1,000, and CPT PV=1,196.00.
Then calculate the ratio of the given price (1,722.25) to the price for a 1,000-face bond: $\frac{1{,}722.25}{1{,}196.00} = 1.4400$ *Thus the actual face amount is 1.44 times 1,000, which is 1,440.*

Answer C

7.
The statement that "The bond can be called at par value 1,100 on any coupon date starting at the end of year 15" shows that the face value *F* is also 1,100. Since the price is 1,722.25, the bond is a premium bond. The minimum yield would be obtained if the bond is called (redeemed) after 15 years or 30 bond periods.

We can obtain this (semi-annual) yield from the BA II Plus calculator using N = 30, PV = -1,722.25, PMT = 44, and FV = 1,100. Then CPT I/Y = 1.608. The semi-annual yield is 1.608%, leading to a nominal annual yield of 3.216%.

Answer A

8.
This is like Problem 6, but this time we have a discount bond since the semi-annual coupon rate of 2% is less than the semi-annual yield rate of 3%. The discount bond is callable in 5 years, but it is priced as if it will be redeemed as late as possible, in 10 years. The problem does not directly give the par value X or the coupon amount $0.02X$, so the financial calculator cannot be used directly. Instead we must set up an equation of value for the price of 1,021.50.

$$1,021.50 = (0.02X)a_{\overline{20}|.03} + Xv^{20} = 0.2975X + 0.5537X = 0.8512X$$

$$X = \frac{1,021.50}{0.8512} = 1,200.07$$

Note: As with Problem 6, this problem can be solved using the calculator's TVM functions. First calculate what the bond's price would be if its face amount were 1,000: set N=20, I/Y=3, PMT=20, FV=1,000, and CPT PV=851.23.
Then calculate the ratio of the given price (1,021.50) to the price for a 1,000-face bond: $\frac{1,021.50}{851.23} = 1.2000$ *The actual face amount is 1.2000 times 1,000, which is 1,200.*

Answer E

9.
Since the price of 1021.50 is less than the par value of 1,100, this is a discount bond and its minimum yield is obtained when it is not called (i.e., it is redeemed after 10 years, or 20 semi-annual periods). The semi-annual coupon is 1,100(0.02) = 22. Thus we can find the semi-annual yield on the BA II plus using N = 20, PV = -1,021.50, PMT = 22, and FV = 1,100. Then CPT I/Y=2.456.

The semi-annual yield is 2.456%. Thus the minimum nominal annual yield is 4.912% convertible semi-annually.

Answer B

10.
First we find the effective rate on the zero coupon bond using the BA II Plus. Set N=12, PV= –624.60, FV=1,000, CPT I/Y = 4. The annual effective rate is 4%, and Susan should buy the bond for the price X that yields 4% effective annually. The semi-annual yield corresponding to an annual effective rate of 4% is $\sqrt{1.04} - 1 = 0.0198$.

Thus Susan will buy the bond at a semi-annual yield of 1.98%. We assume that the redemption value of 1,000 is also the face value of the bond, so that the semi-annual coupon payment on this 6% bond is 30.

Using the BA II Plus, set N = 20, I/Y = 1.98, PMT = 30, FV = 1,000.
Then CPT PV = -1,167.04.

Answer B

11.
The investor will buy at the lower price of 897 to assure the desired yield. To find the yield on the BA II Plus, set N=20, PV=-897, PMT=80, FV=1,050, and CPT I/Y = 9.24.

Answer C

12.
This is an extremely simple financial calculator problem. On the BA II Plus, set N=20, I/Y=3 (for the yield of 6% convertible semi-annually), PMT = 4 (for the 8% coupon rate payable semi-annually, applied to the 100 par value), and PV = -118.20 (the price). Then CPT FV = 106.

Answer D

13.
The purchase price of the bond can be obtained by using the financial calculator with N = 20, I/Y = 3, PMT = 40, and FV = 1,000. CPT PV = 1,148.77. The investor will borrow 1,148.77 at 5%, and at the end of 10 years will repay the loan with a payment of: $1{,}148.77(1.05)^{10} = 1{,}871.23$.

The future value of the reinvested coupons can be obtained from the financial calculator with N = 20, I/Y = 2, PV=0, and PMT = -40. CPT FV = 971.89. The redemption value of the bond is 1,000, so the investor will have 1,971.89 at maturity. After repayment of the loan, the investor will have a net gain of

$$1{,}971.89 - 1{,}871.23 = 100.66$$

Answer B

14.
The semi-annual coupons provide 20 semi-annual payments of 45. These are reinvested at a nominal rate of 7% convertible semi-annually (a 3.5% semi-annual effective rate). On the BA II Plus, set N=20, I/Y = 3.5, PMT = -45, and. CPT FV = 1,272.59. Dan also receives the 1,000 redemption value of the bond in 10 years, for a total of 2,272.59. His original investment was 925, so his semi-annual yield on the investment is

$$\left(\frac{2{,}272.59}{925}\right)^{\frac{1}{20}} - 1 = 0.046$$

The nominal yield convertible semi-annually is $2(0.046) = 0.092$.

Answer C

15.
This can be solved using a financial calculator to find the number of periods (N). The values to enter are PV = -918 (the price paid), PMT = 45 (the semi-annual coupon payment), FV = 1,100 (the redemption value) and I/Y = 5 (5% is the semi-annual effective yield, derived from 10% convertible semi-annually). The computed value of N is 49.35 semi-annual periods. This must be converted to $49.35/2 = 24.675$ years. (Note that choice D will trap the student who does not convert the number of periods (49.35) to the number of years.)

Answer B

16.
On the financial calculator, set N=120 (the number of quarters in 30 years), PV = -850 (the price paid), PMT = 30 (the quarterly coupon payment), FV = 1,000 (the redemption value), and CPT I/Y = 3.539% per quarter.
This must be converted to the nominal annual yield convertible quarterly of $4(3.539\%) = 14.156\%$

Answer C

Section 4.10

Supplemental Exercises

1. A 15-year 1,000 par value bond with 7% semi-annual coupons is priced to yield 6% convertible semi-annually. Find the price.

2. Suppose the bond in Problem 1 is offered at a price of 975. What is the nominal yield convertible semi-annually?

3. The company offering the bond in Problem 1 decides to make it more attractive at that price by increasing the redemption value to 1,050. What is the nominal yield convertible semi-annually for this bond with the new redemption value?

4. A 10-year bond with a face value of 1,000 and 5% semi-annual coupons is sold for 980. What should the redemption value be if the bond is to yield 5.4% convertible semi-annually?

5. A 10-year 1,000 par value bond with 6% semi-annual coupons is priced to yield 6.5% convertible semi-annually. Find the discount for this bond.

6. A 5-year 1,000 par value bond with 7% semi-annual coupons is purchased to yield 6.4% convertible semi-annually. The coupon payments are reinvested in a fund that earns 7.2% convertible semi-annually. What is the annual effective yield on the total investment at the end of the 5-year period?

7. A 10-year 1,000 par value bond with 6% semi-annual coupons is priced to yield 5.6% convertible semi-annually. How much of the premium is amortized in the 8^{th} period?

8. A 1,000 par value bond with 6.5% semi-annual coupons is priced to yield 5.8% convertible semi-annually. The amount of the premium amortized in the 4^{th} period is 2.12. How much premium is amortized in the 9^{th} period?

9. A 10-year 1,000 par value bond with 5% semi-annual coupons is priced to yield 5.6% convertible semi-annually. How much of the discount is amortized in the 6^{th} period?

10. A 10-year 1,000 par value bond with 8% semi-annual coupons is callable in 7 years. At what price should an investor buy the bond to yield 7.2% convertible semi-annually?

11. A 1,000 par value bond with 8% semi-annual coupons has payment dates of May 31 and November 30. The bond matures on November 30, 2010. On May 31, 2007 the coupon payment of 40 is paid. The bond is sold 70 days later with a settlement date of August 9. The bond is sold to yield 7.4% convertible semi-annually. Find the price-plus-accrued, the accrued interest and the price. (There are 183 days from May 31 to November 30.)

Section 4.11

Supplemental Exercise Solutions

1. Using the BA II Plus, set N = 30, I/Y = 3, PMT = 35, and FV = 1,000.
   CPT PV = -1,098. Price is 1,098.

2. To get the new yield, set PV = −975. CPT I/Y = 3.6383 (a 3.6383% semi-annual effective yield). The nominal yield convertible semi-annually is 7.2766%.

3. For the bond with new redemption value, set N = 30, PV = -1,098, PMT = 35, and FV = 1,050. CPT I/Y = 3.097. The nominal yield is 6.194%.

4. To find the redemption value, set N = 20, I/Y = 2.7, PV = -980, and PMT = 25. CPT FV = 1,018.05. The redemption value should be 1,018.

5. To find the discount we must first find the price.
   Set N = 20, I/Y = 3.25, PMT = 30, and FV = 1,000. CPT PV = -963.65.
   Discount = 1,000 – 963.65 = 36.35

6. The accumulation of the reinvested coupon payments is $35\,s_{\overline{10}|0.036} = 412.50$.
   The total accumulation is 1,000 + 412.50 = 1,412.50. The total amount invested is the price of the bond, which is found by setting N = 10, I/Y = 3.2, PMT = 35, and FV = 1,000. CPT PV = -1,025.33.
   To find the annual effective yield, j:
   $(1+j)^5 = 1{,}412.50/1{,}025.33 = 1.3776$ $\qquad j = 1.3776^{1/5} - 1 = 6.6\%$

7. The amount of premium amortized in the 8th period is
   $1{,}000(r-i)v^{n-8+1} = 1{,}000(0.03-0.028)(1/1.028)^{13} = 1.40$.

8. Given that the amount of premium amortized in the 4th period is 2.12, the amount amortized in the 9th period is $2.12(1.029^5) = 2.45$.
   (Note that we didn't need to know the face amount or the coupon rate.)

9. The amount of "principal" paid in the 6th period is:
   $1{,}000(0.025-0.028)(1/1.028)^{(20-6+1)} = -1.98$.
   The amount of discount amortized in the 6th period is 1.98.

10. This is a premium bond, so it is priced assuming it will be called at the earliest redemption date, the end of year 7.
    To get the price using the BA II Plus, set N = 14, I/Y = 3.6, PMT = 40, and FV = 1,000. CPT PV = -1,043.39.
    The price is 1,043.39.

11. Immediately after the coupon is paid on May 31, there are 7 coupon payments remaining. At this point the price of the bond can be obtained by setting N = 7, I/Y = 3.7, PMT = 40 and FV = 1,000. CPT PV = -1,018.21.
    The price-plus-accrued is $1{,}018.21(1.037)^{70/183} = 1{,}032.46$. The accrued interest is 40(70/183) = 15.30. The price is 1,032.46 – 15.30 = 1,017.16.

Module 5

Yield Rate of an Investment

The yield rate of an investment is just the interest rate that the investor earns based on the investment's cash flows. We have already found yield rates for bonds and other investments. In this module we will consider various measures of investment yield. We begin with the internal rate of return.

Section 5.1

IRR: Internal Rate of Return

An investor who is contemplating a potential investment is interested in what must be paid out to make the investment and what will be received in return. Suppose an investor is asked to invest 1,000 and is promised in return a payment of 600 in one year and 550 at the end of the second year. Using the convention that money paid out is negative and money received is positive, the investor could describe this investment as the sequence:

−1,000; 600; 550

The payments made each period are called **cash flows.** The cash flow at time k can be denoted by C_k. In this 2-period investment the initial cash flow is $C_0 = -1000$, and the investment returns at times 1 and 2 are $C_1 = 600$ and $C_2 = 550$. The investor is interested in answering the question: "What interest rate (i.e., yield) am I earning on the money I invested?" The **internal rate of return** answers that question. First we will define IRR and show how to calculate it; then we will talk about *why* IRR measures the investment's yield.

Definition.
Suppose an investment for n periods has cash flows $C_0, C_1, ..., C_n$. An internal rate of return for this investment is a solution for i in the equation:

(5.1)
$$C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + ... + \frac{C_n}{(1+i)^n} = 0$$

We can also write **(5.1)** in the form $C_0 + C_1 v + C_2 v^2 + ... + C_n v^n = 0$.

If we find a root for v, we can immediately find $i = (1/v) - 1$.Thus an IRR problem with n periods is really a matter of finding a root for a polynomial of degree n.

There is one additional constraint that is applied in IRR problems. With a sequence of cash flows $C_0, C_1, ..., C_n$ the worst that can happen is that you invest an amount at time 0 and then end up getting nothing back. (That is, you have a negative cash flow at time 0 ($C_0 < 0$), and then you received nothing in return ($C_1 = C_2 = ... = C_n = 0$).) In that case your return is -100% (or -1). Since $i = -1$ is the worst possible result, in IRR problems we ignore all roots less than −1 This corresponds to a requirement that v be greater than or equal to 0, and that d be less than or equal to 1. (By contrast, δ can be any real number.)

Example (5.2)

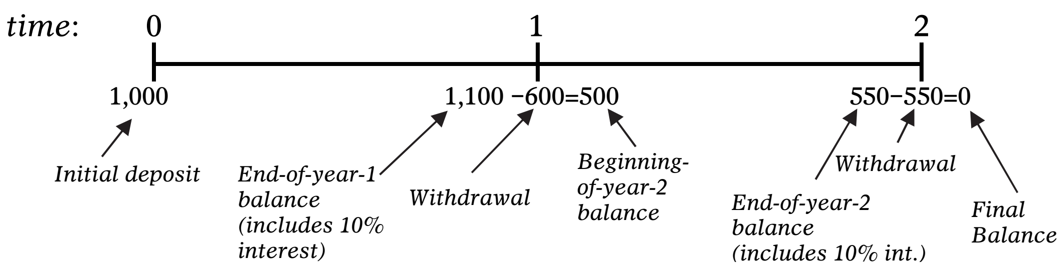
In the investment discussed above (with cash flows -1,000; 600; 550), the internal rate of return is a solution for i in the quadratic equation:

$$-1{,}000 + \frac{600}{(1+i)} + \frac{550}{(1+i)^2} = -1{,}000 + 600v + 550v^2 = 0$$

Using the quadratic formula we see that there are two solutions for v. The solutions are given below along with the corresponding values of i.

$$v = 10/11 \rightarrow i = 0.10 \qquad v = -2 \rightarrow i = -1.50$$

As in many applied problems we have one realistic solution of 10% and one unrealistic solution of -150%.We discard the second root because it is less than -1. The rate earned is an IRR of 10%. To see what a yield of 10% means, let us compare this investment to a bank account earning 10% annually. The timeline below shows the results for this bank account over 2 years:



In one year, the original 1,000 grows to 1,100 at 10% interest. The withdrawal of 600 reduces the balance to 500. In the second year, this 500 grows to 550, which is withdrawn to close the account. Thus the investment behaves exactly like an account earning 10%.

One way to describe the internal rate of return is that it is **the rate of interest at which the present value of all amounts invested is equal to the present value of all the amounts paid back to the investor**. In Example **(5.2)**, for $i = 0.10$:

$$-1{,}000 + \frac{600}{(1+.10)} + \frac{550}{(1+.10)^2} = 0 \quad \text{or, equivalently:} \quad 1{,}000 = \frac{600}{(1.1)} + \frac{550}{(1.1)^2}$$

These equations, with $t = 0$ as the valuation date, show that the combined present value of the payments of 600 and 550 at times 1 and 2 is equal to the present value of the original investment of 1,000 at time 0. In other words, the present value of the amount invested equals the present value of the returns.

Exercise (5.3)

An investor is asked to invest 1,000 and is promised in return a payment of 450 in one year and 630 at the end of the second year. Find the IRR.

Answer: 5%

Note that the cash flow C_k represents *net income* at time k. You may be given information about "revenue" and "expense" at time k instead of being given C_k directly. Then C_k = Revenue at time k - Expense at time k.

For example, in **(5.2)** we were directly given the cash flows: -1,000; 600; 550. This problem could instead have been described as follows:

> An investor spends 1,000 to set up a mining operation for 2 years. In his first year he has revenues of 800 and expenses of 200. In his second year he has revenues of 800 and expenses of 250. Find the IRR of this investment.

This is the same problem as Example **(5.2)**, since the net cash flows are:

−1,000   800−200=600   800−250=550

Calculator Note

The internal rate of return is widely used to evaluate investments. For this reason modern financial calculators like the BA II Plus have a worksheet for entering the cash flows and an IRR key which will calculate the value of i. We have already discussed entering cash flows in module 2. Here we will go through the steps again for the investment in **(5.2)**.

Entering cash flows: Press [CF] to enter the worksheet. Key in [2ND] CLR WORK to remove any numbers left over from prior work. You will see a prompt for the value of CF_0, the cash flow at time 0. Enter the value −1,000. (Don't forget to press ENTER after keying in −1,000.) Scroll down and you will see a prompt for C01, the cash flow at time 1. Enter the number 600. Scroll down again, and there will be a new prompt: "F01=1." This is a request for the number of times (frequency) the cash flow C01 will be repeated. The default value is 1, and if you scroll down, the value of 1 will be assumed with no entry. You will then be prompted for the value of C02. Enter 550.

Calculate the IRR with the following keystrokes. [IRR] [CPT]

The display will show the answer: 10.

IRR problems with more than two periodic payments are difficult to solve directly because they involve higher degree polynomials instead of quadratics. The reader who knows about Galois theory knows that there is no general quadratic formula-type method to find roots for polynomials of degree ≥ 5.

Higher degree polynomials therefore need to be solved approximately using iterative methods such as Newton's method, the secant method, etc. This is how modern financial calculators like the BA II Plus find IRR. Fortunately, we can solve problems for investments with a large number of cash flows using the BA II Plus. Microsoft EXCEL also has an IRR function, which is extensively used for cash flow analysis of real investments. EXCEL spreadsheets were used to create problems and check the answers for this manual.

Exercise (5.4)

An investor is asked to invest 1,000 and is promised in return a payment of 380 in one year, 256 at the end of the second year and 540 at the end of the third year. Find the IRR for this proposed investment.

Answer: 8%

Remember that the term IRR is synonymous with investment "yield." We could have phrased the last problem to ask for the yield on the investment or the "true interest rate earned." Many financial professionals use both of these terms (yield and IRR) interchangeably.

Note that in previous modules we have found yield rates (IRRs) for investments with level payments (PMT) by using the TVM keys (and also by using the Bond worksheet).

Example (5.5)

A lender makes a loan of 15,000 that will be repaid by 4 annual end-of-year payments of 5,000. What is the yield on this investment?

Solution.
Set N = 4, PV = −15,000, PMT = 5,000, FV = 0 and CPT I/Y = 12.59.
The yield for this investment is 12.59%.

In the above problem we could also have asked for the IRR, as it means the same thing. Many students find this confusing. Due to the calculator key structure they think of IRR as something that applies only when cash flows are not level, but this is not the case. For example, you can use the BA II Plus's CF worksheet to solve the problem of Example **(5.5)** by entering $CF_0 = -15{,}000$, CO1 = 5,000, FO1 = 4, and then pressing the IRR key and CPT. The answer of 12.59 is the same as when we used the TVM keys.

Exam problems often make yield questions a bit tougher by asking for nominal rates of return or annual effective yields. Remember that the BA II Plus calculates yields as *effective rates* per period. In the problems we have worked so far in this module, the period has been one year, so the yields have been *annual* effective rates. In the next exercise, the cash flows are *not* annual. This is a type of problem that we have already done in Module 2.

Exercise (5.6)

A lender makes a loan of 24,000 to be repaid with 10 semi-annual payments of 3,500. What is a) her nominal yield convertible semi-annually and b) her annual effective yield?

Answers: a) 15.04% b) 15.61%

Why is the Rate of Return "Internal"?

The investment yields are called "internal" because they do not apply to money after it is paid out. Consider our original problem of finding the IRR for the investment with cash flows -1,000; 600; 550.

Suppose the investor is trying to build a fund for use in two years. Then when she receives the first payment of 600 she will re-invest it. If she has to reinvest it at only 5% one year from now, she will have $1.05(600)+550=1{,}180$ at the end of the second year. The yield on an investment of 1,000 that pays 1,180 in 2 years is an annual effective rate of 8.628%, not 10%. The money that was "internal" to the original investment earned 10% while it was in that investment. But the overall return was affected by the lower rate that was earned on the money that had to be reinvested at 5% (outside of the original investment).

Some analysts prefer to use a "modified IRR" that adjusts for reinvestment. Which method to use for measuring the return really depends on the investor's objectives. Most investors we know use IRR as the primary tool for evaluating an investment. We will focus on it as the primary yield tool here.

The reader should remember that there are reinvestment problems in Module 2 (in the section entitled Reinvestment Problems), and you are required to be able to solve them for the exam.

Uniqueness of the Internal Rate of Return

A polynomial of degree n can have anywhere from 0 to n real roots. In all of the previous problems, the equation of value had only one meaningful solution for the IRR. Confusing situations involving multiple roots can arise, as we will see in the next example. However, in the very common situations where there is an investment at time 0 ($C_0 < 0$) and all subsequent cash flows ($C_1, C_2, ..., C_n$) are positive, there is a unique IRR greater than -1. (And of course there is also a unique IRR when $C_0 > 0$ and $C_1, C_2, ..., C_n$ are all ≤ 0.)

Another method to assure the IRR is unique involves analyzing the investment as a bank account that pays interest at a rate equal to the IRR. Calculate the period-by-period account balances (using the investment's cash flows and the IRR). If the account balance never changes sign (from positive to negative, or vice-versa) during the life of the investment, then the IRR is unique. Examples **(5.7)** and **(5.8)** demonstrate how to apply this test.

Example (5.7)

An investor can invest 100,000 in a mining operation. In one year he will receive a payout of 230,000, but at the end of the second year he must pay 132,000 for cleanup costs. His cash flow sequence is:

−100,000; 230,000; −132,000

The internal rate of return is a solution for i of the quadratic equation:

$$-100{,}000 + 230{,}000v - 132{,}000v^2 = 0$$

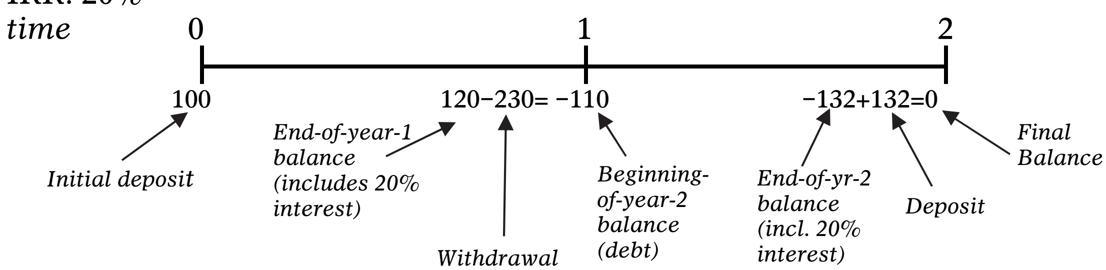
Using the quadratic formula we see that there are two solutions for v. The solutions, along with the corresponding values of i, are:

$$v = 0.90909 \rightarrow i = .10 \text{ and } \qquad v = 0.83\overline{3} \rightarrow i = .20$$

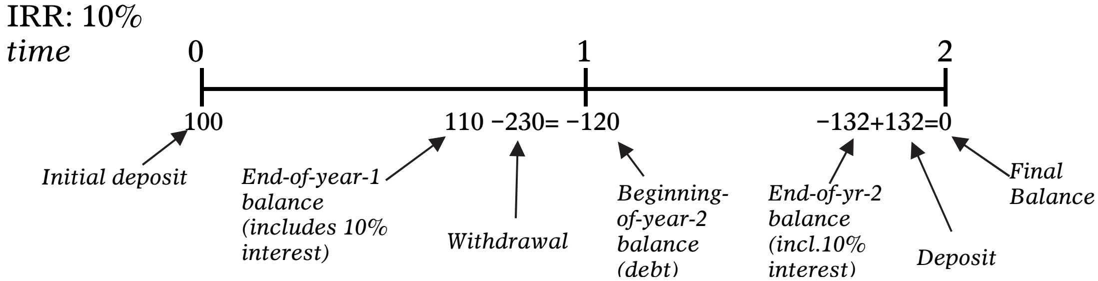
This is confusing. The project appears to earn realistic rates of either 10% or 20%. To see how this can happen, we will provide a savings account analysis for the project at each rate.

(The cash flow amounts have been divided by 1,000 to fit the available space.)

IRR: 20%



IRR: 10%



At either 10% or 20%, if this were a true bank account, the withdrawal of 230 at time 1 would be more than is in the account, so the investor would be in debt to the bank. The investor would then pay back the loan with interest at time 2. Such projects are called **borrowing projects**, and they can lead to multiple rates of return, as in this example.

Because the IRR in a borrowing project is used as both a payout rate (a rate *earned* by the investor) and a borrowing rate (a rate *paid* by the investor, who is then a borrower) it is not a valid basis for evaluating the project. A high IRR is "good" to the extent that the investor *earns* that rate during a part of the investment period, but a high IRR is also "bad" during the period when the investor *pays* that rate.

Neither IRR in Example (5.7) is valid for investment purposes. IRR should be used to analyze an investment only if it is *not* a borrowing project. In other words, do *not* use IRR if the "account balance" changes sign during the life of the investment. (In this case, the account balance was positive after the initial investment of 100 at time 0, then turned negative when 230 was withdrawn at time 1, and of course ended at 0 at time 2.)

Example (5.8)

An investor can invest 250,000 in a mining operation. In one year he will receive a payout of 230,000, but at the end of the second year he must pay 132,000 for cleanup costs. After the cleanup, the land can be used for development, and will be sold after the third year for 250,000. The cash flow sequence is:

−250,000; 230,000; −132,000; 250,000

The internal rate of return is a solution for i of the quadratic equation:

$$-250,000 + 230,000v - 132,000v^2 + 250,000v^3 = 0$$

We cannot easily solve this cubic, but by using the Cash Flow worksheet (CF_0=-250,000, C01=230,000, C02=-132,000, C03=250,000, and IRR CPT), we calculate an IRR of 18.59% for this project. Is this IRR unique? In the following table, we perform a savings account analysis to determine whether 18.59% is the only IRR for these cash flows.

Time	Balance before CF	CF	Balance after CF
0	0	250,000	250,000
1	296,465	-230,000	66,465
2	78,818	132,000	210,818
3	250,000	-250,000	0

The "Balance before CF" in each year is the prior year's "Balance after CF" increased by a factor of 1.1859 to reflect interest at the IRR.

Because the balance in this "savings account" is always positive (or 0), this is *not* a borrowing project, so the IRR of 18.59% is valid. That is, the IRR is always a rate that the investor is *earning* on his investment (his "balance"), and not a rate he is paying on a "borrowed" amount (which would be a negative "balance"). The cash flows change signs multiple times, as happened in Example **(5.7)**. But because *the account balance does not change sign*, we can be certain that the calculated IRR is unique.

Section 5.2

Time-Weighted and Dollar-Weighted Rates of Return

This section presents two methods of measuring the rate of return on an investment fund and evaluating the investment manager's performance. They are the time-weighted and dollar-weighted methods, and they generally do not produce the same value. The next example illustrates these methods.

Example (5.9)

An investment manager had a fund of 100,000 at the start of the year 2016. On June 30th the value of the fund had dropped to 90,000 and a new deposit of 110,000 was received. At year-end the account balance was 220,000. We will measure the return on this fund using both methods.

Time-weighted rate of return

Let j_1 and j_2 represent the rates earned for the first and second halves of the year. For the first half (where 100,000 dropped to 90,000):

$$1+j_1=\frac{90{,}000}{100{,}000}=0.9\rightarrow j_1=-0.10$$

After the new deposit of 110,000 the second half began with 200,000 in the fund and it grew to 220,000.

$$1+j_2=\frac{220{,}000}{200{,}000}=1.1\rightarrow j_2=0.10$$

To get the time-weighted yield j for the entire year, we use the time-weighted return relationship:

$$1+j=(1+j_1)(1+j_2)=0.9(1.1)=0.99 \rightarrow \quad j=-0.01$$

Using the time-weighted rate of return, the fund's rates of return over different periods of the year are compounded to get a return for the entire year. Under this method, the investment manager has a loss of 1% (a time-weighted return of -1%).

Dollar-weighted rate of return

Here we are looking for i, the rate of *simple* interest that would cause the invested dollars to accumulate to 220,000 at year-end if it had been in effect for the entire year. This rate i must satisfy the equation:

$$100{,}000(1+i)+110{,}000\left(1+\frac{i}{2}\right)=220{,}000$$

This is easily solved for i: $155{,}000i=10{,}000\rightarrow i=0.0645$.

Under the dollar-weighted method the investment manager's yield is 6.45%. It is not hard to see why performance looks better under this method. The fund had more money invested during the second half of the year, when performance was better. The dollar-weighted method takes account of the amount of money in the fund during each period of the year; the time-weighted method does not.

Exercise (5.10)

Find the time-weighted and dollar-weighted yields if the original balance of 100,000 declined to 90,000 at mid-year but the deposit made at that point was 10,000 and the final amount in the fund was 110,000.

Answers: Time-weighted: −1% Dollar-weighted: 0%

Note that the dollar-weighted rate of return is based on simple interest. A similar computation could be performed to find a dollar-weighted yield using compound interest. With modern computer and calculator tools, this is not a hard problem to solve. In fact, a dollar-weighted yield based on compound interest would be the IRR for that series of cash flows. Historically the dollar-weighted method evolved using simple interest because the necessary computer tools for compound interest were not available when it was first used.

In the preceding example, we used only the starting amount and one deposit during the fund year to keep it simple. Next we will summarize the general methods used for measurement where the fund may have many deposits or withdrawals. As we introduce these general methods we will refer to Example **(5.9)** at some points to make the general formula concrete.

Time-weighted Rate of Return

Suppose that contributions are made at times $t_1, t_2, \ldots, t_{m-1}$, with the fund year starting at time $t_0 = 0$ and ending at time $t_m = 1$. In Example **(5.9)**, m=2 and there is one contribution at time $t_1 = 0.5$. We will use the notation:

C'_k = contribution at time t_k, where a negative amount is a withdrawal
B'_k = fund value (balance) at time t_k before the contribution C'_k is made

In Example **(5.9)**, the single contribution is $C'_1 = 110{,}000$ while $B'_0 = 100{,}000$, $B'_1 = 90{,}000$ and $B'_2 = 220{,}000$.

We use j_k to denote the effective rate over the period $[t_{k-1}, t_k]$:

(5.11)

$$1 + j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}} = \frac{\text{Current Balance}}{\text{Prior Balance + Prior Contribution}}$$

The time-weighted rate of return j is found by compounding the rates of return for all m time periods:

(5.12)

$$1 + j = (1 + j_1)(1 + j_2)\ldots(1 + j_m)$$

Note that j is an effective rate for the time period $[t_0, t_m]$. We have defined t_0 and t_m to be 0 and 1, so the period is 1 year and j is an *annual* effective rate.

The concept underlying the time-weighted rate of return for a fund is that a dollar invested in the fund at the beginning of the year grows to $(1+j)$ dollars at the end of the year, since it will have experienced each of the effective rates during the year $(j_1, j_2, ..., j_m)$. Similarly, an investor who keeps money invested in the fund throughout the year will find that the investment has grown by a factor of $(1+j)$. On the other hand, money that was deposited *after* the beginning of the year or withdrawn *before* the end of the year could grow by either more or less than $(1+j)$, depending on what part of the year it was in the fund. (In Example **(5.9)**, the first half of the year had a return less than j, and the second half had a return greater than j.)

Dollar-weighted Rate of Return

Let A = initial fund balance, B = final fund balance, and I = interest earned.

In Example **(5.9)**, A = 100,000 and B = 220,000. The interest earned will be calculated below.

We use C_t to denote the deposit or withdrawal at time t, and define C as the total (net) cash contributions: $C = \sum C_t$. In Example **(5.9)** there was only one contribution: 110,000 at time t=0.5. Thus $C = C_{0.5} = 110{,}000$.

Note that in dollar-weighted problems we ordinarily assume that a year has 12 months of equal length, so the deposit on June 30 is assumed to earn interest for exactly one-half year. The calculation of interest is based on the observation that interest income accounts for the difference between the ending amount B and the sum of the starting amount A and the total net contributions C. Thus:

(5.13)

$$I = B - A - C$$

In Example **(5.9)** $I = 220{,}000 - 100{,}000 - 110{,}000 = 10{,}000$.

Then the dollar-weighted yield is defined by

(5.14)

$$i = \frac{I}{A + \sum C_t(1-t)}$$

In Example **(5.9)**, this would give $i = \dfrac{10{,}000}{100{,}000 + 110{,}000(0.5)} = \dfrac{10{,}000}{155{,}000} = 0.0645$.

The answer calculated using Formula **(5.14)** matches the answer found in Example **(5.9)**, because it is based on the same reasoning that we used in **(5.9)**.

Note that the denominator of **(5.14)** is equal to the initial amount of 100,000, plus one-half of the mid-year contribution of 110,000 (since 110,000 was invested for only the second half of the year). Each amount is multiplied by the number of years over which it earned interest: A earned interest for 1 year, and $C_{0.5}$ earned interest for 0.5 years (the second half of the year). If $C_{0.5}$ had been a *negative* cash flow, that would be an amount that *stopped* earning interest at midyear. We would *subtract* that amount times 0.5 to reflect that it did not earn interest in the second half of the year.

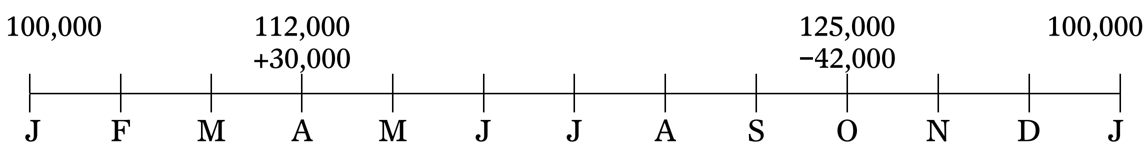
We can think of the dollar-weighted rate of return as being an "average" rate that all of the money invested for some or all of the year earned during the time it was invested. It is exactly like an IRR, except that it is based on simple interest. In fact, if we calculate the IRR for this investment, the semi-annual effective rate is 3.193%, which corresponds to an annual effective rate of 6.49%. This is only slightly different from the dollar-weighted return of 6.45%. For short time periods (1 year or less), the IRR and the dollar-weighted return are nearly equal, unless the rate of return is very large (so that the compounding during the year makes a significant difference).

The following examples and exercises apply the time-weighted and dollar-weighted methods to a somewhat more involved problem.

Example (5.15)

An investment manager had a fund of 100,000 at the start of year 2016. On April 1st that fund had grown to 112,000 and a new deposit of 30,000 was received. On October 1st the fund balance was 125,000 and a withdrawal of 42,000 was made. At year-end, the account balance was 100,000. Calculate the 2016 time-weighted rate of return for this fund.

Here is a timeline of contributions and fund balances for the fund:



The time-weighted return j can be calculated as follows:

$$1+j=(1+j_1)(1+j_2)(1+j_3)=\left(\frac{112{,}000}{100{,}000}\right)\left(\frac{125{,}000}{142{,}000}\right)\left(\frac{100{,}000}{83{,}000}\right)=1.1878$$

$\rightarrow j=0.1878$

Notice that the *dates* of the deposit and the withdrawal do not affect the calculation. Only the *fund balances* and the *amounts of the cash flows* are used in calculating the time-weighted rate of return.

Exercise (5.16)

An investment manager had a fund of 100,000 at the start of year 2016. On May 1st that fund had grown to 108,000 and a new deposit of 20,000 was made.

On December 1st the fund balance was 130,000 and a withdrawal of 12,000 was made. At year-end the account balance was 110,000. Find the 2016 time-weighted rate of return for this fund.

Answer: 2.25%

Example (5.17)

Calculate the 2016 dollar-weighted rate of return for the fund described in Example **(5.15)**.

Solution.

$A = 100,000 \quad B = 100,000 \quad C = 30,000 - 42,000 = -12,000$

$$I = B - A - C = 100,000 - 100,000 - (-12,000) = 12,000$$

$$i = \frac{12,000}{100,000 + \left(1 - \frac{3}{12}\right)30,000 + \left(1 - \frac{9}{12}\right)(-42,000)} = \left(\frac{12,000}{112,000}\right) = 0.1071$$

Notice that the *balances during the year* do not affect the calculation. Only the *beginning and ending balances* and the *dates and amounts of the cash flows* are used in calculating the dollar-weighted rate of return.

By comparing the rates calculated in Examples **(5.15)** and **(5.17)**, we see that the time-weighted return for this fund (18.78%) was greater than the dollar-weighted return (10.71%). This is explained by the fact that the fund had its worst results during the middle period of the year, when the amount invested was the largest. Because the dollar-weighted formula reflects the amount invested, this adverse result had a greater effect on the dollar-weighted return. Like IRR, the dollar-weighted return is the rate earned on all the money in the fund *while it is in the fund*; the time-weighted return is the rate earned on an amount that was in the fund for the whole year.

Exercise (5.18)

Calculate the 2016 dollar-weighted rate of return for the fund described in Exercise (5.16).

Answer: 1.78%

Section 5.3

Net Present Value

The **net present value (NPV)** of a series of cash flows $C_0, C_1, \ldots, C_n$ .at a rate i is just the sum of the present values of the cash flows:

(5.19)

$$NPV = C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_n}{(1+i)^n}$$

Your calculator has an NPV key that you should be familiar with.

Example (5.20)

Find the net present value of the annual cash flow series from Example **(5.2)** at the following rates: $i = 0.09$, $i = 0.10$, and $i = 0.11$.

Solution.
The cash flows were −1,000; 600; 550.

Go to the CF worksheet and enter CF_0 = -−1,000, C01= 600, and C02=550.
Then press the NPV key and you will see a prompt for the interest rate.
Enter 9 for the interest rate, scroll down to NPV and press CPT.
The NPV at 9% is 13.38.
Now scroll up and down to change I and CPT NPV at 10% and 11%.
The NPV at 10% is 0, and the NPV at 11% is −13.07.

The above problem could easily have been phrased to ask for the *present value* instead of the *net present value*. The meaning is the same. The word "net" simply emphasizes that there could be negative cash flows included in the calculation (as was the case here).

Note that we found in Example **(5.2)** that the IRR of this investment was 10%. Now in Example **(5.20)** we see that the NPV of this investment at 10% is 0. This relationship always holds. In fact, some texts *define* IRR as "a solution of the equation $\text{NPV}(i) = 0$."

NPV is sometimes used to compare two or more investments. A company (or an investor) may set a minimum rate of return (a "hurdle rate") that an investment must achieve. That rate is then used to calculate the present values (NPVs) for several alternative investments to determine which one generates the greatest value. There are, of course, other considerations, such as the riskiness of each investment and what other resources (besides capital) may be required. But NPV can be a very useful tool, particularly for comparing investments that have similar characteristics.

Example (5.21)

A corporation is considering two alternative projects. Project A requires an initial investment of 1,000,000 at time 0 and is expected to generate returns of 200,000 per year (at the end of each year) for 9 years. Project B also requires an initial investment of 1,000,000 at time 0, and it is expected to return 300,000 per year for 5 years.

The corporation is able to borrow 1,000,000 at a 10% annual effective interest rate. Using this 10% "cost of capital" to calculate the NPVs for Projects A and B, which project has the larger net present value?

Solution.
For Project A, press CF and set CF_0=-1,000,000; C01=200,000; F01=9.
Then Press NPV, set I=10, scroll down, and press CPT.
Result: NPV=151,804.76

For Project B, press CF and set CF_0=-1,000,000; C01=300,000; F01=5.
Then Press NPV, set I=15, scroll down, and press CPT.
Result: NPV=137,236.03

Project A has a larger net present value based on a 10% cost of capital. Using this criterion, the corporation would choose Project A.

Exercise (5.22)

If the corporation in Example **(5.21)** uses a 15% required return in its calculations, what is the NPV for each project, and which project will the corporation choose?

Answers: NPV(A)=-45,683.22 NPV(B)=5,646.33 Project B

In Exercise **(5.22)**, Project A has a negative NPV at 15%, indicating that its IRR is less than 15%. (In fact, it is 13.7%.) Project B's NPV at 15% is quite small, indicating that its IRR is only slightly greater than 15%. (In fact, it is 15.2%.) For any series of cash flows, the NPV calculated at the IRR will equal zero. An NPV calculated at a *lower* interest rate will be *positive*, and one calculated at a *higher* interest rate will be *negative*. (This assumes that the IRR is unique and we are using the sign convention that money invested is entered as a negative cash flow, and returns are entered as positive cash flows.)

Section 5.4

Formula Sheet

Internal Rate of Return

Given investment cash flows $C_0, C_1, \ldots, C_n$, an internal rate of return is a solution for i of the equation:

$$C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_n}{(1+i)^n} = 0 \text{ or } C_0 + C_1 v + C_2 v^2 + \ldots + C_n v^n = 0.$$

The internal rate of return is the rate of interest at which the present value of all amounts invested is equal to the present value of all amounts returned to the investor.

There may be multiple IRR solutions if the investment is a "borrowing project."

Time-weighted Rate of Return

C'_k = Contribution or withdrawal at time t_k

B'_k = Fund Balance at time t_k before the contribution C'_k is made.

j_k = Effective rate earned over the period $[t_{k-1}, t_k]$

$$1 + j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}}$$

The time-weighted rate of return j is found by calculating:

$$1 + j = (1 + j_1)(1 + j_2)\ldots(1 + j_m)$$

Dollar-weighted Rate of Return

A = Initial fund balance B = Final fund balance. I = Interest earned

C_t = Contribution or withdrawal at time t.

$C = \sum C_t$ = Total (net) cash contributions

$B = A + C + I \rightarrow I = B - A - C$

The dollar-weighted rate of return i is found by calculating:

$$i = \frac{I}{A + \sum C_t(1-t)}$$

Section 5.5

Basic Review Problems

1. An investor is asked to invest 1,100 and is promised in return a payment of 500 in one year and 700 at the end of the second year. Find the IRR.

2. An investor is asked to invest 11,000 and is promised in return a payment of 4,000 after one year, 5,000 after the second year and 4,500 after the third year. Find the IRR for this investment.

3. A lender invests 20,000 to make a loan which will be repaid with 3 annual end-of-year payments of 8,000. What is her yield (IRR) on this investment?

4. An investment manager had a fund of 100,000 at the start of the year 2016. On February 1st that fund had dropped to 98,000 and a withdrawal of 10,000 was made. On September 1st the fund balance was 100,000 and a new deposit of 10,000 was made. At year-end the account balance was 105,000. Find the time-weighted and dollar-weighted rates of return. Also calculate the IRR (as an annual effective rate).

Section 5.6

Basic Review Problem Solutions

1. Use the calculator's CF worksheet with $CF_0 = -1{,}100$, $C01 = 500$ and $C02 = 700$. Then press IRR CPT. The yield is 5.67%. (You could also do this one as a quadratic but it will take more time.)

2. Calculator. Use the CF worksheet with $CF_0 = -11{,}000$, C01=4,000. C02=5,000, and C03=4,500. Then press IRR CPT. The yield is 10.75%.

3. Set PV = -20,000, N=3, PMT=8,000, FV=0, and CPT I/Y = 9.70.
   The yield is 9.70%.

4. Time-weighted return

   $$1+j=(1+j_1)(1+j_2)(1+j_3)=\left(\frac{98{,}000}{100{,}000}\right)\left(\frac{100{,}000}{88{,}000}\right)\left(\frac{105{,}000}{110{,}000}\right)=1.063\rightarrow j=0.063$$

   Dollar-weighted return.

   $B=100{,}000 \qquad A=105{,}000 \qquad C=-10{,}000+10{,}000=0$

   $I=B-A-C=105{,}000-100{,}000-0=5{,}000$

   $$i=\frac{5{,}000}{100{,}000+\left(1-\frac{1}{12}\right)(-10{,}000)+\left(1-\frac{8}{12}\right)(10{,}000)}=\left(\frac{5{,}000}{94{,}166.67}\right)=0.05310$$

   IRR

   CF_0=-100,000, C01=10,000, F01=1, C02=0, F02=6, C03=-10,000, F03=1, C04=0, F04=3, C05=105,000, F05=1.
   Press IRR CPT to find IRR=0.43223.
   The *monthly* effective interest rate is 0.43223%.
   The *annual* effective rate is $1.0043223^{12}-1=5.312\%$.

   Note that the IRR and the dollar-weighted rate of return are nearly the same (5.312% and 5.310%). As mentioned earlier, the only difference between these two measures is that IRR is based on compound interest, and dollar-weighted return is based on simple interest. In this example, where the annual rate of return is about 5%, the compounding within a one-year period has no significant effect.

Section 5.7

Sample Exam Problems

1. **(2005 Exam FM Sample Questions #5)**
An association had a fund balance of 75 on January 1 and 60 on December 31. At the end of every month during the year, the association deposited 10 from membership fees. There were withdrawals of 5 on February 28, 25 on June 30, 80 on October 15, and 35 on October 31.

Calculate the dollar-weighted (money-weighted) rate of return for the year.

(A) 9.0% (B) 9.5% (C) 10.0% (D) 10.5% (E) 11.0%

2. **(Nov 05 #1)**
An insurance company earned a simple rate of interest of 8% over the last calendar year based on the following information:

Assets, beginning of year	25,000,000
Sales revenue	X
Net investment income	2,000,000
Salaries paid	2,200,000
Other expenses paid	750,000

All cash flows occur at the middle of the year.

Calculate the effective yield rate.

(A) 7.7% (B) 7.8% (C) 7.9% (D) 8.0% (E) 8.1%

3. (2005 Exam FM Sample Questions #19)
You are given the following information about the activity in two different investment accounts:

Account K

Date	Fund Value Before Activity	Activity Deposit	Activity Withdrawal
January 1, 1999	100.0		
July 1, 1999	125.0		X
October 1, 1999	110.0	$2X$	
December 31, 1999	125.0		

Account L

Date	Fund Value Before Activity	Activity Deposit	Activity Withdrawal
January 1, 1999	100.0		
July 1, 1999	125.0		X
December 31, 1999	105.8		

During 1999, the dollar-weighted (money-weighted) return for investment account K equals the time-weighted return for investment account L, which equals i. Calculate i.

(A) 10% (B) 12% (C) 15% (D) 18% (E) 20%

4. (2005 Exam FM Sample Questions #23)
Project P requires an investment of 4000 at time 0. The investment pays 2000 at time 1 and 4000 at time 2.

Project Q requires an investment of X at time 2. The investment pays 2000 at time 0 and 4000 at time 1.

The net present values of the two projects are equal at an interest rate of 10%. Calculate X.

(A) 5400 (B) 5420 (C) 5440 (D) 5460 (E) 5480

5. (2005 Exam FM Sample Questions #32)
An investor pays $100,000 today for a 4-year investment that returns cash flows of $60,000 at the end of each of years 3 and 4. The cash flows can be reinvested at 4.0% per annum effective.

If the rate of interest at which the investment is to be valued is 5.0%, what is the net present value of this investment today?

(A) −1398 (B) −699 (C) 699 (D) 1398 (E) 2629

6. (2005 Exam FM Sample Questions #45)
You are given the following information about an investment account:

Date	Value Immediately Before Deposit	Deposit
January 1	10	
July 1	12	X
December 31	X	

Over the year, the time-weighted return is 0%, and the dollar-weighted (money weighted) return is Y.

Calculate Y.

(A) −25% (B) −10% (C) 0% (D) 10% (E) 25%

7. (May 05 #7)
Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these cash flows is 364.46 at an annual effective rate of interest i.

Calculate i.

(A) 10% (B) 11% (C) 12% (D) 13% (E) 14%

8. (May 05 #16)
At the beginning of the year, an investment fund was established with an initial deposit of 1000. A new deposit of 1000 was made at the end of 4 months. Withdrawals of 200 and 500 were made at the end of 6 months and 8 months, respectively. The amount in the fund at the end of the year is 1560.

Calculate the dollar-weighted (money-weighted) yield rate earned by the fund during the year.

(A) 18.57% (B) 20.00% (C) 22.61% (D) 26.00% (E) 28.89%

9. (May 05 #21)
A discount electronics store advertises the following financing arrangement:

"We don't offer you confusing interest rates. We'll just divide your total cost by 10 and you can pay us that amount each month for a year."

The first payment is due on the date of sale and the remaining eleven payments at monthly intervals thereafter.

Calculate the effective annual interest rate the store's customers are paying on their loans.

(A) 35.1% (B) 41.3% (C) 42.0% (D) 51.2% (E) 54.9%

Section 5.8

Sample Exam Problem Solutions

1.
We have:

A = initial fund balance = 75
B = final fund balance = 60
$C = \sum C_t$ = total (net) cash contribution = $10(12) - 5 - 25 - 80 - 35 = -25$

We then calculate:

$I = B - A - C$ = 60 – 75 – (-25) = 10.

We will use the standard convention of assuming all months are of equal length (and will assume that October 15 occurs in the middle of October). For example, for the end-of-month payment in January, $t = \dfrac{1}{12}$ and $1 - t = \dfrac{11}{12}$.

The calculation of i requires us to find:

$$A + \sum C_t(1-t)$$
$$= 75 + 10\left(\frac{11}{12} + \frac{10}{12} + \ldots + \frac{1}{12}\right) - 5\left(\frac{10}{12}\right) - 25\left(\frac{6}{12}\right) - 80\left(\frac{2.5}{12}\right) - 35\left(\frac{2}{12}\right)$$
$$= 75 + \frac{10}{12}\left(\frac{11 \cdot 12}{2}\right) - \frac{470}{12} = 90.8\overline{3}$$

$$i = \frac{I}{A + \sum C_t(1-t)} = \frac{10}{90.8\overline{3}} = 0.1101$$

Answer E

2.
The company earns 8% interest on its initial assets of 25,000,000 for a full year, and on the net amount added at midyear for half of a year. The amount added at midyear is $X - 2{,}200{,}000 - 750{,}000 = X - 2{,}950{,}000$.

Net investment income is 2,000,000. Thus

$$0.08(25{,}000{,}000) + 0.50(0.08)(X - 2{,}950{,}000) = 2{,}000{,}000$$
$$2{,}000{,}000 + (0.04)(X - 2{,}950{,}000) = 2{,}000{,}000$$
$$X = 2{,}950{,}000$$

The net amount added at midyear is $X - 2{,}950{,}000 = 0$. Hence investment income consists only of earnings of 2,000,000 on the original 25,000,000 invested for one year at 8%. Therefore, the effective yield must be 8%. (Note that 2,000,000 is 8% of 25,000,000.)

Answer D

Note: This problem is rather odd, in that it gives us a "simple rate of interest" of 8% and asks us to calculate an "effective yield rate," which turns out to be 8%. Basically, it provides the dollar-weighted rate of return and asks us to find the IRR, and we discover that the two are equal. As has been discussed, these two measures tend to be very close together when we as looking at a 1-year period and low-to-moderate interest rates (such as 8%).

But in this case the IRR is <u>exactly</u> the same as the dollar-weighted rate of return Why? It turns out that the IRR is 8% because the net cash flow at midyear (time 0.5) is 0. So the only cash flow that earns interest is the initial balance of 25,000,000. Whether it earns simple interest at 8% or compound interest at 8%, in one year it will produce 2,000,000 of interest. So the IRR (at compound interest) equals the given rate of 8% simple interest.

What if the midyear cash flow had <u>not</u> been 0? In that case, there would have been some money that earned interest for a half-year at 8% simple interest, and it would have grown by a factor of 1.04. That is equivalent to an annual effective rate of 8.16% $(= 1.04^2 - 1)$*. So if there had been a <u>positive</u> cash flow at midyear, the IRR would have been a sort of "weighted average" of 8% on the 25,000,000 and 8.16% on the midyear cash flow, resulting in an IRR <u>above</u> 8%. But if there had been a <u>negative</u> midyear cash flow, it would have resulted in an IRR <u>below</u> 8%. Since the midyear cash flow was 0, the IRR is simply 8%.*

3.
For the <u>dollar-weighted account K</u> we have:

A = initial fund balance = 100
B = final fund balance = 125
C = total (net) cash contribution = $2X - X = X$

$$I = B - A - C = 25 - X$$

$$i = \frac{25 - X}{100 - X\left(\frac{6}{12}\right) + 2X\left(\frac{3}{12}\right)} = \frac{25 - X}{100} \rightarrow 1 + i = \frac{125 - X}{100}$$

For <u>the time-weighted account L</u> we have only 2 time periods to consider:

$$1 + j_1 = \left(\frac{125}{100}\right) \qquad 1 + j_2 = \left(\frac{105.8}{125 - X}\right)$$

The time-weighted rate is given by

$$1 + i = (1 + j_1)(1 + j_2) = \left(\frac{125}{100}\right)\left(\frac{105.8}{125 - X}\right) = \frac{132.25}{125 - X}.$$

Since the value of i is the same for both accounts we have:

$$\frac{132.25}{125 - X} = \frac{125 - X}{100} \rightarrow (125 - X)^2 = 13{,}225 \rightarrow X = 10$$

It follows that:

$$1 + i = \frac{125 - 10}{100} = 1.15 \quad \text{and} \quad i = 0.15$$

Answer C

4.
The net present value of P at 10% is $-4{,}000 + \frac{2{,}000}{1.1} + \frac{4{,}000}{1.1^2} = 1{,}123.97$
(This can also be found on the BA II Plus using the NPV function on the calculator with $CF_0 = -4{,}000$, $C01 = 2{,}000$, $C02 = 4{,}000$, and I = 10.)

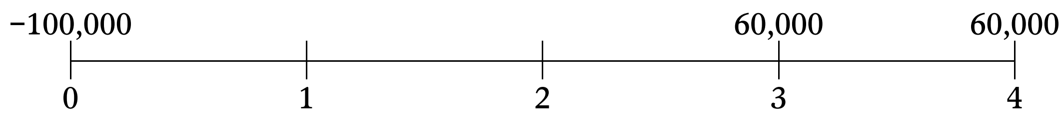
The net present value of Q is: $2{,}000 + \frac{4{,}000}{1.1} - \frac{X}{1.1^2}$.

Thus: $2{,}000 + \frac{4{,}000}{1.1} - \frac{X}{1.1^2} = 1{,}123.97 \rightarrow X = 5{,}460$

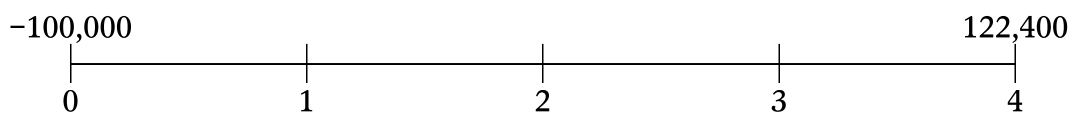
Answer D

5.

The timeline for the investment is:



There is only one cash flow to reinvest: the amount of 60,000 at time 3. At a 4% rate it grows to 60,000(1.04) = 62,400 at time 4. The total amount returned to the investor at time 4 is then 122,400. The resulting situation for the investor is that he invests 100,000 and gets back 122,400 in 4 years. His timeline is:



The NPV of the investor's cash flows at 5% is: $-100{,}000+\dfrac{122{,}400}{1.05^4}=698.78$

Answer C

6.

We can use the fact that the time-weighted yield is 0% to find X:

$$1+0=\left(\frac{12}{10}\right)\left(\frac{X}{12+X}\right)$$
$$120+10X=12X$$
$$X=60$$

We can then use the value of X to find the dollar-weighted yield. For this calculation we need:

A = initial fund balance = 10
B = final fund balance=X
C = total cash contribution (net) = X

Then we can find I (interest earned) using the relation:

$$I=B-A-C=X-10-X=-10\,.$$

The deposit of $X=60$ was made at time t= ½. Thus the dollar-weighted return is given by:

$$Y=\frac{-10}{10+\left(\frac{1}{2}\right)60}=-0.25$$

The dollar-weighted rate of return is -25%.

Answer A

7.
The equation of value here is: $364.46 = 100 + 200v + 100v^2$
Thus we need to solve the quadratic: $0 = -264.46 + 200v + 100v^2$
The only valid root is $v = 0.90908$, which gives $i = 0.10$.

Answer A

Calculator note: The quadratic we solved is the IRR equation for the cash flow sequence −264.46, 200, 100. If you recognize this, you can enter these values in the CF worksheet and compute the IRR. Of course, the answer is still 10%

8.
This is a standard dollar-weighted yield question.

$A = 1{,}000$, $B = 1{,}560$, C=1,000-200-500=300

$I = 1{,}560 - 1{,}000 - 300 = 260$

$$i = \frac{260}{1{,}000 + \left(1 - \frac{4}{12}\right)(1{,}000) + \left(1 - \frac{6}{12}\right)(-200) + \left(1 - \frac{8}{12}\right)(-500)} = \frac{260}{1{,}400} = 0.1857$$

Answer A

9.
No price is specified. We will find the monthly yield assuming a price of 100 and monthly payments of 10 at the beginning of each month for a year. On the BA II Plus in BGN mode, set N=12, PV=100, PMT=-10, and CPT I/Y = 3.5032. The monthly interest rate is 3.5032%, The annual effective rate is:

$(1.035032)^{12} - 1 = 0.512$

Alternatively, we can enter PV as the *net* loan at time 0 (the purchase price of 100, minus the initial payment of 10). This allows us to solve the problem with the calculator in END mode. The entries are: N=11, PV=90, and PMT=-10. Then CPT I/Y = 3.5032, as before.

Answer D

Section 5.9

Supplemental Exercises

1. For an investment of 15,000, an investor is promised return payments of 6,000 in one year, 7,000 in two years and 7,000 in three years. Find the IRR for these cash flows.

For Problems 2 and 3 use the following account summary.

Date	Balance Before Activity	Deposits	Withdrawals
January 1	1,000		
March 1	1,020		50
July 1	990	70	
November 1	1,100		120
December 31	1,050		

2. Find the time-weighted yield for this account.

3. Find the dollar-weighted yield for this account.

4. You are given the following account summary.

Date	Balance Before Activity	Deposits	Withdrawals
January 1	2,000		
April 1	2,060		X
September 1	2,010	300	
December 31	2,405		

The time-weighted yield is 11.11%. Find X.

5. You are given the following account summary.

Date	Balance Before Activity	Deposits	Withdrawals
January 1	1,000		
March 1	1,020	60	
T	1,110		100
December 31	1,050		

The dollar-weighted yield is 8.852%. Find the date T.

6. The following two investment projects have the same net present value at $i = 8\%$.
   (1) Invest 5,000 now and receive 3,000 in 1 year and 4,000 in 2 years.
   (2) Invest 2,500 now and receive 2,000 in 1 year and K in 2 years.

   Find K.

7. An investment of 20,000 now is projected to return 5,000 in one year, 6,000 in two years, 7,000 in three years and 10,000 in four years. What is the net present value of the investment's cash flows at $i = 10\%$?

Section 5.10

Supplemental Exercise Solutions

1. To solve using the BA II Plus, press the CF key, then enter:
   CF_0 = -15,000, C01 = 6,000, C02 = 7,000 and C03 = 7,000
   Then IRR CPT produces: 15.44%

2. Applying the formula for the time-weighted rate of return we have:
   $1 + j = (1020/1000)(990/970)(1100/1060)(1050/980) = 1.157$
   $j = 15.7\%$

3. The interest earned is I = 1,050 – 1,000 – 70 + 50 + 120 = 150.
   $i = 150/[1{,}000 - 50(10/12) + 70(6/12) - 120(2/12)] = 0.154$ or 15.4%

4. $1 + i = 1.1111 = (2{,}060/2{,}000)(2{,}010/[2{,}060 - X])(2{,}405/2{,}310)$
   $2{,}060 - X = 1{,}940$ (to nearest dollar) $\Rightarrow$ $X = 120$

5. The interest earned I = 1,050 – 1,000 – 60 + 100 = 90.
   Let x be the fraction of a year remaining after time T. That is, x is the amount of time after the withdrawal of 100. So the formula for the dollar-weighted rate of return should reflect that 100 was *not* earning interest for a period of length x.
   $$i = 0.08852 = 90/[1{,}000 + 60(5/6) - 100x]$$
   $$1{,}050 - 100x = 90/0.8852 = 1{,}016.72 \quad \Rightarrow \quad x = 0.333$$
   After the 100 was withdrawn, it did not earn interest for the rest of the year, a period of 0.333 years, or 4 months.
   Therefore, the withdrawal occurred 4 months before the end of the year, so it occurred on Sept. 1. The date T is September 1.

6. The net present value for the first project is:
   $$\text{NPV} = -5{,}000 + 3{,}000/1.08 + 4{,}000/1.08^2 = 1{,}207.13$$
   The net present value for the second project is:
   $$\text{NPV} = 1{,}207.13 = -2{,}500 + 2{,}000/1.08 + K/1.08^2$$
   $K/1.08^2 = 1{,}855.28 \Rightarrow K = 2{,}164$

7. To find the NPV using the BA II Plus, first press the CF key, then enter CF_0 = -20,000, C01 = 5,000, C02 = 6,000, C03 = 7,000 and C04 = 10,000. Then press NPV and enter I = 10. Scroll down to NPV and press CPT.
   NPV = 1,593.47

Module 6

The Term Structure of Interest Rates

Section 6.1

Spot Rates and the Yield Curve

The interest rate earned on an investment, or the interest rate charged for a loan will depend in part on the time to maturity of the investment or loan. Anyone who has purchased a certificate of deposit has observed that the interest rate paid depends on the term of the CD. Typically, the longer the term, the higher the rate, although this does not have to be the case.

Similarly, individuals looking for mortgage loans generally find that they can get a lower interest rate on a 15-year loan than on a 30-year loan. The relationship between yield and maturity is referred to as the **term structure of interest rates.** The term structure may be observed by looking at the rates for **zero coupon bonds** based on United States Treasury securities. This requires a bit of explanation.

If a bond has a **zero coupon**, that means no coupon payments will be made to the bondholder. The only payments involved are the original investment and the final repayment of the redemption value at maturity. The price of a two-year zero-coupon bond with a redemption value of 1,000 and an annual yield of 3% would be:

$$P = \frac{1000}{1.03^2} = 942.60$$

In practice, investors buy bonds at prices that give them the yield they desire. Thus, if investors are willing to pay 942.60 for a two-year zero-coupon bond, we could look at this price and calculate the implied two-year annual effective rate:

$$942.60 = \frac{1000}{(1+i)^2} \rightarrow (1+i)^2 = \frac{1,000}{942.60} = 1.0609 \rightarrow (1+i) = 1.0609^{0.5} = 1.03 \rightarrow i = 3\%$$

Investors require higher interest rates on bonds issued by firms that are considered risky, because there is a greater chance they will default (i.e., a greater chance that the issuing firm will not make all of the bond's payments). Thus if we looked at market prices for two-year zero-coupon bonds, we would find different interest rates for bonds issued by different firms, based on the perceived risk that each firm will default.

The United States government is generally regarded as having a lower risk of default than any other borrower, so U.S. government bonds are used as the standard to which all other bonds are compared. In fact, the interest rates on short-term Treasury bills are sometimes referred to as **risk-free rates**.

A bond's riskiness is indicated by its interest rate. If a company's bonds yield 8% and similar U.S. government bonds yield 5 ½%, the 2 ½% difference in yield is a risk premium. It is what investors demand to account for the difference in risk. This will be discussed in more detail in Module 8, when we examine the factors that determine market interest rates.

Except for short-term Treasury bills, which have no coupons, the United States government does not directly issue zero-coupon bonds. Investment bankers buy U.S. government coupon bonds and break them down into single-payment components called **Treasury STRIPS**. To create a two-year Treasury STRIP (zero-coupon) bond, they buy a large dollar amount of Treasury coupon bonds, and resell the coupon payment due in 2 years as a 2-year Treasury STRIP. The annual interest rate on the n-year Treasury STRIP is called the ***n*-year spot rate,** and the series of spot rates for various maturities is called the **yield curve.**

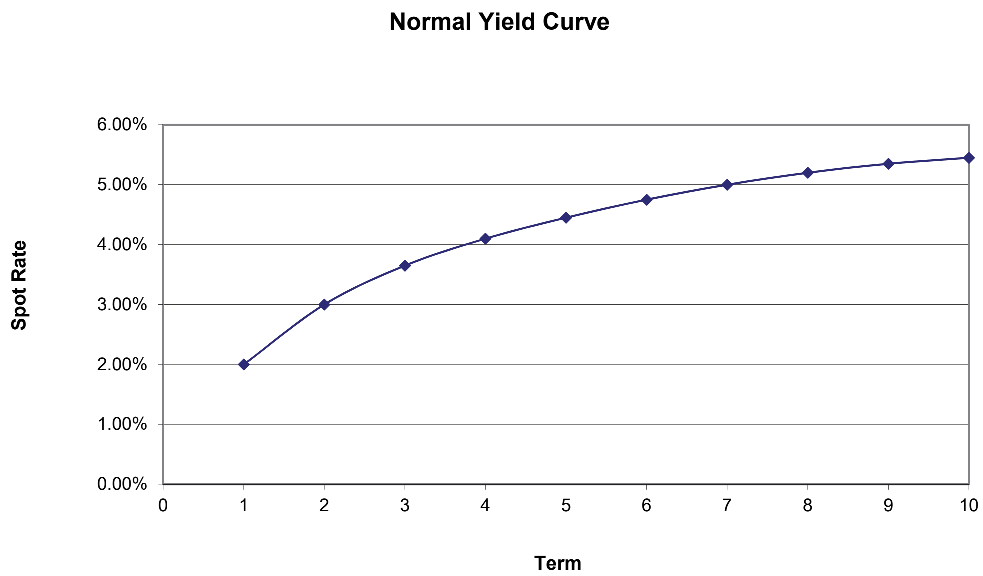
The following table contains a hypothetical set of spot rates for terms of 1 to 10 years.

Table (6.1) Yield Curve Example

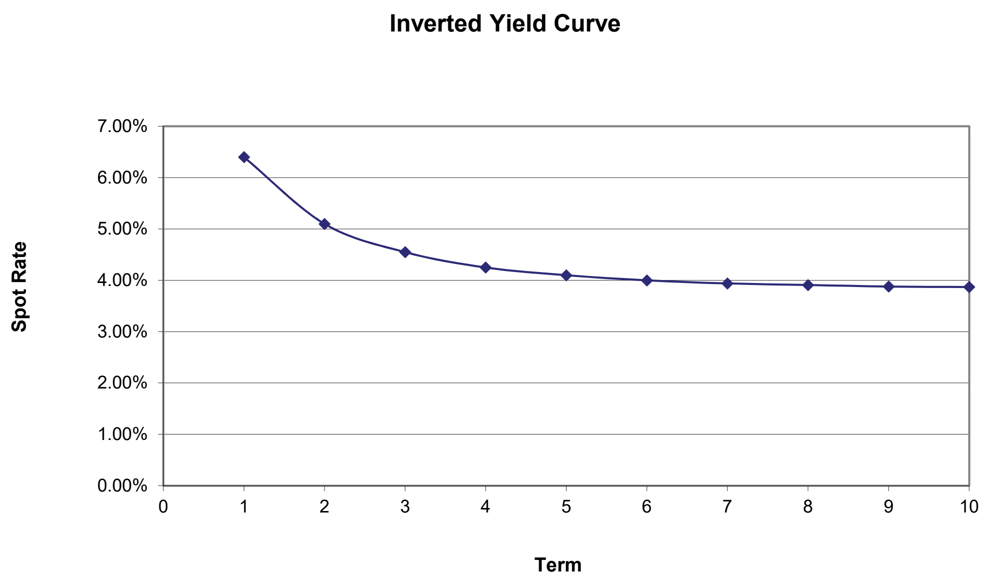
Term	Spot Rate
1	2.00%
2	3.00%
3	3.65%
4	4.10%
5	4.45%
6	4.75%
7	5.00%
8	5.20%
9	5.35%
10	5.45%

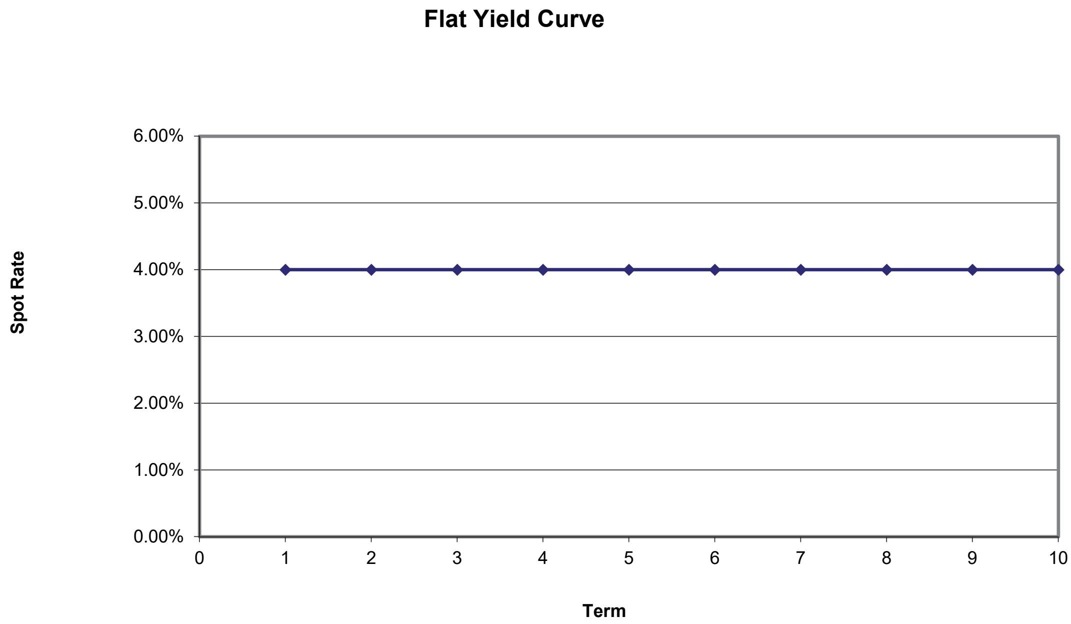
To see the actual current spot rates for zero-coupon Treasury bonds, consult the U.S. Treasury website: https://www.treasury.gov (search for "yield curve").

The yield curve shown on the following page is based on the above sample rates for terms of 1 to 10 years.

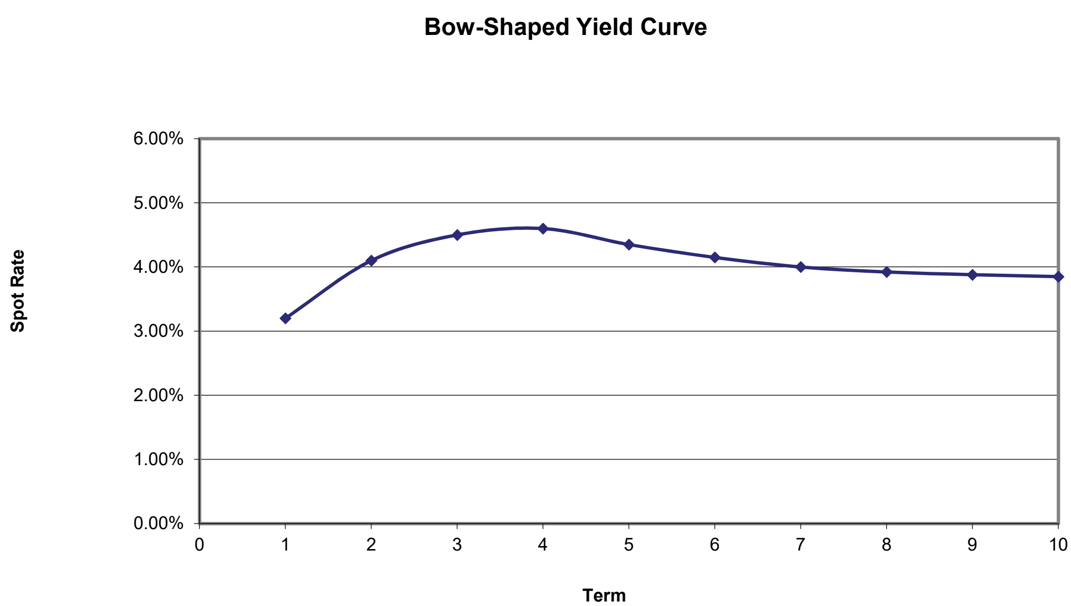


At most times, lenders demand higher rates of interest for longer-term loans, so the increasing yield curve above is referred to as a normal yield curve. In times when current rates are high but lenders anticipate that rates will drop in the future, this situation can reverse, resulting in an inverted yield curve or a flat yield curve, as shown in the following two graphs.





Another shape that occurs infrequently is a bow-shaped yield curve, where medium-term interest rates are higher than both the long- and short-term rates. (A yield curve can also be described as "bow-shaped" if the medium-term rates are *lower* than both the long-term and short-term rates.)



These various shapes of yield curves will be discussed in more detail in Module 8 when we consider the factors that affect the yield curve.

We will use the notation s_n for the n-year spot rate, expressed as an annual effective rate. Thus the n-year accumulation factor, $a(n)$, equals $(1+s_n)^n$. As explained above, s_n is also the annual effective yield for a zero-coupon bond maturing in n years.

Interest rates from the yield curve can be used to price a coupon bond, as shown in the following example.

Example (6.2)

A four-year annual-coupon 1,000 par bond has a coupon rate of 3%. Thus its payments are:

Year	1	2	3	4
Payment	30	30	30	1030

We can value this bond by calculating the present value of each payment at the appropriate spot rate from the yield curve, and summing the present values. Using the sample yield curve in table **(6.1)**, the price of this bond is:

$$P=\frac{30}{1.02}+\frac{30}{1.03^2}+\frac{30}{1.0365^3}+\frac{1030}{1.041^4}=961.70.$$

Now that we have found the bond's price using spot rates from the yield curve, we can find its yield to maturity, which is the IRR for the cash flow sequence:

$$-961.70 \quad 30 \quad 30 \quad 30 \quad 1030$$

This can be done using the BA II Plus.
Set N = 4, PV = -961.70, PMT = 30, FV = 1,000, and CPT I/Y = 4.0565.
The yield to maturity is 4.0565%.
This single interest rate of 4.0565% can be thought of as a complex weighted average of the 4 spot rates that apply to the bond's 4 cash flows. Note that the yield is closest to the 4-year spot rate of 4.10%, since by far the largest cash flow occurs at time 4.

Note: This calculation could also have been done using the BA II Plus's Cash Flow worksheet. However, since the cash flows follow a pattern that can be represented with the TVM worksheet's PV, PMT, and FV values, that is the simpler way to calculate the bond's yield. (The calculator's Bond worksheet could also have been used, but TVM is the simplest way to solve this problem.)

Exercise (6.3)

A three-year annual-coupon 1,000 par bond has a coupon rate of 3.2%. Use the yield curve in Table **(6.1)** to find the price P and then use this price to find the yield to maturity.

Answers: Price = 969.05 Yield to maturity = 4.0538%

Just as the spot rates in a yield curve can be used to calculate prices for coupon-paying bonds, it is also possible to use coupon bond prices to calculate the spot rates for a yield curve. In the case of zero-coupon bonds, of course, each bond's price and maturity value allows us to calculate the spot rate for that bond's term. But the next example demonstrates how we can also find the spot rates of the yield curve from the prices of coupon bonds. The process of calculating spot rates from coupon bond prices and yields is called **bootstrapping**. This process is illustrated by the following example.

Example (6.4)

Given the following information about annual-coupon bonds with terms of 1 to 5 years, we will calculate a spot rate yield curve for terms of 1 to 5 years.

Term	Coupon Rate	Price per 100 Face	Yield to Maturity
1	2.50%	100.00	2.500%
2	4.00%	101.76	3.079%
3	3.50%	100.54	3.308%
4	4.00%	101.81	3.517%
5	3.00%	97.19	3.625%

Note: The above table gives both the price and the yield for each bond, but it is not necessary to be given both price and yield. If we are given one, we can find the other.

The 1-year spot rate equals the yield to maturity of the 1-year bond. Since the bond makes all of its payments (a total of 102.50) at time 1, its yield can easily be found from its price (100.00) and its payment at maturity (102.50). The 1-year bond's yield and the 1-year spot rate are:

$$s_1 = \frac{102.50}{100} - 1 = 2.500\%$$

(Also, from the fact that the bond is selling at par, we could have concluded that its yield is equal to its coupon rate.)

The 2-year bond pays coupons of 4 at times 1 and 2. The coupon paid at time 1 has a present value (at time 0) of $\frac{4}{1.025}=3.9024$. So the coupon at time 1 accounts for 3.9024 of the 2-year bond's price. That means that the coupon at time 2 plus the maturity value account for the remainder of the price. So the coupon and maturity value at time 2 have a value of:

101.76 – 3.9024 = 97.8576.

We can now solve for the 2-year spot rate as follows:

$$\frac{104.00}{(1+s_2)^2}=97.8576 \quad \rightarrow \quad s_2=\left(\frac{104.00}{97.8576}\right)^{0.5}-1=3.0907\%$$

The 2-year spot rate is 3.0907%.

Similarly, we can use the price of the 3-year bond and the 1- and 2-year spot rates to calculate the 3-year spot rate:

$$\begin{aligned} PV(\text{pmt at time 3}) &= \text{Price} - PV(\text{pmts at times 1 and 2}) \\ &= 100.54-\left[\frac{3.50}{(1+s_1)}+\frac{3.50}{(1+s_2)^2}\right] \\ &= 100.54-\left[\frac{3.50}{1.025}+\frac{3.50}{1.030907^2}\right]=93.8321 \end{aligned}$$

$$93.8321=\frac{103.50}{(1+s_3)^3} \qquad s_3=\left(\frac{103.50}{93.8321}\right)^{1/3}-1=0.033228$$

The 3-year spot rate is 3.3228%.

Using the same method for the 4- and 5-year spot rates, we have:

$$101.81-\left[\frac{4.00}{1.025}+\frac{4.00}{1.030907^2}+\frac{4.00}{1.033228^3}\right]=90.5174$$

$$s_4=\left(\frac{104.00}{90.5174}\right)^{1/4}-1=0.035322$$

$$97.19-\left[\frac{3.00}{1.025}+\frac{3.00}{1.030907^2}+\frac{3.00}{1.033228^3}+\frac{3.00}{1.035322^4}\right]=86.1095$$

$$s_5=\left(\frac{103.00}{86.1095}\right)^{1/5}-1=0.036471$$

Thus, the 4-year and 5-year spot rates are 3.5322% and 3.6471%, respectively.

The following table shows the bond data *and* the spot rates:

Term	Coupon Rate	Price per 100 Face	Yield to Maturity	Spot Rate
1	2.50%	100.00	2.500%	2.500%
2	4.00%	101.76	3.079%	3.090%
3	3.50%	100.54	3.308%	3.323%
4	4.00%	101.81	3.517%	3.532%
5	3.00%	97.19	3.625%	3.647%

As we noted in Example **(6.2)**, each bond yield in the table is slightly lower than the spot rate of the same term. Here we see that the difference between the n-year coupon bond yield and the n-year spot rate increases as n increases. This happens because the bond's maturity value becomes a smaller and smaller proportion of the bond's total value as n increases, and therefore the n-year spot rate has less and less effect on the bond's overall yield.. (The maturity value becomes a less significant part of the bond's total price, both because there are more coupons on a longer-term bond and also because the maturity value is discounted for more years).

Exercise (6.5)

Suppose that bond prices and coupon rates are as shown in Example **(6.4)** except that the price for the 1-year bond with 2.5% coupon is 100.50. Use the bootstrapping method of Example **(6.4)** to find the spot rates for terms of 1 to 5 years.

Answers: 1.9900% 3.1010% 3.3289% 3.5373% 3.6503%

Section 6.2

Forward Rates

A **forward rate** is an interest rate agreed upon today (at time 0) that will be applied to a loan that is going to occur over a specified future period. For example, a borrower and lender might arrange a loan with a term of n years beginning at a time m years from today. In that case, the loan interest rate for that period (from time m to time m+n) would be described as "the ***m*-year-forward, *n*-year rate**." We will use the expression $i_{m,m+n}$ to represent this rate.

Using the same terminology and notation, we would use $i_{n,n+1}$ to represent the "n-year-forward, 1-year rate." Single-period forward rates (i.e., rates that apply to a 1-year-long future period) are extremely common, so the usual practice is to say simply "the ***n*-year forward rate**." In that case, it is understood that the rate applies to a future 1-year period beginning at time n.

Note: Interest theory texts use a variety of different notations for forward rates. As an example, the n-year forward rate could be written as $i_0(n,n+1)$, *where the subscript "0" indicates that the rate is determined as of time 0, based on the prevailing interest rates at that time. For our purposes, spot rates and forward rates will generally be determined as of time 0, so the 0 subscript will not be used. Occasionally, a problem may require the use of a spot rate or forward rate as of a specified time other than time 0 (e.g., time t). In that case, it should be understood that the subscripts represent times measured from time t. (For example, if we calculate* $i_{n,n+1}$ *(the n-year forward rate) as of time t, then it is an interest rate for the period beginning at t+n, and ending at t+n+1.)*

The values of the spot rates in a yield curve *imply* certain values for forward rates. These are called the **implied forward rates**. For example, we can calculate the implied (n-1)-year forward rate, $i_{n-1,n}$, if we are given the spot rates s_{n-1} and s_n. The relationship among these values is:

$$(1+s_n)^n = (1+s_{n-1})^{n-1} \cdot (1+i_{n-1,n})$$

The left and right members of this equation represent two ways to make an n-year investment. The left side is the n-year accumulation function, $a(n)$, based on an investment at the n-year spot rate, s_n. The right side is the n-year accumulation function based on an (n-1)-year investment at the (n-1)-year spot rate, followed by a 1-year investment (arranged at time 0) at the (n-1)-year forward rate. These two values must be equal. If they were not equal, all investors would choose to invest in the way that produced the larger value. This would increase the demand for that investment, driving up its price and resulting in a lower rate of return, until the returns for the two investment methods became equal. This is an example of the **law of one price**, which will be applied in the following examples relating spot rates and forward rates.

Example (6.6)

Using the spot rates from Table **(6.1)**, calculate the one- and two-year forward rates.

<u>One-year forward rate</u>: We are given $s_1 = 0.02$ and $s_2 = 0.03$. There are two ways to calculate the accumulation factor for a two-year investment:

a) If we assume 1 is invested for 2 years at time 0, the 2-year spot rate, $s_2 = 0.03$, applies. The 2-year accumulation factor is $1.03^2 = 1.0609$.

b) If at time 0 we invest 1 for one year at the spot rate of $s_1 = 0.02$, and also agree to reinvest the first year accumulation at the 1-year forward rate $i_{1,2}$, the 2-year accumulation factor is $1.02 \cdot (1 + i_{1,2})$.

By the law of one price, these two accumulation factors must be equal, so we have:

$$1.02(1 + i_{1,2}) = 1.03^2 \rightarrow (1 + i_{1,2}) = \frac{1.03^2}{1.02} = 1.0401 \rightarrow i_{1,2} = 0.0401$$

<u>Two-year forward rate</u>: We are given $s_2 = 0.03$ and $s_3 = 0.035$. Two ways to find the accumulation factor for a three-year investment are:

a) Invest for the entire 3 years at the 3-year spot rate $s_3 = 0.035$. The accumulation factor is $a(3) = 1.035^3$.

b) Invest for 2 years at the 2-year spot rate of $s_2 = 0.03$, and also agree to reinvest the 2-year accumulation at the 2-year forward rate $i_{2,3}$. The resulting accumulation factor is $a(3) = 1.03^2 \cdot (1 + i_{2,3})$.

Equating these two expressions for $a(3)$, we have:

$$1.03^2 (1 + i_{2,3}) = 1.035^3 \rightarrow (1 + i_{2,3}) = \frac{1.035^3}{1.03^2} = 1.0451 \rightarrow i_{2,3} = 0.0451$$

From these examples, the general pattern should be clear:

(6.7)

$$1 + i_{n-1,n} = \frac{(1 + s_n)^n}{(1 + s_{n-1})^{n-1}} \quad \text{or, equivalently:}$$

$$(1 + s_n)^n = (1 + s_{n-1})^{n-1} (1 + i_{n-1,n})$$

Note: Our definition of forward rates assumes that they will be expressed as annual effective rates. However, following the same principles, forward rates could be expressed as (for example) quarterly effective rates. We will see examples of this when we study interest rate swaps in Module 9.

Exam problems usually involve single-period forward rates, as in Formula **(6.7)**. However, for completeness, we include here the generalized formulas that relate spot rates and forward rates:

(6.8)

$$(1+i_{m,m+n})^n = \frac{(1+s_{m+n})^{m+n}}{(1+s_m)^m} \quad \text{or, equivalently:}$$

$$(1+s_{m+n})^{m+n} = (1+s_m)^m \cdot (1+i_{m,m+n})^n$$

Exercise (6.9)

Find the 3-year forward rate $i_{3,4}$ for the yield curve of Table **(6.1)**.

Answer: 0.0551

Note that it is also possible to recover the spot rates if we are given the sequence of forward rates. Recognizing that the 0-year forward rate $i_{0,1}$ is equal to s_1 (i.e., the interest rate for the period from time 0 to time 1 is the 1-year spot rate), we have:

$$(1+i_{0,1})(1+i_{1,2}) = (1+s_2)^2$$

$$(1+i_{0,1})(1+i_{1,2})(1+i_{2,3}) = (1+s_3)^3$$

And, in general:

(6.10)

$$(1+i_{0,1})(1+i_{1,2})\dots(1+i_{n-1,n}) = (1+s_n)^n$$

In words, an *n*-year accumulation at the first *n* single-period forward rates is equal to an *n*-year accumulation at the *n*-year spot rate.

When a problem involves converting between spot rates and forward rates, it is sometimes easiest to calculate the accumulation factors, $a(n)$, from the given values, and then use those factors to find the required values. This method is applied in Example **(6.12)** on the next page using the following formulas, which relate accumulation factors to spot rates and forward rates:

(6.11)

For spot rates:

$$a(n) = (1+s_n)^n \quad \text{or} \quad s_n = [a(n)]^{1/n} - 1$$

For forward rates:

$$a(n) = a(n-1)\cdot(1+i_{n-1,n}) \quad \text{or} \quad i_{n-1,n} = \frac{a(n)}{a(n-1)} - 1$$

Example (6.12)

Given the following forward rates, calculate the spot rates for terms of 1 to 5 years.

n	$i_{n-1,n}$
1	5.0%
2	6.2%
3	6.8%
4	7.3%
5	7.7%

As mentioned above, an efficient way to find the spot rates (especially when the calculations are being done in a computer spreadsheet) is to calculate the accumulation factors, and then use the accumulation factors to find the spot rates. To do this, we will start with the formula for computing successive accumulation factors from the forward rates:

$$a(n) = a(n-1) \cdot (1 + i_{n-1,n})$$

Using this formula (and the fact that $a(0)$ is 1), the values of $a(n)$ have been entered in the table below. From the values for $a(n)$, the spot rates have been calculated using:

$$s_n = \left[a(n)\right]^{1/n} - 1$$

n	$i_{n-1,n}$	$a(n)$	s_n
1	5.0%	1.0500	5.000%
2	6.2%	1.1151	5.598%
3	6.8%	1.1909	5.997%
4	7.3%	1.2779	6.322%
5	7.7%	1.3763	6.596%

Exercise (6.13)

Using the following set of spot rates, which happen to have the same numerical values as the forward rates of Example **(6.12)**, calculate the corresponding forward rates, $i_{n-1,n}$, for $n = 1$ to 5.

n	s_n
1	5.0%
2	6.2%
3	6.8%
4	7.3%
5	7.7%

Answers: 5.000% 7.414% 8.010% 8.814% 9.315%

Section 6.3

Formula Sheet

Definitions:

The annual effective interest rate for an n-year zero-coupon bond is called the ***n*-year spot rate** (s_n). The series of spot rates for all possible terms is called the **yield curve.**

To value a bond, calculate the present value of each of its payments using the appropriate spot rates, and sum the present values.

The yield curve consists of the yield rates for zero-coupon bonds of various terms (i.e., spot rates). The interest rates of the yield curve can also be calculated from the prices (or from the yields) of *coupon* bonds by the process of **bootstrapping** (see Example **(6.4)**).

The ***m*-year-forward, *n*-year rate**, $i_{m,m+n}$ is the annual effective interest rate, determined at time 0, that applies to a loan for the n-year period from time m to time $m+n$.

The term ***n*-year forward rate** refers to $i_{n,n+1}$, the interest rate (determined at time 0) for a 1-year loan whose term begins at time n.

Formulas:

$$a(n) = (1+s_n)^n \qquad s_n = \left[a(n)\right]^{1/n} - 1$$

$$a(n) = a(n-1)\cdot(1+i_{n-1,n}) \qquad i_{n-1,n} = \frac{a(n)}{a(n-1)} - 1$$

$$(1+i_{0,1})(1+i_{1,2})\ldots(1+i_{n-1,n}) = (1+s_n)^n = a(n)$$

$$1+i_{n-1,n} = \frac{(1+s_n)^n}{(1+s_{n-1})^{n-1}} \qquad (1+s_n)^n = (1+s_{n-1})^{n-1}(1+i_{n-1,n})$$

$$(1+i_{m,m+n})^n = \frac{(1+s_{m+n})^{m+n}}{(1+s_m)^m} \qquad (1+s_{m+n})^{m+n} = (1+s_m)^m(1+i_{m,m+n})^n$$

Section 6.4
Basic Review Problems

Problems 1. and 2. use rates from the following yield curve table:

Term	Spot Rate
1	5.00%
2	4.50%
3	4.00%
4	4.00%
5	4.00%

1. A 3-year annual-coupon 1,000 par bond has a coupon rate of 4%. Use the yield curve above to find the bond's price P, and then use that price to find its yield to maturity.

2. Calculate the 1-year and 2-year forward rates.

3. Based on the spot rates, forward rates, and accumulation factors shown in the following table, fill in the missing values.
(All of the interest rates in the table are annual effective rates.)

n	Spot Rate, s_n	Forward Rate, $i_{n-1,n}$	Accumulation Factor, $a(n)$
1	3.40%	3.40%	1.0340
2		4.40%	1.0795
3	4.30%		
4			1.1994
5			1.2732
6		6.45%	

Section 6.5

Basic Review Problem Solutions

1. The bond's payments are:

Year	1	2	3
Payment	40	40	1040

To value the bond, take the present value of each payment at the appropriate spot rate and sum the present values. Using the given yield curve the price P is:

$$P = \frac{40}{1.05} + \frac{40}{1.045^2} + \frac{1,040}{1.04^3} = 999.28\,.$$

We can find the yield to maturity by solving for the yield rate (IRR) of the cash flow sequence: −999.28, 40, 40, 1040.

On the BA II Plus, set PV=-999.28, PMT = 40, N=3, FV=1,000, and CPT I/Y=4.026. The yield to maturity is 4.026%.

2. $$1 + i_{1,2} = \frac{(1+s_2)^2}{(1+s_1)^1} = \frac{1.045^2}{1.05} = 1.04002 \rightarrow i_{1,2} = 4.002\%$$

$$1 + i_{2,3} = \frac{(1+s_3)^3}{(1+s_2)^2} = \frac{1.04^3}{1.045^2} = 1.03007 \rightarrow i_{2,3} = 3.007\%$$

Note: The given yield curve is "inverted" (i.e., the spot rates decrease as the term increases. As a result, the forward rates are lower than the spot rates used to calculate them. With a normal yield curve, a forward rate would be larger than the corresponding spot rate.

3. Working a problem like this is like solving a puzzle, where each square that we fill in gives us information to find the value for another square. On an exam, the problem would likely ask for a particular value (such as $i_{3,4}$) that would require calculating a number of other values first.

We can begin by filling in some of the values for spot rates and accumulation factors, since if we know one of these values for a given term, we can calculate the other. Thus we have:

$$s_2 = \left(a(2)\right)^{1/2} - 1 = 1.0795^{1/2} - 1 = 0.03899$$

$$a(3) = \left(1 + s_3\right)^3 = 1.043^3 = 1.13463$$

$$s_4 = \left(a(4)\right)^{1/4} - 1 = 1.1994^{1/4} - 1 = 0.04650$$

$$s_5 = \left(a(5)\right)^{1/5} - 1 = 1.2732^{1/5} - 1 = 0.04949$$

Next, if we know two adjacent accumulation functions, we can calculate the forward rate that "connects" them. This gives us the missing forward rates:

$$i_{2,3} = \frac{a(3)}{a(2)} - 1 = \frac{1.13463}{1.0795} - 1 = 0.05107$$

$$i_{3,4} = \frac{a(4)}{a(3)} - 1 = \frac{1.1994}{1.13463} - 1 = 0.05709$$

$$i_{4,5} = \frac{a(5)}{a(4)} - 1 = \frac{1.2732}{1.1994} - 1 = 0.06153$$

Then we can calculate $a(6)$ from the values of $a(5)$ and $i_{5,6}$:

$$a(6) = a(5) \cdot \left(1 + i_{5,6}\right) = 1.2732 \cdot 1.0645 = 1.35532$$

Finally, s_6 can be calculated from the value of $a(6)$:

$$s_6 = \left(a(6)\right)^{1/6} - 1 = 1.35532^{1/6} - 1 = 0.05198$$

Here is the completed table with the computed values underlined:

n	**Spot Rate, s_n**	**Forward Rate, $i_{n-1,n}$**	**Accumulation Factor, $a(n)$**
1	3.40%	3.40%	1.0340
2	<u>3.90%</u>	4.40%	1.0795
3	4.30%	<u>5.11%</u>	<u>1.1346</u>
4	<u>4.65%</u>	<u>5.71%</u>	1.1994
5	<u>4.95%</u>	<u>6.15%</u>	1.2732
6	<u>5.20%</u>	6.45%	<u>1.3553</u>

Section 6.6

Sample Exam Problems

1. (Fall 05 Sample Problems #33)
You are given the following information with respect to a bond:
par amount: 1000
term to maturity: 3 years
annual coupon rate: 6% payable annually

Term	Annual Spot Interest Rate
1	7%
2	8%
3	9%

Calculate the value of the bond.

(A) 906 (B) 926 (C) 930 (D) 950 (E) 1000

2. (Fall 05 Sample Problems #34)
You are given the following information with respect to a bond:
par amount: 1000
term to maturity: 3 years
annual coupon rate: 6% payable annually

Term	Annual Spot Interest Rate
1	7%
2	8%
3	9%

Calculate the annual effective yield rate for the bond if the bond is sold at a price equal to its value.

(A) 8.1% (B) 8.3% (C) 8.5% (D) 8.7% (E) 8.9%

3. (May 05 #10)
Yield rates to maturity for zero coupon bonds are currently quoted at 8.5% for one-year maturity, 9.5% for two-year maturity, and 10.5% for three-year maturity. Let i be the one-year forward rate for year two implied by current yields of these bonds.

Calculate i.

(A) 8.5% (B) 9.5% (C) 10.5% (D) 11.5% (E) 12.5%

4. (Nov 05 #6)
Consider a yield curve defined by the following equation:

$$i_k = 0.09 + 0.002k - 0.001k^2$$

where i_k is the annual effective rate of return for zero coupon bonds with maturity of k years.

Let j be the one-year effective rate during year 5 that is implied by this yield curve.

Calculate j.

(A) 4.7% (B) 5.8% (C) 6.6% (D) 7.5% (E) 8.2%

5. (Nov 05 #15)
You are given the following term structure of spot interest rates:

Term (in years)	Spot interest rate
1	5.00%
2	5.75%
3	6.25%
4	6.50%

A three-year annuity-immediate will be issued a year from now with annual payments of 5000. Using the forward rates, calculate the present value of this annuity a year from now.

(A) 13,094 (B) 13,153 (C) 13,296 (D) 13,321 (E) 13,401

6. (Nov 05 #19)
Which of the following statements about zero-coupon bonds are true?

I. Zero-coupon bonds may be created by separating the coupon payments and redemption values from bonds and selling each of them separately.

II. The yield rates on stripped Treasuries at any point in time provide an immediate reading of the risk-free yield curve.

III. The interest rates on the risk-free yield curve are called forward rates.

(A) I only
(B) II only
(C) III only
(D) I, II, and III
(E) The correct answer is not given by (A), (B), (C), or (D).

Section 6.7

Sample Exam Problem Solutions

1.
We will use the spot rate for a term of n years to find the present value of a payment at time n. The annual coupon for this bond is 60, and its final payment is 1,060. The present value of the bond's payments is:

$$\frac{60}{1.07}+\frac{60}{1.08^2}+\frac{1{,}060}{1.09^3}=926.03$$

Answer B

2.
We have already seen (in Problem 1.) that the value of this bond is 926.03. Thus we need to find the yield i for an investor who buys it at this price.

We can solve for i using the financial calculator. On the BA II Plus, enter N=3, PV=-926.06, PMT=60, and FV=1,000. Then CPT I/Y = 8.92.

Answer E

3.
We are given $s_2=0.095$ and $s_1=0.085$. We are asked to find $i_{1,2}$.

$$1+i_{1,2}=\frac{(1+s_2)^2}{(1+s_1)^1}=\frac{1.095^2}{1.085}=1.1051\rightarrow i_{1,2}=0.1051$$

Answer C

4.
Year 5 extends from time n=4 to n=5. We are looking for the implied forward rate $i_{4,5}$. This is defined by $(1+s_5)^5=(1+s_4)^4(1+i_{4,5})$.

We are given that $s_k=0.09+0.002k-0.001k^2$. (The problem uses the notation i_k for spot rates where we are using s_k). Thus:

$$s_4=0.09+0.002(4)-0.001(4^2)=0.082$$

$$s_5=0.09+0.002(5)-0.001(5^2)=0.075$$

$$(1.075)^5=(1.082)^4(1+i_{4,5})\quad\rightarrow\quad i_{4,5}=0.047$$

Answer A

5.

The problem tells us to "use forward rates" to calculate a present value as of one year from now. Basically, it asks us to calculate $i_{1,1}$, $i_{1,2}$, and $i_{1,3}$ and treat those as the spot rates that will be in effect at time 1. This is a valid approach to solving the problem. However, the present value of the annuity as of time 1 (based on the spot rates we are given) will be the same regardless of whether we use forward rates or apply a different method. To demonstrate this, we will first solve the problem using forward rates, and then apply a different method that you may find more efficient.

Let j_n be the implied n-year spot rate as of time 1, based on the given spot rates at time 0. *(Note that $j_n = i_{1,n}$, so this solution is using forward rates.)*

$$1 + j_1 = 1 + i_{1,2} = \frac{1.0575^2}{1.05} = 1.06505$$

$$\left(1 + j_2\right)^2 = \left(1 + i_{1,3}\right)^2 = \left(\frac{1.0625^3}{1.05}\right) = 1.14235$$

$$\left(1 + j_3\right)^3 = \left(1 + i_{1,4}\right)^3 = \left(\frac{1.065^4}{1.05}\right) = 1.22521$$

The implied present value of the annuity at time 1 is:

$$PV_1 = 5{,}000\left[\frac{1}{1 + j_1} + \frac{1}{\left(1 + j_2\right)^2} + \frac{1}{\left(1 + j_3\right)^3}\right]$$
$$= 5{,}000\left[\frac{1}{1.06505} + \frac{1}{1.14235} + \frac{1}{1.22521}\right] = 13{,}152.50$$

The second method, which doesn't require calculating forward rates, is to find the annuity's value at time 0 and accumulate that value one year to time 1:

$$PV_1 = PV_0 \cdot \left(1 + s_1\right) = 5{,}000 \cdot \left(1.0575^{-2} + 1.0625^{-3} + 1.0650^{-4}\right) \cdot 1.05 = 13{,}152.50$$

Answer B

6.

I is true. Treasury STRIPS are created in this fashion.
II is true.
III is false. The yield curve interest rates are called spot rates.

Note: Although "yield curve" generally means the <u>spot rate</u> yield curve, it is possible to construct a forward rate yield curve, or (as we will discuss in Module 8) a par bond yield curve. However, this exam problem, which focuses on zero-coupon bond yields, is clearly talking about the spot rate yield curve.

Answer E

Section 6.8

Supplemental Exercises

For Problems 1, 2, and 3 use the following yield curve:

Term	Spot Rate
1	5%
2	6%
3	7%
4	8%

1. Calculate the price for a 4-year 1,000 par bond with 5% annual coupons.

2. Based on the price calculated for the bond in Problem 1, what is its annual effective yield to maturity?

3. Based on the above yield curve, find the 3-year forward rate.

For Problems 4 and 5, the spot rates of the yield curve are based on the following formula: $s_k = 0.085 + 0.003k - 0.0015k^2$

4. Calculate the price of a 3-year 1,000 par bond with 6% annual coupons.

5. Find the 3-year forward rate implied by this yield curve.

6. You are given the following n-year forward rates:

n	Forward Rate $\left(i_{n,n+1}\right)$
0	3.0%
1	4.4%
2	4.8%
3	5.6%

   Find s_4.

7. Using the spot rates implied by the forward rates in Problem 6, calculate the price for a 4-year 1,000 par bond with 5% annual coupons.

8. Find the yield to maturity of the bond in Problem 7.

Section 6.9

Supplemental Exercise Solutions

1. The price of the bond is
   $50/1.05 + 50/1.06^2 + 50/1.07^3 + 1{,}050/1.08^4 = 904.72$

2. To find the yield using the BA II Plus calculator, set
   N = 4, PMT = 50, PV = −904.72 and FV = 1,000.
   Then CPT I/Y = 7.868%

3. $1 + i_{3,4} = (1 + s_4)^4/(1 + s_3)^3 = 1.08^4/1.07^3 = 1.111$
   $i_{3,4} = 11.1\%$

4. First calculate the spot rates for terms of 1, 2 and 3 years.
   $s_1 = 0.085 + 0.003 – 0.0015 = 0.0865$
   $s_2 = 0.085 + 0.006 – 0.0060 = 0.0850$
   $s_3 = 0.085 + 0.009 – 0.0135 = 0.0805$

   The price of the bond is
   $60/1.0865 + 60/1.085^2 + 1{,}060/1.0805^3 = 946.49$

5. For this problem we also need s_4.
   $s_4 = 0.085 + 0.012 – 0.024 = 0.073$

   Then:
   $1 + i_{3,4} = 1.073^4/1.0805^3 = 1.0508$
   $i_{3.4} = 5.08\%$

6. $(1 + s_4)^4 = (1 + i_{0,1})(1 + i_{1,2})(1 + i_{2,3})(1 + i_{3,4})$
   $= (1.03)(1.044)(1.048)(1.056) = 1.1900$
   $1 + s_4 = 1.1900^{1/4} = 1.0445 \Rightarrow s_4 = 4.45\%$

7. The price of the bond is given by
   $50/(1 + s_1) + 50/(1 + s_2)^2 + 50/(1 + s_3)^3 + 1{,}050/(1 + s_4)^4$
   From Problem 6 we know that $(1 + s_4)^4 = 1.1900$

   $1 + s_1 = 1.03$, $(1 + s_2)^2 = (1.03)(1.044) = 1.0753$
   $(1 + s_3)^3 = (1.03)(1.044)(1.048) = 1.1269$

   $P = 50/1.03 + 50/1.0753 + 50/1.1269 + 1{,}050/1.1900 = 1{,}021.77$

8. To find the yield to maturity using the BA II Plus calculator,
   set N = 4, PMT = 50, PV = -1,021.77 and FV = 1,000.
   Then CPT I/Y = 4.395.
   The yield to maturity is 4.395%.

"Midterm" #2

Questions

1. A 30-year bond has a face amount (and redemption value) of 10,000. It pays semi-annual coupons at an annual coupon rate of 4.8%. This bond is purchased on its 6th anniversary at a price that results in a 6% yield to maturity (a nominal rate, convertible semi-annually).

What is the amount of principal in the coupon payment received on the bond's 9th anniversary?

A) -33.30 B) -25.60 C) -16.83 D) 25.60 E) 33.30

2. The price of a 5-year 1,000-face bond with 7% annual coupons is calculated using the following table of spot rates. Based on the calculated price, what is the bond's (level) yield to maturity?

Term(years)	Yield
1	4.75%
2	5.00%
3	5.25%
4	5.50%
5	6.00%

A) 5.1% B) 5.3% C) 5.5% D) 5.7% E) 5.9%

3. A bond will mature for 10,000 on March 1, 2028. It pays semi-annual coupons at an 8% (annual) rate.

On November 12, 2013, the bond is trading at a price such that its yield to maturity is 7.2% (convertible semi-annually). What is the bond's market price as of November 12, 2013?
(Use "actual" days (365-days/year), and use "simple interest" to calculate accrued interest.)

A) 10,698 B) 10,705 C) 10,713 D) 10,727 E) 10,864

4. An association had a balance of 200 on Dec. 31 of Year X-1. Then in Year X, it had quarterly deposits of 25 on the last day of each quarter. The association also had withdrawals of 30 on Feb. 28 and 60 on June 30, and ended Year X with a balance of 250 on Dec 31. Calculate the association's dollar-weighted rate of return for Year X.

A) 18% B) 19% C) 20% D) 21% E) 22%

5. A 30-year 10,000-face bond pays 3.5% annual coupons and matures at par. It is purchased at a price such that it will yield 5% annually for the first 15 years and 8% annually thereafter. Calculate the price of the bond.

A) 4,978 B) 6,090 C) 6,371 D) 6,590 E) 7,858

6. The prices per 100 of maturity value for zero-coupon Treasury bonds are 98.20 for 1-year bonds, 95.37 for 2-year bonds, 92.05 for 3-year bonds, and 88.68 for 4-year bonds.

Let $i_{1,2}$ be the one-year-forward, one-year rate for Treasuries, i.e., the rate for the period from t=1 to t=2, and let $i_{2,4}$ be the two-year-forward, two-year rate, i.e., the annual effective rate for the period from t=2 to t=4.

What is the value of $i_{1,2} + i_{2,4}$?

A) 0.065 B) 0.066 C) 0.067 D) 0.068 E) 0.069

7. A proposed project requires an initial investment of 5 million and a second investment of 3 million at the beginning of year 3. It will generate a return of 2 million at the end of three years and returns of 4 million at the end of years 5 and 6.

What is the internal rate of return for the project?

A) 5.2% B) 5.3% C) 5.4% D) 5.5% E) 5.6%

8. A bond has a par value and redemption value of 1,000. It has an annual coupon rate of 12% payable semi-annually, and is priced to yield 10% convertible semi-annually. The bond's term is n years. If the term of the bond were doubled, its price would increase by 50. Calculate n.

A) 5 B) 7 C) 10 D) 12 E) 14

9. Suppose a yield curve for spot rates is given by the following equation: $s_t = 0.08 - 0.001t + 0.002t^2$. What would be the forward rate $(i_{4,7})$ for a loan originating at time t=4, with a term of 3 years (expressed as an annual effective rate)?

A) 7% B) 11% C) 17% D) 21% E) 26%

10. A 15,000 par value 12-year bond with 6% annual coupons is bought at a discount to yield a 7.5% annual effective rate. What is the interest portion of the 7^{th} coupon?

A) 1,001 B) 1,012 C) 1,046 D) 1,166 E) 1,175

11. Investments A and B have equal rates of return (IRRs).

Investment A requires an initial investment of 4,000 at t=0. The return consists of payments of 500 at the end of each year for 9 years (at t=1 through t=9), plus a payment of 2,000 at t=10.

Investment B requires the investor to deposit 1,000 at t=0. The return consists of 8 annual payments of amount X, the first payment occurring at t=1.

What is the value of X?

A) 250 B) 252 C) 255 D) 258 E) 260

Solutions

1. The principal in a coupon payment can be calculated by the formula:

$$PRin_k = F(r-i)v^{n-k+1} = 10{,}000 \cdot (0.024 - 0.030) \cdot 1.03^{-(60-18+1)} = -16.83$$

Note: The purchase date (6th anniversary) does not affect the amount of principal and interest in each coupon payment. We need only the face amount (F), coupon rate (r), yield (i), term (n), and the number of the coupon payment (k).

Answer:* *C

2. Calculate the price of a 5-year 1,000-face bond with 7% annual coupons.

$$P = \frac{70}{1.0475} + \frac{70}{1.05^2} + \frac{70}{1.0525^3} + \frac{70}{1.055^4} + \frac{1{,}070}{1.06^5} = 1{,}046.4280$$

Solve for i:

$$1{,}046.4280 = \frac{70}{(1+i)} + \frac{70}{(1+i)^2} + \frac{70}{(1+i)^3} + \frac{70}{(1+i)^4} + \frac{1070}{(1+i)^5}$$

This can be done with the BA II Plus, using the TVM worksheet, or the Cash Flow worksheet (and IRR function), or the Bond worksheet.
i = 5.9008%

Answer:* *E

3. This problem can be solved using formulas or the Bond worksheet.

By formulas:
Calculate the price as of the previous coupon date (Sep 1, 2013).

$$Fr \cdot a_{\overline{29}|} + F \cdot v^{29} = 10{,}000 \cdot 0.04 \cdot \frac{1-1.036^{-29}}{0.036} + 10{,}000 \cdot 1.036^{-29} = 10{,}712.71$$

Calculate the fraction of a coupon period from Sep 1 to Nov 12:

$$\frac{\text{days from 9/1/13 to 11/12/13}}{\text{days from 9/1/13 to 3/1/14}} = \frac{72}{181} = 0.39779$$

Calculate the total price on the settlement date (Nov. 12, 2013):

$$10{,}712.71 \cdot 1.036^{0.39779} = 10{,}864.49$$

Calculate the accrued interest (accrued coupon) as of Nov 12, 2013:

$$10{,}000 \cdot 0.04 \cdot 0.39779 = 159.12$$

Subtract accrued interest from the total price to find the market price:

$$10{,}864.49 - 159.12 = 10{,}705.37$$

Using the Bond worksheet:
Settlement Date (SDT) = 11-12-2013
Annual Coupon Payments (CPN) = 800
Redemption Date (RDT) = 3-01-2028
Redemption Value (RV) = 10,000
Actual Days (365-day year): ACT
Coupons per Year: 2/Y
Yield (YLD) = 7.2%
CPT PRI = 10,705.37

Answer: B

4.

$$i = \frac{250 - 200 - (4(25) - 90)}{200 - 30(1-2/12) + 25(1-3/12) - 35(1-6/12) + 25(1-9/12)}$$
$$= 0.2191781$$

Answer: E

5. We need to find the present value of the coupons plus the present value of the maturity value, reflecting a rate change during the term of the bond.

$$P = 350a_{\overline{15}|0.05} + v_{0.05}^{15} 350a_{\overline{15}|0.08} + v_{0.05}^{15} v_{0.08}^{15}(10{,}000)$$
$$= 3{,}632.8803 + 1{,}441.0395 + 1{,}516.3665 = 6{,}590.2863$$

This can be done efficiently on the BA II Plus's TVM worksheet as follows:
N=15, I/Y=8, PMT=350, FV=10,000. CPT PV=-6,148.23
This number (-6,148.23) is the present value at t=15 of all payments *after* t=15. Then press +/- and FV. (This makes 6,148 a cash flow at the end of the first 15 years.)
Change I/Y to 5 and CPT PV=6,590.29.

***Answer:* D**

6. From the prices per 100 for zero-coupon bonds, we can calculate the spot rates needed for this problem, which are s_1, s_2, and s_4.

1-year zero-coupon bond: $s_1 = 0.9820^{-1} - 1 = 0.01833$
2-year zero-coupon bond: $s_2 = 0.9537^{-1/2} - 1 = 0.02399$
4-year zero-coupon bond: $s_4 = 0.8868^{-1/4} - 1 = 0.03049$

Then we calculate the required forward rates:

$$i_{1,2} = \frac{(1+s_2)^2}{1+s_1} - 1 = \frac{1.02399^2}{1.01833} - 1 = 0.02967$$

$$i_{2,4} = \left[\frac{(1+s_4)^4}{(1+s_2)^2}\right]^{\frac{1}{2}} - 1 = \left[\frac{1.03049^4}{1.02399^2}\right]^{\frac{1}{2}} - 1 = 0.03703$$

$$i_{1,2} + i_{2,4} = 0.02967 + 0.03703 = 0.0667$$

Or, using the relationship between zero-coupon bond prices and values of the accumulation function $(a(t))$, you can calculate the answer as follows:

$$i_{1,2} = \frac{0.9820}{0.9537} - 1 = 0.02967 \qquad i_{2,4} = \left(\frac{0.9537}{0.8868}\right)^{\frac{1}{2}} - 1 = 0.03703$$

$$i_{1,2} + i_{2,4} = 0.02967 + 0.03703 = 0.0667$$

***Answer:* C**

7. First, note that the 3 million investment at the beginning of year 3, actually happens at t = 2 (which is the *beginning* of the 3rd year). The cash flows (in millions) are CF_0 = -5, C01 = 0, C02 = -3, C03 = 2, C04 = 0, C05 = 4, and C06 = 4 (or enter F05 = 2 and don't enter C06). Then press IRR and CPT to get an internal rate of return of 5.40073%

***Answer:* C**

8. For the bond of n years ($2n$ coupons), let the price be $P1$. Then for the bond of $2n$ years ($4n$ coupons), the price is $P1$ +50, and we have:

$$P1 = 60a_{\overline{2n}|0.05} + 1{,}000v^{2n}$$
$$P1 + 50 = 60a_{\overline{4n}|0.05} + 1{,}000v^{4n}$$

Substitute the first equation into the second equation and solve for n.

$$60a_{\overline{2n}|0.05} + 1{,}000v^{2n} + 50 = 60a_{\overline{4n}|0.05} + 1{,}000v^{4n}$$
$$60\frac{(1-v^{2n})}{0.05} + 1{,}000v^{2n} + 50 = 60\frac{(1-v^{4n})}{0.05} + 1{,}000v^{4n}$$
$$1{,}200(1-v^{2n}) + 1{,}000v^{2n} + 50 = 1{,}200(1-v^{4n}) + 1{,}000v^{4n}$$
$$-200v^{2n} + 50 = -200v^{4n}$$
$$200v^{4n} - 200v^{2n} + 50 = 0$$
$$v^{4n} - v^{2n} + 0.25 = 0 \quad \textit{(note that this is a quadratic in } v^{2n}\textit{)}$$
$$(v^{2n} - 0.5)^2 = 0$$
$$v^{2n} = 0.5$$
$$n = \frac{\ln(0.5)}{2\ln v} = 7.1033$$

Note: A term of 7.1033 years for a coupon bond doesn't make sense. However, a term of 7 yields an increase in the bond's price of 49.99, which is close enough.

Another approach, based on the premium-discount formula, is to recognize that the n-year bond has a *premium* of $10 \cdot a_{\overline{2n}|5\%}$ (With a semi-annual coupon of 50, it would be selling at par, based on a yield of 5% per coupon period. With a coupon of 60, it sells at a premium equal to the present value of $2n$ payments of 10 (= 60 – 50).) Similarly, the $2n$-year bond has a premium of $10 \cdot a_{\overline{4n}|5\%}$. The difference between these two values is 50, leading to:

$$10 \cdot a_{\overline{4n}|5\%} - 10 \cdot a_{\overline{2n}|5\%} = 50$$
$$\frac{1-v_{0.05}^{4n}}{0.05} - \frac{1-v_{0.05}^{2n}}{0.05} = 5$$
$$v_{0.05}^{2n} - v_{0.05}^{4n} = 0.25$$

This is equivalent to the quadratic in the above solution.

Answer: B

9. The problem is asking for the value of $i_{4,7}$.

$$(1+i_{4,7})^3 = \frac{(1+s_7)^7}{(1+s_4)^4}$$

$$s_7 = 0.08 - 0.001\cdot 7 + 0.002\cdot 49 = 0.171$$

$$s_4 = 0.08 - 0.001\cdot 4 + 0.002\cdot 16 = 0.108$$

$$(1+i_{4,7})^3 = \frac{(1+s_7)^7}{(1+s_4)^4} = \frac{(1.171)^7}{(1.108)^4} = \frac{3.01924}{1.50716} = 2.00327$$

$$i_{4,7} = 2.00327^{1/3} - 1 = 0.260607$$

Answer: E

10. Principal portion of the kth coupon: $PRin_k = F(r-i)v^{n-k+1}$

So, principal of the 7th coupon: $PRin_7 = 15{,}000(.06-.075)v^{12-7+1} = -145.79$

The coupon amount is 900, so the interest portion of the 7th coupon is:

$Int_7 = Total\ Pmt_7 - Prin_7 = 900 - (-145.79) = 1{,}045.79$

Alternatively, we can simply calculate the bond's book value at time 6 and multiply by 7.5% to find the amount of interest due at the end of the 7th year:

N=6 (the number of *remaining payments*), I/Y=7.5, PMT=900, FV=15,000.
CPT PV=13,943.88. This is the book value at time 6, with 6 payments left.
$Int_7 = 13{,}943.88\cdot 7.5\% = 1{,}045.79$

Answer: C

11. We will first calculate the IRR for Investment A. This can be done using the CF worksheet, but it is simpler with the TVM keys:
N=10, PV=-4,000, PMT=500, FV=1,500. (Note that the payment at time 10 is 2,000, consisting of the FV of 1,500, plus the last PMT of 500.)
Then CPT I/Y=8.233%.

Therefore, the IRR for Investment B is also 8.233%, which means that the present value of B's cash flows at 8.233% must be 0. In other words, the investment of 1,000 at time 0 has the same present value (at 8.233%) as the 5 payments of X at times 1 to 5. We can represent this in the calculator's TVM variables as follows:

Set N=5, I/Y=8.233, PV=-1,000, FV=0, and CPT PMT=252.00.

The value of X is 252.

Answer: B

Module 7

Asset-Liability Management

Section 7.1

Introduction to Matching Assets and Liabilities

Insurance companies collect premiums from their customers and then invest those premiums. The premiums and the interest earned on them are the company's **assets.** These assets will be used to pay claims as they occur. The company is obligated to pay future claims on policies for which it has collected premiums. These future cash outflows are **liabilities**. Each insurance company is required to make sure that its assets and liabilities are managed in such a way as to ensure that cash will be available to pay claims when they occur.

In this chapter, we will first give some examples of asset-liability management for some very simple situations to illustrate the basic ideas. Then we will examine the concepts of duration and immunization, which are used for the more complex situations that occur in reality. The examples used here are based on actuarial examination problems.

Example (7.1)

A company must pay liabilities of 2,000 and 3,000 at the end of years 1 and 2, respectively. The only investments available to the company are the following two zero-coupon bonds:

Maturity (years)	Annual Effective Yield	ParValue
1	5%	1,000
2	6%	1,000

The company can cover its liabilities exactly by buying two of the 5% one-year zero-coupon bonds and three of the 6% two-year zero-coupon bonds. This is called **cash flow matching** or **dedication**. Next we will find the cost of the bonds required to match the cash flows.

For its year 1 liability, the company buys 5% one-year zero-coupon bonds with a total maturity value of 2,000. The cost is $\frac{2{,}000}{1.05}=1{,}904.76$.
For its year 2 obligation, the company buys 6% two-year zero-coupon bonds with a total maturity value of 3,000. The cost is $\frac{3{,}000}{1.06^2}=2{,}669.99$.
The total cost to match the liabilities is $1{,}904.76+2{,}669.99=4{,}574.75$.

Exercise (7.2)

Suppose the liabilities in the above problem were 1,000 in one year and 2,000 in two years. Find the cost of matching those liability cash flows.

Answer: 2,732.37

The matching process is a bit more complicated when the bonds are coupon bonds, as the next example shows.

Example (7.3)

Joe has liabilities that require a payment of 1,000 six months from now and another payment of 1,000 one year from now. There are 2 available investments:

1) a 6-month bond with a face amount of 1,000, a 6% nominal annual coupon rate payable semi-annually, and a 5% nominal annual yield rate convertible semi-annually
2) a 1-year bond with a face amount of 1,000, a 7% nominal annual coupon rate payable semi-annually, and an 8% nominal annual yield rate convertible semi-annually

We will first look at the amount of each bond to buy.
Note: Problems like this assume that you can purchase fractions of bonds.

The payment of 1,000 one year from now must be provided by the 1-year bond, as the other bond will have matured at 6 months. The total payments for the 1-year bond at the end of the year consist of a coupon of 35 and the redemption value of 1,000 for a total of 1,035. To cover a liability of 1,000 due in one year, the fraction of a 1-year bond that Joe must buy is $\frac{1,000}{1,035} = 0.96618$.

This fraction (0.96618) of a 1-year bond will also make a coupon payment of 33.82 $(= 0.96618 \cdot 35)$ in 6 months. The total amount due at 6 months is 1,000, so the additional amount needed from a 6-month bond is $1,000 - 33.82 = 966.18$.

The total payments at month 6 from a 6-month bond consist of a coupon of 30 plus the redemption value of 1,000 for a total of 1,030. To provide the required amount (966.18), the fraction of this bond that is needed is $\frac{966.18}{1,030} = 0.93804$.

Now we can look at the *cost* of the bonds required to match the liabilities. We have seen that Joe must purchase 0.93804 of the 6-month bond and 0.96618 of the 1-year bond. We can find the price of each bond using the financial calculator.

<u>6-month</u>: N = 1, I/Y = 2.5% (yield per semiannual period), PMT = 30, FV = 1,000.
This gives a price of: PV = −1,004.88.

<u>1-year</u>: N = 2, I/Y = 4% (yield per semiannual period), PMT = 35, FV = 1,000.
This gives a price of: PV = −990.57.

The total cost to purchase the required bonds is:

$$0.93804(1,004.88) + 0.96618(990.57) = 1,899.69$$

Exercise (7.4)

Jane has liabilities that require payments of 1,000 six months from now and 2,000 one year from now. There are two available investments:

1) a 6-month bond with a face amount of 1,000, a 4% nominal annual coupon rate payable semi-annually, and a 5% nominal annual yield rate convertible semi-annually

2) a 1-year bond with a face amount of 1,000, a 6% nominal annual coupon rate payable semi-annually, and an 8% nominal annual yield rate convertible semi-annually

Find the amount of each bond to purchase and the total cost of the bonds needed to match the liability cash flows exactly.

Answers: Buy 1.94175 1-year bonds and 0.92328 6-month bonds. Cost = 2,823.90

Section 7.2
Duration

The reality of investments for an insurance company or a bank is much more complex than the previous cash flow matching examples. There are thousands of claim liabilities, thousands of accounts, and thousands of bonds and other investments available to buy. The company may have to sell bonds to meet unexpected liabilities at various times, and it also faces interest rate risk. **Interest rate risk** occurs because the market value of the company's investments decreases when interest rates rise, and increases when interest rates decline.

The concept of **duration** gives an investment manager a measure of the company's interest rate risk and a tool for managing that risk by matching assets and liabilities for the entire portfolio. There are two closely related types of duration: Macaulay duration and modified duration. There is a simple way to describe the Macaulay duration of an investment: it is the *weighted average time* at which the investment's payments will occur.

> It is worthwhile to review the concept of weighted average. Let $x_1,\ldots,x_n$ be a set of n real numbers and $w_1,\ldots,w_n$ be a set of n positive real numbers such that $w_1+\ldots+w_n=1$. The weighted average of $x_1,\ldots,x_n$ with the weights $w_1,\ldots,w_n$ is the sum $\bar{x}=x_1w_1+\ldots+x_nw_n$.
>
> For example, the weighted average of the numbers 1, 2, and 3 with weights 0.5, 0.3, 0.2, respectively, is 0.5(1) + 0.3(2) + 0.2(3) = 1.7.

For an investment that has n cash flows $CF_{t_1}, CF_{t_2}, \ldots, CF_{t_n}$, which occur at times t_1, t_2, …, t_n, the Macaulay duration, D_{mac}, is a weighted average of the times of payment: t_1, t_2, …, t_n.

The weights are based on the present values of the cash flows. The total **present value** (or **price**) of this investment is:

$$P = CF_{t_1}(1+i)^{-t_1} + CF_{t_2}(1+i)^{-t_2} + \ldots + CF_{t_n}(1+i)^{-t_n}$$

The weight for the payment at time t_k is just the present value of that payment $\left(CF_{t_k}(1+i)^{-t_k}\right)$, divided by the total present value (P).

(7.5)

$$w_k = \frac{CF_{t_k}(1+i)^{-t_k}}{P} = \frac{CF_{t_k}(1+i)^{-t_k}}{\sum_{k=1}^{n} CF_{t_k}(1+i)^{-t_k}}$$

It is clear that $w_1+\ldots+w_n=1$.

Using the weights w_k from **(7.5)**, the **Macaulay duration, D_{mac}**, is defined as:

(7.6)

$$\begin{aligned} D_{\text{mac}} &= t_1 \cdot w_1 + t_2 \cdot w_2 + \ldots + t_n \cdot w_n \\ &= \left[\frac{t_1 \cdot CF_{t_1} (1+i)^{-t_1} + t_2 \cdot CF_{t_2} (1+i)^{-t_2} + \ldots + t_n \cdot CF_{t_n} (1+i)^{-t_n}}{P} \right] \\ &= \frac{\sum_{k=1}^{n} t_k \cdot CF_{t_k} (1+i)^{-t_k}}{\sum_{k=1}^{n} CF_{t_k} (1+i)^{-t_k}} \end{aligned}$$

For n equally-spaced cash flows, the formula for D_{mac} is:

(7.7)

$$D_{\text{mac}} = 1 \cdot w_1 + 2 \cdot w_2 + \ldots + n \cdot w_n = \frac{\sum_{t=1}^{n} t \cdot CF_t (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}$$

Note: The above analysis assumed a finite number of discrete cash flows. A similar analysis can be applied to an infinite series of cash flows (a perpetuity) or to continuous cash flows (using integrals).

Example (7.8)

An investment that pays 1,000 in one year, 2,000 at the end of the second year, and 3,000 at the end of the third year has been purchased to yield an annual effective rate of $i = 0.10$. Find its Macaulay duration.

The present value is:

$$P = \frac{1{,}000}{1.1} + \frac{2{,}000}{1.1^2} + \frac{3{,}000}{1.1^3} = 909.09 + 1{,}652.89 + 2{,}253.94 = 4{,}815.92$$

The weights for calculating the duration are:

$$w_1 = \frac{909.09}{4{,}815.92} = 0.18877,\ w_2 = \frac{1{,}652.89}{4{,}815.92} = 0.34321,\ w_3 = \frac{2{,}253.94}{4{,}815.92} = 0.46802$$

The Macaulay duration is the weighted average time of the cash flows:

$$D_{\text{mac}} = 0.18877(1) + 0.34321(2) + 0.46802(3) = 2.27925$$

Exercise (7.9)

An investment pays 1,000 in one year, 2,000 at the end of the second year and 3,000 at the end of the third year. An investor has purchased it to yield an annual effective rate $i = 0.08$. Find the Macaulay duration.

Answer: 2.28983

Note: The cash flows in Exercise **(7.9)** *are the same as in Example* **(7.8)**, *but the Macaulay duration is slightly different. Duration depends mainly on the amount and timing of the cash flows, but it is also affected by the interest rate. Since the interest rate in Exercise* **(7.9)** *is not the same as in Example* **(7.8)**, *the durations for these two situations are not the same. The weights used in the weighted average are the* <u>*present values*</u> *of the cash flows, and these present values depend on the interest rate used to calculate them. Because Macaulay duration is a function of the interest rate, we will sometimes write it as* $D_{\text{mac}}(i)$.

<u>A comment on notation:</u>
We have used D_{mac} as the notation for Macaulay duration, because that notation is used in the Exam FM study note on duration. The recommended texts for Exam FM use various notations for Macaulay duration, including D, $MacD$, and $\bar{d}$.

You have seen that Macaulay duration is a weighted average payoff time for an investment. It may not be immediately obvious why this tells you something about interest rate risk. We will see how duration relates to interest rate risk in the next section, where we will study *modified* duration.

Section 7.3

Modified Duration

The interest rate risk that worries an investment manager is the change in market value that occurs when interest rates change. We can measure the sensitivity of asset prices to changes in interest rates by looking at $\frac{dP}{di}$, the derivative of price P with respect to interest rate i. We illustrate this in the next example.

Example (7.10)

We return to the investment of Example **(7.8)**. The investment pays 1,000 in one year, 2,000 at the end of the second year, and 3,000 at the end of the third year. The price P at rate i is:

$$P(i) = \frac{1{,}000}{1+i} + \frac{2{,}000}{(1+i)^2} + \frac{3{,}000}{(1+i)^3} = 1{,}000(1+i)^{-1} + 2{,}000(1+i)^{-2} + 3{,}000(1+i)^{-3}$$

Thus, $\frac{dP}{di} = P'(i) = (-1)1{,}000(1+i)^{-2} + (-2)2{,}000(1+i)^{-3} + (-3)3{,}000(1+i)^{-4}$.

The investment in Example **(7.8)** was purchased to yield 10%. At $i = 0.10$:

$$\frac{dP}{di} = (-1)1{,}000(1.1)^{-2} + (-2)2{,}000(1.1)^{-3} + (-3)3{,}000(1.1)^{-4} = -9{,}978.83$$

Note that the derivative is negative, since P (the present value of the cash flows) is a *decreasing* function of i (i.e., the price will decrease if i increases). Stated mathematically, the derivative of $(1+i)^{-n}$ is $-n \cdot (1+i)^{-(n+1)}$, which is always negative (assuming that n and $(1+i)$ are both greater than 0).

The **modified duration D_{mod}** (also referred to as the **volatility**) is defined as the *negative* of this derivative, divided by the price. D_{mod} represents the rate of change in price as a percent of price. The negative of the derivative is used in the formula for D_{mod} so that modified duration will be a positive number.

(7.11)

$$D_{mod} = \frac{-\left(\frac{dP}{di}\right)}{P}$$

Example (7.12)

The investment in Example **(7.8)** had a yield of 10% at a price P = 4,815.92. Thus the modified duration at $i = 0.10$ is:

$$D_{\text{mod}} = \frac{-\left(\frac{dP}{di}\right)}{P} = -\frac{-9,978.83}{4,815.92} = 2.07205$$

Example **(7.10)** developed a formula for $\frac{dP}{di}$ based on a particular set of cash flows. We can write a general formula for a series of n equally-spaced cash flows as follows:

$$\frac{dP}{di} = (-1)CF_1(1+i)^{-2} + (-2)CF_2(1+i)^{-3} + \ldots + (-n)CF_n(1+i)^{-(n+1)} = -\sum_{t=1}^{n}\left[t \cdot CF_t(1+i)^{-(t+1)}\right]$$

Substituting this expression for $\frac{dP}{di}$ into Formula **(7.11)**, we have:

(7.13)
$$D_{\text{mod}} = \frac{-\left(\frac{dP}{di}\right)}{P} = \frac{-\left(-\sum_{t=1}^{n}\left[t \cdot CF_t(1+i)^{-(t+1)}\right]\right)}{P(i)} = \frac{\sum_{t=1}^{n}\left[t \cdot CF_t(1+i)^{-(t+1)}\right]}{\sum_{t=1}^{n}\left[CF_t(1+i)^{-t}\right]}$$

Except for the exponent in the numerator, this is identical to Formula **(7.7)** for Macaulay duration, leading to the following important relation between D_{mod} and D_{mac}:

(7.14)
$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i}$$

We can use Formula **(7.14)** and the value of D_{mac} from Example **(7.8)** to verify the answer in Example **(7.12)**:

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{2.27925}{1.1} = 2.07205$$

It is important to memorize the relationship in (7.14) and be able to use it. We developed this equation based on a formula for equally-spaced cash flows; however, (7.14) is valid for *any* pattern of cash flows (including continuous cash flows or an infinite series of cash flows). The only requirement is that *a single interest rate* (i) is used for valuing all of the cash flows. In other words, **(7.14)** assumes a *flat yield curve*, where one interest rate i applies to all the cash flows.

Exercise (7.15)

In Exercise **(7.9)** we found the Macaulay duration D_{mac} of an investment when $i = 0.08$. Find the modified duration D_{mod} using derivatives (as in Formula **(7.13)**), and verify it using **(7.14)**.

Answer: 2.12022

Formula **(7.14)** gives a nice insight into the behavior of asset values when interest rates change. D_{mod} measures the proportionate change in price when the interest rate changes; D_{mac} is the weighted average time when the cash flows occur. Since the two measures are proportional, we see that a longer-term investment (with its larger value of D_{mac}) is more sensitive to interest rate changes than a shorter-term investment (as indicated by the larger value of D_{mod}). This is intuitively reasonable, because the value of a payment that will be received next week will be changed only slightly by a variation in the interest rate, whereas the value of a payment that will be received 10 years from now will be affected quite significantly by a change in interest rate.

A comment on notation:
We have used the notation D_{mod} for modified duration because it is used in the Exam FM study note on duration. The various texts for Exam FM use DM, $ModD$, or $\bar{v}$ (where $\bar{v}$ stands for **volatility**).

To gain a deeper understanding of the duration concept, it is useful to consider how the duration of an asset is affected by various changes in the factors that determine its duration. Let's consider how a bond's duration is affected by its term, by its coupon rate, by a change in yield, and by the passage of time. These observations apply to both Macaulay duration and modified duration.

Term: If two bonds have identical coupon rates and yields, but one has a longer term, the bond with the longer term has a longer duration. The longer bond will have additional coupon payments that occur later than those of the shorter bond. And its maturity value will be paid later than that of the shorter bond. So naturally the longer bond has a longer duration.

Coupon rate: If two bonds are identical in term and yield, but have different coupon rates, the bond with the higher coupon rate has a shorter duration. The larger coupons will constitute a larger share of the bond's value and will therefore have larger weights in the duration calculation. Since the coupons are paid before the maturity value, greater weight on the coupons (and less on the maturity value) means the bond's duration will be shorter. A bond with a lower coupon rate, on the other hand, will have a longer duration. The extreme example is the *zero*-coupon bond, where 100% of the weight is on the final payment at time n, so its Macaulay duration is n.

Yield: If two bonds have identical terms and coupon rates, the bond with the higher yield has a shorter duration. A higher yield reduces the present value of each coupon payment and of the maturity value. But the value of the later payments is reduced proportionately more than the value of the earlier payments, thus increasing the relative weight on early payments and reducing the weight on later payments. The result is that the bond with the higher yield has a shorter duration.

Passage of time: As a bond ages, the duration of each of its payments decreases. So in general we can say that a bond's duration decreases with the passage of time. It is worth noting, however, that each time a coupon payment occurs, the bond's duration increases on the payment date (and then afterward it continues to decrease). That is because as a coupon's payment date approaches, that coupon's duration becomes extremely short (e.g., a Macaulay duration of 1 day on the day before the coupon is paid). Once the coupon payment is made, the bond's duration is calculated based on the durations of all its *remaining* payments, *excluding* the coupon payment that just occurred. This causes the bond's duration to jump up a bit when a coupon is paid because that coupon (with its very short duration) is dropped from the duration calculation.

Section 7.4

Helpful Formulas for Duration Calculations

For an n-payment level annuity, the formula for Macaulay duration is:

(7.16) Macaulay Duration of a level-payment investment:

$$D_{\text{mac}} = \frac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}$$

We can see why this is true by looking at an example.

Example (7.17)

An investment pays 1,000 at the end of each year for the next 3 years. At a rate i, the price is $P = 1{,}000 \cdot a_{\overline{3}|i}$. Macaulay duration is given by:

$$D_{\text{mac}} = \frac{1(1{,}000)v + 2(1{,}000)v^2 + 3(1{,}000)v^3}{1{,}000a_{\overline{3}|i}} = \frac{v + 2v^2 + 3v^3}{a_{\overline{3}|i}} = \frac{(Ia)_{\overline{3}|i}}{a_{\overline{3}|i}}$$

At a rate $i = 0.06$, we have:

$$D_{\text{mac}} = \frac{(Ia)_{\overline{3}|}}{a_{\overline{3}|}} = \frac{5.242}{2.673} = 1.96$$

Exercise (7.18)

An investment pays 2,000 at the end of each year for the next 5 years. Find the Macaulay duration at rate $i = 0.06$.

Answer: 2.88

There is also a formula for the Macaulay duration of a coupon bond:

(7.19) Macaulay duration of a coupon bond with face value F, coupon Fr for n periods, and redemption value C:

$$D_{\text{mac}} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{Fr(a_{\overline{n}|}) + Cv^n} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{\text{Bond Price}}$$

Note: In Formula **(7.19)**, *n is the number of coupon periods, and the interest functions use the effective interest rate per coupon period. The resulting value for Macaulay duration is expressed in coupon periods. For a bond with semi-annual coupons, we must divide the result by 2 to find D_{mac} expressed in years.*

For an annual-coupon "par" bond, where F=C and r=i in the above formula, we can demonstrate that $D_{\text{mac}} = \ddot{a}_{\overline{n}|\,i}$. Substituting F for C and i for r, we have:

(7.20) Macaulay duration of an annual-coupon "par" bond:

$$D_{\text{mac}} = \frac{Fi(Ia)_{\overline{n}|} + nFv^n}{Fi(a_{\overline{n}|}) + Fv^n} = \frac{i \cdot \dfrac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i} + nv^n}{i \cdot \dfrac{1-v^n}{i} + v^n} = \frac{\ddot{a}_{\overline{n}|}}{1} = \ddot{a}_{\overline{n}|}$$

In the case of an n-year par bond with coupons payable m times per year, an analysis similar to **(7.20)** shows that $D_{mac} = \ddot{a}^{(m)}_{\overline{n}|}$. (In this case, D_{mac} is expressed in years.) Since $\ddot{a}^{(m)}_{\overline{n}|} < \ddot{a}_{\overline{n}|}$, the Macaulay duration of a par bond with m coupons per year is less than that of a similar annual-coupon par bond. This is reasonable because, on average, the coupons will be paid *earlier* when there are m coupon payments per year than when the entire annual coupon payment occurs at the end of each year.

Example (7.21)

An annual-coupon par bond has a face value of 1,000, a coupon rate of 5%, and three years to maturity. At a yield of $i = 0.06$:

$$D_{\text{mac}} = \frac{50(Ia)_{\overline{3}|0.06} + 3(1{,}000)v^3}{50(a_{\overline{3}|0.06}) + (1{,}000)v^3} = \frac{50(5.242247) + 3{,}000(.839619)}{50(2.673012) + 1{,}000(.839619)} = 2.86$$

Exercise (7.22)

An annual-coupon par bond has a face value of 1,000, a coupon rate of 6%, and 5 years to maturity. Find D_{mac} for a yield of $i = 0.05$.

Answer: 4.47

Applying Formula **(7.19)** to a zero-coupon bond gives an intuitively obvious result. If the coupon rate r is zero, we have:

(7.23) Macaulay duration of a zero-coupon bond:

$$D_{\text{mac}} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{Fr(a_{\overline{n}|}) + Cv^n} = \frac{nCv^n}{Cv^n} = n$$

This confirms that the Macaulay duration for a zero-coupon bond maturing in n years is n.

Section 7.5

Using Duration to Approximate Change in Price

Once we know the modified duration (or Macaulay duration) for a set of cash flows, we can use a Taylor series approximation to estimate the change in price that would result from a given change in interest rate. For a single asset, of course, it is simple enough to calculate the exact price at the new interest rate. However, when a portfolio includes tens of thousands of assets and we want to consider a range of possible interest rates, it is impractical to calculate an exact value for the entire portfolio at each possible interest rate, so approximation methods are useful.

Taylor Series

Given the value of a function $f(x)$ and its derivatives for one particular value of x (call it x_0), the Taylor series provides a formula to approximate the value of the function for other values of x near x_0. The Taylor series of $f(x)$ for a neighborhood around x_0 is:

(7.24)

$$f(x)=f(x_0)+f'(x_0)\cdot(x-x_0)+\frac{f''(x_0)\cdot(x-x_0)^2}{2!}+....+\frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!}+...$$

Using only the value of the function, $f(x_0)$, and its first derivative, $f'(x_0)$, the first two terms of the Taylor series give us a first-order approximation of the function:

(7.25)

$$f(x)\approx f(x_0)+f'(x)\cdot(x-x_0)$$

We can add the third term of the Taylor series (involving $f''(x)$) to create a second-order approximation for $f(x)$:

(7.26)

$$f(x)\approx f(x_0)+f'(x)\cdot(x-x_0)+\frac{f''(x)\cdot(x-x_0)^2}{2!}$$

In this section and the next, we will use the Taylor series to develop formulas for estimating the price (present value) of an asset after a given change in the interest rate. This section will use Macaulay duration and modified duration to develop first-order approximations. The next section will define convexity and develop a second-order approximation that uses both duration and convexity.

First-Order Modified Approximation

$P(i)$, the price of an asset at interest rate i, is a function of i. Given the price of the asset at one interest rate, i_0, and the derivatives of the price function at that interest rate, we can approximate the price of the asset at other interest rates. The first-order Taylor series approximation for $P(i)$ is:

(7.27)
$$P(i) \approx P(i_0) + P'(i_0) \cdot (i - i_0)$$

By manipulating **(7.11)**, we can write an expression for $P'(i)$:

$$P'(i) = \left(\frac{dP}{di}\right) = -D_{\text{mod}} \cdot P(i)$$

This expression involving modified duration can be substituted into **(7.27)** to produce the **first-order modified approximation** for $P(i)$:

(7.28)
$$P(i) \approx P(i_0) - D_{\text{mod}}(i_0) \cdot P(i_0) \cdot (i - i_0)$$

Note: D_{mod} in this expression is shown as $D_{\text{mod}}(i_0)$ to make it clear that modified duration is a function of the interest rate, and that we are using the value of D_{mod} calculated using an interest rate of i_0.

Formula **(7.28)** is sometimes written as an equation for the *change* in price, $\Delta P = P(i) - P(i_0)$, resulting from a given change in interest rate $\Delta i = i - i_0$:

(7.29)
$$\Delta P \approx -D_{\text{mod}}(i_0) \cdot P(i_0) \cdot \Delta i$$

It should be clear from **(7.28)** and **(7.29)** that long-duration investments undergo greater price changes than do short-duration investments when interest rates change. To a first approximation, the change in price is *proportional to* the modified duration of the investment. Because of this important relationship, long durations are viewed as a sign of greater volatility.

Formula **(7.28)** gives a first-order approximation to price change using modified duration. It treats the price function, $P(i)$, as a linear function with slope equal to $P'(i_0)$. This is reasonably accurate for values of i near i_0, but it is less accurate for larger changes in the interest rate, as the following example demonstrates.

Example (7.30)

Consider an annuity-immediate with 20 annual payments of 100. At an annual effective interest rate of 10%, the present value of this annuity is:

$$100 \cdot a_{\overline{20}|10\%} = 100 \cdot \frac{1-1.1^{-20}}{0.1} = 851.36$$

We will use **(7.28)** to develop a formula to approximate this annuity's price at various interest rates. We will then use this formula to estimate the price of the annuity at a series of interest rates, and will compare the resulting *estimated* prices to the *actual* prices at those interest rates.

Using **(7.16)** to calculate the Macaulay duration for this level-payment annuity, we have:

$$D_{\text{mac}}(10\%) = \frac{(Ia)_{\overline{20}|10\%}}{a_{\overline{20}|10\%}} = \frac{\ddot{a}_{\overline{20}|10\%} - \dfrac{20}{1.1^{20}}}{0.10} \div 8.5136 = 7.5081$$

The modified duration is then:

$$D_{\text{mod}}(10\%) = \frac{D_{\text{mac}}(10\%)}{1+i} = \frac{7.5081}{1.1} = 6.8255$$

Using **(7.28)** to create an expression for the approximate price at a given interest rate i, we have:

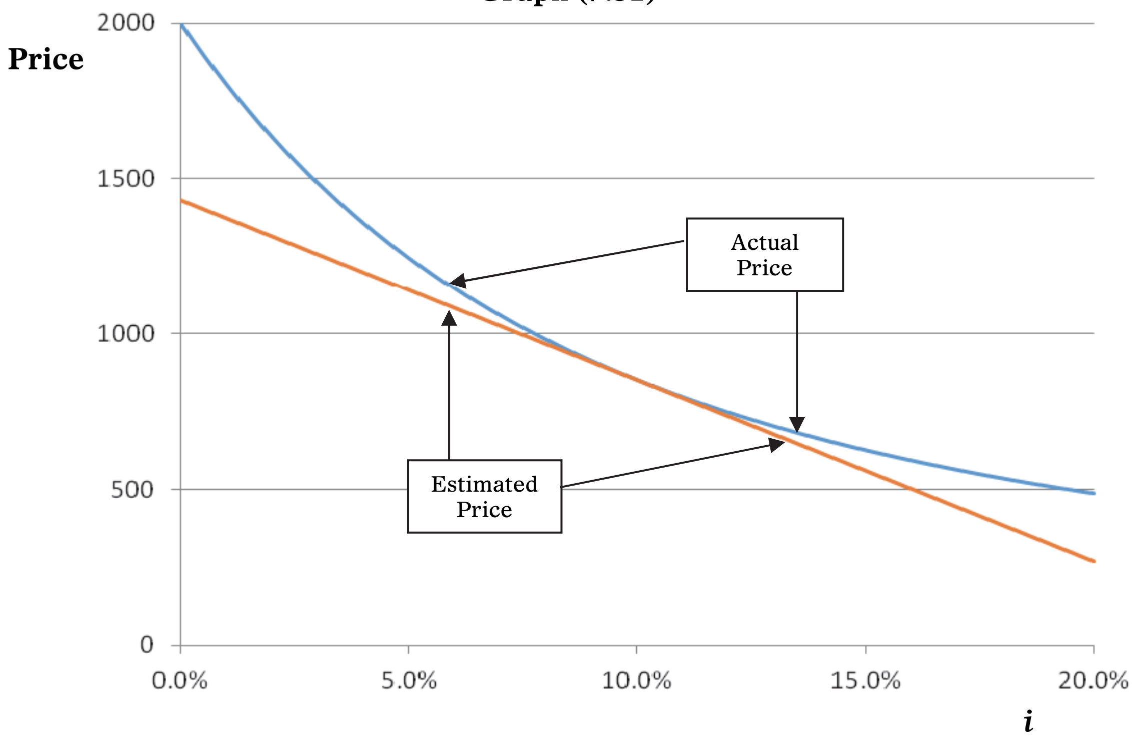
$$\begin{aligned} P(i) &\approx P(i_0) - D_{\text{mod}}(i_0) \cdot P(i_0) \cdot (i - i_0) \\ &= 851.36 - 6.8255 \cdot 851.36 \cdot (i - i_0) = 851.36 - 5{,}810.95 \cdot (i - 0.10) \end{aligned}$$

The following table shows, for various values of i, the *estimated* price (present value) for this annuity, and also the *actual* price at each of those interest rates.

i	Actual $P(i)$	Estimated $P(i)$	Estimate – Actual	% error
6.0%	1,146.99	1,083.79	-63.20	-5.510%
9.0%	912.85	909.47	-3.38	-0.371%
9.9%	857.20	857.17	-0.03	-0.004%
10.0%	851.36	851.36	0.00	0%
10.1%	845.58	845.55	-0.03	-0.004%
11.0%	796.33	793.25	-3.08	-0.388%
14.0%	662.31	618.92	-43.39	-6.552%

For each value of i in the preceding table, the error (Estimate – Actual) is negative. That is, the estimated value for $P(i)$ is less than the actual value. Also, as the change in interest rate $(i - i_0)$ increases in absolute value, the difference between the estimated price and the actual price increases. We can see why this is the case by considering the following graph:

Graph (7.31)



The curved line in the graph shows the *actual* values of $P(i)$ as a function of i. The straight line, which is tangent to the curve at the point where i equals 10%, represents the first-order modified approximation for $P(i)$, based on the values of $P(i_0)$ and $P'(i_0)$, at $i_0 = 10\%$. Because the actual function has positive convexity (that is, its second derivative is positive), the tangent line (the approximation) lies *below* the curve (except at $i = 10\%$). This means that the first-order modified approximation of $P(i)$ is always *less* than the actual value of $P(i)$. This is true for any asset whose cash flows are fixed (i.e., where the cash flows will not change when the interest rate changes).

Example (7.32)

We return again to the investment of **(7.8)**. The investment pays 1,000 in one year, 2,000 at the end of the second year, and 3,000 at the end of the third year.

Assume that this asset is purchased to yield $i = 0.10$. We have already seen that the price is:

$$P(0.10) = \frac{1,000}{1.1} + \frac{2,000}{1.1^2} + \frac{3,000}{1.1^3} = 4,815.9279\,.$$

Now suppose that the yield decreases by 0.001 to 0.099. We can estimate the price at this new lower yield by applying **(7.28)**. Using the value of $D_{\text{mod}}(0.10)$ that we found in Exercise **(7.12)**, we have:

$$\begin{aligned} P(i) &\approx P(i_0) - D_{\text{mod}} \cdot P(i_0) \cdot (i - i_0) \\ &= 4815.9279 - (2.07205) \cdot (4,815.9279) \cdot (-0.001) = 4,825.9067 \end{aligned}$$

The actual new price is:

$$P(0.099) = \frac{1,000}{1.099} + \frac{2,000}{1.099^2} + \frac{3,000}{1.099^3} = 4,825.9227$$

Our estimate of the price at $i = 9.9\%$ is very slightly smaller than the actual price at that interest rate. This is consistent with the fact that the first-order modified approximation of price is always less than the actual price.

Exercise (7.33)

Assume that the investment of **(7.9)** (which pays 1,000 in one year, 2,000 at the end of the second year and 3,000 at the end of the third year), was purchased to yield $i = 0.08$. Using the first-order modified approximation, estimate the new price if the interest rate increases to 0.082. Compare this estimated price to the actual price of the investment at $i = 0.082$.

Answer: Estimate = 5,000.80 Actual = 5,000.87 (Estimate < Actual)

Example (7.34)

An annual-coupon corporate bond is priced to yield 6.5% annually. It has a price of 969.56 and a Macaulay duration of $D_{\text{mac}} = 6.5572$. If the interest rate (yield) increases to 7.0%, then the first-order modified approximation of the new price is calculated as follows:

$$\begin{aligned} P(0.07) &\approx P(0.065) - \frac{D_{\text{mac}}(0.065)}{1.065} \cdot P(0.065) \cdot (0.07 - 0.065) \\ &= 969.56 - \frac{6.5572}{1.065} \cdot (969.56)(.005) = 939.71 \end{aligned}$$

Exercise (7.35)

An annual-coupon corporate bond yields 6% annually at its current price of 965.35. The bond's Macaulay duration is $D_{\text{mac}} = 3.7177$. Use **(7.28)** to estimate what the bond's price will be if its yield increases by 1.00%.

Answer: 931.49

In all of the above formulas and calculations involving modified duration and first-order modified approximations, the interest rates have been annual effective interest rates (or effective rates per period). In the event that we need to estimate the change in price for a given change in a *nominal* interest rate, $i^{(m)}$, the modified duration is calculated using the following formula:

(7.36)

$$D_{\text{mod}}\left(i^{(m)}\right) = \frac{P'\left(i^{(m)}\right)}{P\left(i^{(m)}\right)} = \frac{D_{\text{mac}}\left(i^{(m)}\right)}{1+\dfrac{i^{(m)}}{m}}$$

In the above formula, the price $P\left(i^{(m)}\right)$ and Macaulay duration $D_{\text{mac}}\left(i^{(m)}\right)$ are shown as functions of the nominal interest rate $i^{(m)}$. But the price and Macaulay duration can be calculated at *any interest rate that is* *<u>equivalent</u>* *to* $i^{(m)}$ without altering their values. For example, we could have applied an annual effective rate $i = \left[\left(1+\dfrac{i^{(m)}}{m}\right)^m - 1\right]$ to the same set of cash flows, and the resulting values of P and D_{mac} would be the same. The modified duration, $D_{\text{mod}}\left(i^{(m)}\right)$, however,is defined as $\dfrac{P'\left(i^{(m)}\right)}{P\left(i^{(m)}\right)}$, so it represents the rate of change of price *with respect to a specific measure of interest rate* ($i^{(m)}$). This means that D_{mod} *does* depend on which measure of interest rate we are using, and **(7.36)** provides a formula to calculate modified duration for nominal interest rates.

Note: To prove that Formula **(7.36)** *is valid, you can replace* $P'\left(i^{(m)}\right)$ *with an alternative expression,* $P'\left(i^{(m)}\right) = \dfrac{dP\left(i^{(m)}\right)}{di^{(m)}} = \dfrac{dP(i)}{di}\cdot\dfrac{di}{di^{(m)}} = P'(i)\cdot\dfrac{di}{di^{(m)}}$*, and then simplify the resulting equation.*

Formula **(7.36)** leads to the following formula to estimate the price of an asset after a change in the *nominal* interest rate at which it is valued:

(7.37)

$$P\left(i^{(m)}\right) \approx P\left(i_0^{(m)}\right) - D_{\text{mod}}\left(i_0^{(m)}\right)\cdot P\left(i_0^{(m)}\right)\cdot\left(i^{(m)} - i_0^{(m)}\right)$$

If $m = 1$ in Formula **(7.36)**, then $i^{(m)} = i^{(1)}$, which is the annual effective rate i. In that case, $D_{\text{mod}}(i) = \dfrac{D_{\text{mac}}(i)}{1 + \dfrac{i}{1}}$, which is equivalent to Formula **(7.14)**.

If m is infinite, then $i^{(m)} = i^{(\infty)} = \delta$, where δ is the force of interest. In that case, **(7.36)** becomes: $D_{\text{mod}}(\delta) = \dfrac{D_{\text{mac}}}{1 + \dfrac{\delta}{\infty}} = D_{\text{mac}}$. Therefore, $\dfrac{P'(\delta)}{P(\delta)} = D_{\text{mod}}(\delta) = D_{\text{mac}}$. In other words, Macaulay duration is the first derivative of price with respect to the force of interest, expressed as a percentage of price.

Example (7.38)

A 5-year bond with a face amount of 1,000 pays semi-annual coupons at a 6% annual rate. Its nominal annual yield is 8% convertible semi-annually. Using the first-order modified approximation, estimate what the bond's price would be at a (nominal) yield of 8.2%.

Solution.
The bond's current price is:

$$P(8\%) = 30 \cdot a_{\overline{10}|4\%} + (1{,}000) \cdot 1.04^{-10} = 918.89$$

Its Macaulay duration is:

$$D_{\text{mac}} = \frac{\sum_{t=1}^{10}\left(\frac{t}{2} \cdot 30 \cdot 1.04^{-t}\right) + 5 \cdot (1{,}000) \cdot 1.04^{-10}}{\sum_{t=1}^{10}\left(30 \cdot 1.04^{-t}\right) + (1{,}000) \cdot 1.04^{-10}} = \frac{15 \cdot (Ia)_{\overline{10}|4\%} + 3{,}377.82}{30 \cdot a_{\overline{10}|4\%} + 675.56}$$

$$= \frac{629.88 + 3{,}377.82}{243.33 + 675.56} = 4.36146$$

The modified duration (at a nominal yield rate of $i^{(2)} = 8\%$) is:

$$D_{\text{mod}}\left(i^{(2)}\right) = \frac{D_{\text{mac}}}{1 + \frac{i^{(2)}}{2}} = \frac{4.36146}{1.04} = 4.19371$$

The estimated price at a yield of $i^{(2)} = 8.2\%$ is:

$$P(8.2\%) \approx P(8\%) - D_{\text{mod}}\left(i^{(2)}\right) \cdot P(8\%) \cdot (0.082 - 0.08) = 911.18$$

Note: The approximation slightly understates the actual price of 911.22.

Exercise (7.39)

A 10-year bond with a face amount of 1,000 pays semi-annual coupons at an 8% annual rate. Its nominal annual yield is 6% convertible semi-annually. Using the first-order modified approximation, estimate what the bond's price would be at a (nominal) yield of 5%. Compare this to the bond's actual price at a 5% yield.

Answers: Approximation is 1,230.04. Actual price is 1,233.84; estimate is lower than actual.

First-Order Macaulay Approximation

Now we will consider another formula for approximating the price of an asset after a change in interest rate. This formula, which uses Macaulay duration and the Taylor series, usually gives more accurate estimates than the first-order modified approximation method described above.

Consider an asset consisting of a series of positive cash flows that occur between time 0 and time n. As previously discussed, the *present value* of the asset at time 0 is a *decreasing* function of the interest rate used to calculate its present value. (This function is the curved line in Graph **(7.31)**.) As the interest rate increases, the present value of the asset decreases, so the derivative with respect to interest rate is negative.

On the other hand, the *accumulated* value of the cash flows (the future value of the asset at time n) is an *increasing* function of interest rate. That is, the accumulated value of the cash flows is larger at a higher interest rate than at a lower interest rate, so its derivative with respect to interest rate is positive.

There is, however, an *intermediate* valuation date *between* time 0 and time n for which this derivative is 0. We will call this date "D," and the value of the cash flows as of this date (based on the initial interest rate i_0) will be represented by $V(i_0)$. Then $V'(i_0)=0$, so for small changes in the interest rate (values of i close to i_0), the asset's value as of time D at rate i will be almost the same as at rate i_0: $V(i) \approx V(i_0)$ for i close to i_0.

The study note for this topic ("Using Duration and Convexity to Approximate Change in Present Value") provides a demonstration that this intermediate valuation date D (where the derivative is 0) is the Macaulay duration of the cash flows, $D_{\text{mac}}(i_0)$.

Given that the asset's present value (or price) is $P(i_0)$ and its Macaulay duration is $D_{\text{mac}}(i_0)$, we can write an expression for $V(i_0)$, its value at interest rate i_0 as of time D (that is, at $t = D_{\text{mac}}(i_0)$) and also for $V(i)$, the value at rate i:

$$V(i_0) = P(i_0)\cdot(1+i_0)^{D_{\text{mac}}(i_0)} \qquad V(i) = P(i)\cdot(1+i)^{D_{\text{mac}}(i_0)}$$

Since $V'(i_0) = 0$, the first-order Taylor series approximation gives us:

$$V(i) \approx V(i_0) + V'(i_0)\cdot(i-i_0) = V(i_0) + 0\cdot(i-i_0) = V(i_0)$$

We thus have $V(i) \approx V(i_0)$, which is equivalent to:

$$P(i)\cdot(1+i)^{D_{\text{mac}}(i_0)} \approx P(i_0)\cdot(1+i_0)^{D_{\text{mac}}(i_0)}$$

Rearranging the terms in this equation, we have a formula for estimating $P(i)$ using i, i_0, $P(i_0)$, and the Macaulay duration at i_0:

(7.40)

$$P(i) \approx P(i_0)\cdot\left(\frac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}$$

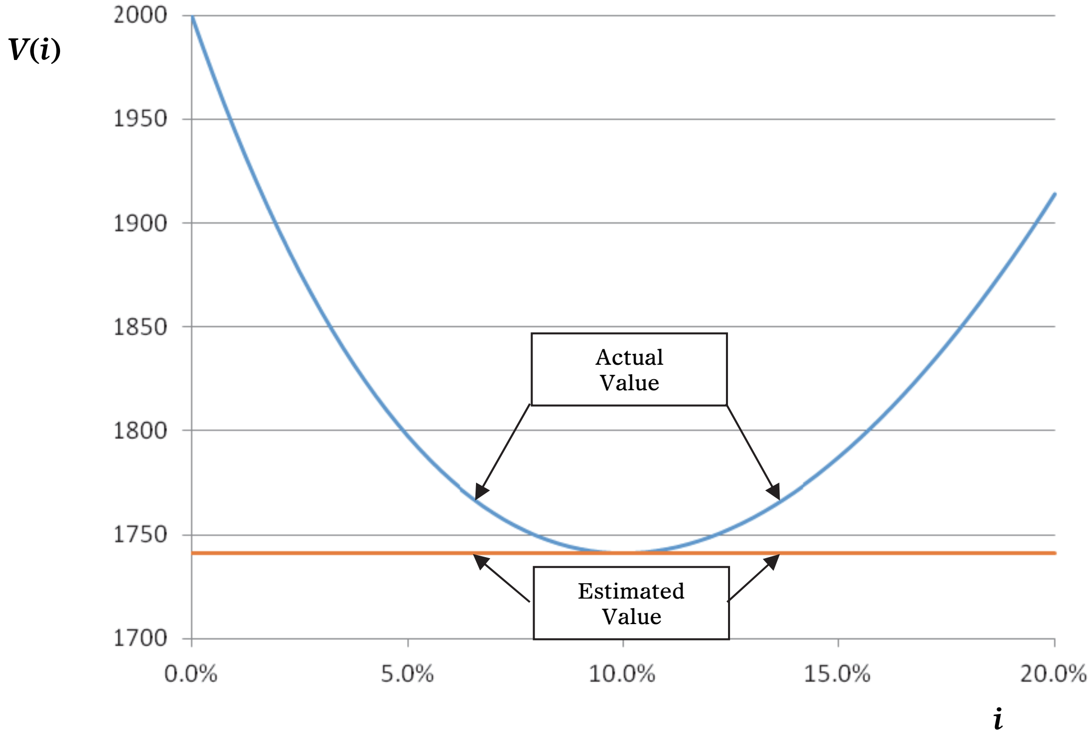
To examine the accuracy of this **first-order Macaulay approximation**, again consider a 20-year annuity-immediate with level payments of 100, valued at a 10% annual effective rate. The following table compares the actual prices at various interest rates to estimated prices that were calculated using the first-order Macaulay approximation (Equation **(7.40)**), and also to our earlier estimates using the first-order *modified* approximation.

i	Actual $P(i)$	D_{mac} est'd $P(i)$	Error in D_{mac} est.	D_{mod} est'd $P(i)$	Error in D_{mod} est.	D_{mac} error $\div D_{\text{mod}}$ error
6.0%	1,146.99	1,124.33	-22.66	1,083.79	-63.20	35.9%
9.0%	912.85	911.78	-1.07	909.47	-3.38	31.7%
9.9%	857.20	857.19	-0.01	857.17	-0.03	30.5%
10.0%	851.36	851.36	0.00	851.36	0.00	-
10.1%	845.58	845.57	-0.01	845.55	-0.03	30.4%
11.0%	796.33	795.43	-0.90	793.25	-3.08	29.2%
14.0%	662.31	651.10	-11.21	618.92	-43.39	25.9%

In this example, the first-order Macaulay approximation is significantly more accurate than the first-order modified approximation. The errors in the Macaulay approximations are only about one-third as large as the errors in the modified approximations.

As was the case for first-order modified approximations, the first-order Macaulay approximations always *underestimate* the actual value of $P(i)$. This is seen in the following graph, which shows the actual and estimated values of $V(i)$ (the value of the annuity as of time $t = D_{\text{mac}}(i_0)$).

Graph (7.41)



The curved line shows the actual values for $V(i)$. The horizontal straight line represents the estimated values, based on the values of $V(i_0)$ and $V'(i_0)$ at $i_0 = 10\%$. Since $V'(10\%) = 0$, the line of estimated values has a slope of 0.

Note: Because the estimated values of $V(i)$ at time $t = D_{\text{mac}}(i_0)$ are smaller than the actual values, the estimated present values at time 0 (the estimated prices) will also be smaller than the actual prices at rates other than $i_0 = 10\%$.

Example (7.42)

Consider again the investment in Example **(7.32)**, which pays 1,000 in one year, 2,000 at the end of the second year, and 3,000 at the end of the third year, and is purchased to yield $i = 0.10$. The yield then changes to 0.099. We can estimate the price at this new yield by applying **(7.40)**:

$$P(0.099) \approx P(0.10) \cdot \left(\frac{1.10}{1.099}\right)^{D_{\text{mac}}(0.10)} = 4,815.9279 \cdot \left(\frac{1.10}{1.099}\right)^{2.27925} = 4,825.9216$$

The actual new price (from **(7.32)**) is 4,825.9227.

The first-order Macaulay approximation is even more accurate than the first-order modified approximation (4,825.9067), and again the estimated price at $i = 0.099$ is slightly *less* than the actual price at that interest rate.

Exercise (7.43)

Again consider the situation of Exercise **(7.33)**. An investment that pays 1,000 in one year, 2,000 at the end of the second year and 3,000 at the end of the third year is purchased to yield $i = 0.08$.

Using the first-order Macaulay approximation, estimate the price of this investment if the yield increases to 0.082. Compare this estimate to the actual price at $i = 0.082$. Also compare the accuracy of this estimate to that of the first-order modified approximation that was calculated in Exercise **(7.33)**.

Answer: Mac approx. = 5000.8691; Actual = 5000.8740; mod approx = 5,000.80
Mac approx. < Actual; Mac approx. is more accurate than mod approx.

Example (7.44)

We can now apply the first-order Macaulay approximation to the facts of Example **(7.34)**, which involved an annual-coupon corporate bond priced to yield 6.5% annually. Its price is 969.56, and its Macaulay duration is $D_{\text{mac}} = 6.5572$. If the yield increases to 7.0%, the bond's price would decrease to approximately:

$$P(0.070) \approx P(0.065) \cdot \left(\frac{1.065}{1.070}\right)^{D_{\text{mac}}(0.065)} = 969.56 \cdot \left(\frac{1.065}{1.070}\right)^{6.5572} = 940.23$$

In Example **(7.34)** our first-order modified approximation for this bond's price at 7.0% was 939.71.

This problem was created using an 8-year bond with a 6% coupon rate. Its actual price at 7.0% is 940.29. The first-order Macaulay approximation of 940.23 provides a more accurate estimate than the first-order modified approximation.

Exercise (7.45)

The annual-coupon bond in Exercise **(7.35)** has an annual effective yield of 6%, a price of 965.35, and a Macaulay duration of $D_{\text{mac}} = 3.7177$.

(a) Compute a first-order Macaulay approximation for the price of the bond based on a 1.00% increase in yield.

(b) Is your answer relatively close to the approximate value calculated in Exercise **(7.35)** using the first-order modified approximation?

(c) Given that this is a 4-year bond, find its coupon rate and determine which approximation is more accurate.

Answers: (a) 932.23 (b) fairly close; they differ by 0.74 (c) 5%; Macaulay is more accurate

Section 7.6

Approximations Using Duration and Convexity

In Section 7.3, we learned that modified duration is the first derivative of price with respect to interest rate, expressed as a percent of price. Similarly, **Modified Convexity** is the *second* derivative of price with respect to interest rate, expressed as a percent of price:

(7.46)

$$C_{\text{mod}} = \frac{P''(i)}{P(i)}$$

For a series of n equally-spaced cash flows, the formula for C_{mod} is:

(7.47)

$$C_{\text{mod}} = \frac{\sum_{t=1}^{n} t(t+1)\cdot CF_t\,(1+i)^{-(t+2)}}{P(i)} = \frac{\sum_{t=1}^{n} t(t+1)\cdot CF_t\,(1+i)^{-(t+2)}}{\sum_{t=1}^{n} CF_t\,(1+i)^{-t}}$$

Note: As was mentioned in connection with modified duration, this formula is based on equally spaced cash flows, but we could develop a similar modified convexity formula for any pattern of cash flows, including an infinite series of cash flows or continuous cash flows.

A comment on notation:
We have used the notation C_{mod} for modified convexity because it is used in the Exam FM study note on duration. The various texts for Exam FM may use $\bar{c}$, *the word "Convexity*," or no symbol.

Using the definition of modified convexity in **(7.46)**, we can write an expression for $P''(i)$, which can then be used in the second-order Taylor series:

$$P''(i) = C_{\text{mod}} \cdot P(i)$$

Using the first three terms of the Taylor series, we have the second-order modified approximation for $P(i)$, which involves both modified duration and modified convexity:

(7.48)

$$P(i) \approx P(i_0) - D_{\text{mod}}(i_0) \cdot P(i_0) \cdot (i - i_0) + C_{\text{mod}}(i_0) \cdot P(i_0) \cdot \frac{(i - i_0)^2}{2}$$

The modified duration $\left(D_{\text{mod}}(i_0)\right)$ and modified convexity $\left(C_{\text{mod}}(i_0)\right)$ in Formula **(7.48)** are shown as functions of i_0 to indicate that they are calculated using an interest rate of i_0. The first two terms of this formula are the first-order modified approximation of present value: $P(i_0) - D_{\text{mod}}(i_0) \cdot P(i_0) \cdot (i - i_0)$. The third term $\left(+C_{\text{mod}}(i_0) \cdot P(i_0) \cdot \frac{(i - i_0)^2}{2}\right)$ is an adjustment to reflect the curvature (convexity) of $P(i)$. In effect, Formula **(7.48)** uses a quadratic to approximate $P(i)$, so it is more accurate (near i_0) than the linear first-order approximation of Equation **(7.28)**.

Note: For investments whose cash flows are fixed (i.e., their timing and amount are not affected by changes in market interest rates), C_{mod} is always positive (unlike D_{mod}, which is always negative). This is because the second derivative of $(1+i)^{-n}$ is positive: $(-n)\left(-[n+1]\right) \cdot (1+i)^{-(n+2)} = \left(n^2 + n\right) \cdot (1+i)^{-(n+2)} > 0$, assuming n and $(1+i)$ are both greater than 0. This means that the third term of Formula **(7.48)** *is always a positive adjustment to the first-order modified approximation. This is to be expected since, as noted earlier, the first-order approximation always understates the actual value (at interest rates other than i_0).*

Example (7.49)

Consider again an annuity-immediate with 20 annual payments of 100. At an annual effective interest rate of 10%, the present value and modified duration of this annuity, as calculated in Example **(7.30)**, are:

$$P(0.10) = 851.36 \qquad D_{\text{mod}} = 6.8255$$

The modified convexity for this asset (calculation not shown here) is 75.9068. Using **(7.48)** to create an expression for a second-order approximation to the price at a given interest rate i, we have:

$$P(i) \approx P(0.10) - D_{\text{mod}}(0.10) \cdot P(0.10) \cdot (i - 0.10) + C_{\text{mod}}(0.10) \cdot P(0.10) \cdot \frac{(i-0.10)^2}{2}$$

$$= 851.36 - 6.8255 \cdot (851.36) \cdot (i - 0.10) + 75.9068 \cdot (851.36) \cdot \frac{(i-0.10)^2}{2}$$

$$= 851.36 - 5{,}810.9523 \cdot (i - 0.10) + 64{,}623.74 \cdot \frac{(i-0.10)^2}{2}$$

The following table shows, for various values of i, the 2nd-order modified approximation of the price (present value) of this annuity. Also shown are the *actual* prices at these interest rates, and the errors in the 2nd-order modified approximation. For comparison, the errors in the 1st-order modified and 1st-order Macaulay approximations are also shown. Of these three methods, the 2nd-order modified approximation is the most accurate, and the 1st-order Macaulay approximation is the second-most accurate.

i	Actual $P(i)$	2nd-order D_{mod} est.	2nd-order D_{mod} error	1st-order D_{mod} error	1st-order D_{mac} error
6.0%	1,146.99	1,135.49	-11.50	-63.20	-22.06
9.0%	912.85	912.70	-0.15	-3.38	-1.07
9.9%	857.20	857.20	0.00	-0.03	-0.01
10.0%	851.36	851.36	0.00	0.00	0.00
10.1%	845.58	845.58	0.00	-0.03	-0.01
11.0%	796.33	796.48	0.15	-3.08	-0.90
14.0%	662.31	670.62	8.31	-43.39	-11.21

In addition to modified convexity, there is also **Macaulay convexity**, C_{mac}. The formula for Macaulay convexity is slightly simpler than the one for modified convexity:

(7.50)

$$C_{\text{mac}} = \frac{\sum_{t=1}^{n} t^2 \cdot CF_t (1+i)^{-t}}{P(i)} = \frac{\sum_{t=1}^{n} t^2 \cdot CF_t (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}$$

There is a simple algebraic relationship involving C_{mod}, C_{mac}, and D_{mac}:

(7.51)

$$C_{\text{mod}} = \frac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2}$$

Formulas **(7.50)** and **(7.51)** will be used to find modified convexity in the Example and Exercise below, and will also be used for immunization calculations in Section 7.8.

Note: When the term "convexity" is used alone, without specifying "modified" or "Macaulay," it generally indicates <u>modified</u> convexity. This is different from the use of "duration"; when the word "duration" is used alone, without specifying modified or Macaulay, it typically means <u>Macaulay</u> duration.

Example (7.52)

Consider again the annual-coupon corporate bond of Examples **(7.34)** and **(7.44)**. At its current yield of 6.5%, it has a price of 969.56, a Macaulay duration of 6.5572, and a Macaulay convexity of 48.3926. We can use the second-order modified approximation to estimate what the price would be if the yield increased to 7.0%.

We will first calculate the modified duration and modified convexity:

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{6.5572}{1.065} = 6.1570$$

$$C_{\text{mod}} = \frac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2} = \frac{48.3926 + 6.5572}{(1.065)^2} = 48.4470$$

Substituting these values into the second-order modified approximation formula, we have:

$$P(0.070) \approx P(0.065) - 6.1570 \cdot P(0.065) \cdot (0.070 - 0.065)$$

$$+48.4470 \cdot P(0.065) \cdot \frac{(0.070 - 0.065)^2}{2}$$

$$= 969.56 \cdot \left(1 - 6.1570 \cdot (0.005) + 48.4470 \cdot (0.005^2 / 2)\right) = 940.30$$

As mentioned in Example **(7.44)**, the bond's actual price at a 7.0% yield is 940.29. The second-order modified approximation provides a more accurate estimate than either the first-order modified approximation (939.71) or the first-order Macaulay approximation (940.23).

Exercise (7.53)

The bond that was analyzed in Exercises **(7.35)** and **(7.45)** yields 6% (an annual effective rate) and has a price of 965.35. The bond's Macaulay duration is $D_{\text{mac}} = 3.7177$ and its Macaulay convexity is $C_{\text{mac}} = 14.4095$.

Calculate a second-order modified approximation for the bond's price at a yield of 7.0%. Compare this approximation to the first-order modified and first-order Macaulay approximations, and also to the actual price at 7.0% (as shown in Exercise **(7.45)**.

Answers: 932.27 The second-order modified approximation is extremely accurate; Macaulay is second-best, and first-order modified is the least accurate.

Note: Section 6 of the study note on Duration and Convexity contains a formula for a second-order Macaulay approximation of present value, using Macaulay duration and Macaulay convexity. Since this is not part of the Exam FM syllabus, it will not be discussed in this manual.

Section 7.7

The Duration of a Portfolio

Up to this point we have concentrated on finding the duration for a single asset. It is more common for an investor to own a "portfolio" consisting of a number of different investments. Since duration is used to estimate sensitivity to changes in interest rates, a portfolio investor would like to know the portfolio's duration. We will begin by looking at a simple example where the portfolio in question has only two assets, both priced at par.

Example (7.54)

An investor can buy two annual-coupon bonds:

Bond a is an annual-coupon bond with a term of 4 years and a coupon rate of 5%. It is priced at 1,000 to yield 5%. Its modified duration is $D^a_{\text{mod}} = 3.5460$.

Bond b is an annual-coupon bond with a term of 8 years and a coupon rate of 7%. It is priced at 1,000 to yield 7%. Its modified duration is $D^b_{\text{mod}} = 5.9713$.

The investor buys 3 of Bond a for 3,000 and 2 of Bond b for 2,000. By performing a first-order modified approximation separately for each bond, she can estimate the change in her portfolio's value if interest rates on both bonds increase by 0.10%:

Bond a: $$P^a(0.051) \approx P^a(0.05) - (D^a_{\text{mod}}) \cdot P^a(0.05)(0.051 - 0.05)$$
$$= 3{,}000 - (3.5460)(3{,}000)(0.001) = 2{,}989.36$$

Bond b: $$P^b(0.071) \approx P^b(0.07) - (D^b_{\text{mod}}) \cdot P^b(0.07)(0.071 - 0.07)$$
$$= 2{,}000 - (5.9713)(2{,}000)(0.001) = 1{,}988.06$$

For the entire portfolio with original value $P_a(0.05) + P_b(0.07) = 5000$,

her approximation of the value after the interest rates increase is:

$$P^a(0.051) + P^b(0.071) \approx 2{,}989.36 + 1{,}988.06 = 4{,}977.42$$

Now we will perform this approximation by finding a modified duration for the portfolio and applying the first-order modified approximation formula to the total value of the portfolio.

Just as the duration of an asset is a weighted average of the durations of its individual cash flows, it can be shown that the duration of a portfolio is the weighted average of the durations of its individual assets. The weights are based on the prices (present values) of the assets. In this example, the present values of the two assets are 3,000 and 2,000, so the weights applied to their durations are 0.6 and 0.4.

Because we are calculating the *modified* duration of the portfolio $\left(D_{\text{mod}}^{\text{portfolio}}\right)$, we apply these weights to the *modified* durations of the individual assets.

$$D_{\text{mod}}^{portfolio} = \frac{3{,}000 \cdot D_{\text{mod}}^{a} + 2{,}000 \cdot D_{\text{mod}}^{b}}{5{,}000} = 0.6 \cdot 3.5460 + 0.4 \cdot 5.9713 = 4.5161$$

Then, using $P^{tot}(orig)$ to represent the portfolio's original value and $P^{tot}(+0.001)$ to represent its value after the 0.1% increase in interest rates, we have:

$$\begin{aligned} P^{tot}(+0.001) &\approx P^{tot}(orig) - (D_{\text{mod}}^{portfolio}) \cdot P^{tot}(orig)\left(0.001\right) \\ &= 5{,}000 - \left(4.5161\right)\left(5{,}000\right)\left(.001\right) = 4{,}977.42 \end{aligned}$$

This is, of course, the same value we calculated by estimating the values of the two bonds individually and adding their estimated values.

It is very important to understand that the single duration for the entire portfolio can be used as a *rough measure of interest sensitivity* for the entire portfolio. But if we use it to calculate *changes* in the portfolio's value, we must assume that the interest rate for every asset in the portfolio changes by the same amount. (In the above calculation, the yield for each bond was assumed to increase by 0.001, so it was appropriate to approximate the portfolio's new value using $D_{\text{mod}}^{portfolio}$.)

Note: In the preceding example, we did not use the Macaulay approximation method to estimate the portfolio's price at the new interest rates. Although the Macaulay approximation method is generally more accurate than the modified approximation method, its formula cannot be applied if the assets in the portfolio have different interest rates. The modified approximation method, on the other hand, can be applied, as long as it is reasonable to assume that the change in interest rate will be the same for all assets in the portfolio. It is important to calculate the portfolio's modified duration by finding the weighted average of the modified durations of the individual assets. It is not possible to calculate the portfolio's modified duration by finding its Macaulay duration and dividing by $\left(1+i\right)$, because there is no single value of i for the portfolio.

The reasoning we applied in Example **(7.54)** works in general. Suppose that a portfolio consists of m investments with present values $X_1, X_2, ..., X_m$, and that the modified durations of these investments are $D^1_{\text{mod}}, D^2_{\text{mod}}, ..., D^m_{\text{mod}}$. The modified duration of the portfolio is the weighted average of the modified durations of the individual investments, with each investment having a weight equal to its percent of the total portfolio value:

(7.55)

$$D^{portfolio}_{\text{mod}} = w_1 \cdot D^1_{\text{mod}} + w_2 \cdot D^2_{\text{mod}} + ... + w_m \cdot D^m_{\text{mod}}$$
$$\text{where } w_k = \left(\frac{X_k}{X_1 + X_2 + ... + X_m} \right)$$

In summary, if it can be assumed that all investments in a portfolio will undergo the same shift in interest rate, then we can calculate the modified duration of the portfolio as the weighted average of the modified durations of the individual investments, and use the portfolio's modified duration to estimate the change in the portfolio's value when interest rates change. In calculating the portfolio's modified duration, the weight for each investment is equal to its percentage of the portfolio's total present value.

Example (7.56)

An investor has a portfolio containing 30,000 worth of a 2-year bond with a modified duration of 1.96; 20,000 worth of a 3-year bond with a modified duration of 2.88; and 50,000 worth of a 5-year bond with a modified duration of 4.59. Thus her portfolio has weights of 30% in 2-year bonds, 20% in 3-year bonds and 50% in 5-year bonds. The modified duration of the entire portfolio is:

$$0.30(1.96) + 0.20(2.88) + 0.50(4.59) = 3.46$$

Exercise (7.57)

An investor has a portfolio containing 10,000 worth of a 2-year bond with a modified duration of 1.95; 40,000 worth of a 4-year bond with a modified duration of 3.71; and 50,000 worth of a 6-year bond with a modified duration of 5.50. Find the modified duration of the entire portfolio.

Answer: 4.43

When all investments undergo the same interest rate shift, Δi, we say that there is a **parallel shift in the yield curve.** Yield curve shifts are not always parallel, and when they are not, this weighted average approach to portfolio duration will be less accurate in determining interest rate sensitivity for the portfolio. However, it is common to use such value-weighted averages of modified duration (or Macaulay duration) as a *rough* indication of a portfolio's volatility.

Section 7.8

Immunization

In Section 7.1 we dealt with the simple situation where the cash flows of assets and liabilities could be matched exactly, but we pointed out that in many cases exact matching is not possible. **Immunization** is a method of protecting a portfolio against the adverse effects of interest rate changes in these more complex cases.

Suppose that the current interest rate for valuing assets and liabilities is i_0. The present value of assets and the present value of liabilities, both at rate i_0, will be denoted by $PV^A(i_0)$ and $PV^L(i_0)$, respectively.

We will begin with a portfolio of assets whose value is just sufficient to support a particular set of liabilities. In other words, we know the liabilities and their present value at interest rate i_0, and we will choose a set of assets that has the same present value at i_0. This is our first condition for immunization:

(7.58)

Immunization Condition 1

Present Values are equal:
$$PV^A(i_0) = PV^L(i_0)$$

Now suppose that there is a small change in the interest rate from i_0 to a value i near i_0. We would like for this change to cause the present value of the assets to be *greater* than the present value of liabilities, thus increasing the surplus:

$$PV^A(i_0) > PV^L(i) \qquad \text{Increase in Surplus } = PV^A(i) - PV^L(i)$$

We can create this situation, where a small change in interest rate has a positive effect on surplus, by selecting assets with duration and convexity that bear a certain relation to the duration and convexity of the liabilities.

We know that if the assets' modified duration is *greater* than the liabilities' modified duration, then if interest rates increase (to a new rate i that is slightly greater than i_0) the assets' value will *decrease* by a *greater* amount than the liabilities' value. This is not what we want. So the assets' modified duration cannot be *greater* than that of the liabilities.

On the other hand, if the assets' modified duration is *less* than the liabilities' modified duration, then an increase in the interest rate will cause the assets' value to decrease by a *smaller* amount than the decrease in the value of the liabilities. That would be a favorable result. But now a *decrease* in the interest rate would boost liability values more than asset values, so we would not be protected against a decrease in interest rates. This means that the assets' modified duration cannot be *less* than that of the liabilities.

Therefore, in order to protect the portfolio against both increases and decreases in the interest rate, the modified duration of the assets can't be greater than or less than the modified duration of the liabilities. Instead, we need a portfolio of assets whose modified duration is *equal to* the modified duration of the liabilities. Then a small change in interest rates, either up or down, will have approximately the same effect on assets and liabilities. Our first condition for immunization was that the assets and liabilities must have equal present values. Now our second condition is that the assets and liabilities must have equal modified durations:

(7.59)

Immunization Condition 2

Modified Durations are equal:

$$D_{\text{mod}}^{A}\left(i_0\right)=D_{\text{mod}}^{L}\left(i_0\right)$$

If the assets and liabilities have equal present values $\left(PV^{A}\left(i_0\right)=PV^{L}\left(i_0\right)\right)$ and equal modified durations $\left(D_{\text{mod}}^{A}\left(i_0\right)=D_{\text{mod}}^{L}\left(i_0\right)\right)$, then a first-order modified approximation of their values after a change in interest rate will produce identical estimates for $PV^{A}\left(i\right)$ and $PV^{L}\left(i\right)$. Using Formula **(7.28)**, we see that all present values and durations in the following two equations are equal:

$$PV^{A}\left(i\right)\approx PV^{A}\left(i_0\right)-D_{\text{mod}}^{A}\left(i_0\right)\cdot PV^{A}\left(i_0\right)\cdot\left(i-i_0\right)$$
$$PV^{L}\left(i\right)\approx PV^{L}\left(i_0\right)-D_{\text{mod}}^{L}\left(i_0\right)\cdot PV^{L}\left(i_0\right)\cdot\left(i-i_0\right)$$

Equal modified durations assure that the assets and liabilities will have approximately equal values after a small change in interest rate. However, the first-order modified approximation does not reflect the curvature (convexity) of the *PV* functions at i_0. The second-order modified approximation (Equation **(7.48)**) does reflect convexity. It consists of the first-order approximation shown above, plus a convexity term. For our assets and liabilities, the convexity terms are:

$$C_{\text{mod}}^{A}\left(i_0\right)\cdot PV^{A}\left(i_0\right)\cdot\frac{\left(i-i_0\right)^2}{2}$$
$$C_{\text{mod}}^{L}\left(i_0\right)\cdot PV^{L}\left(i_0\right)\cdot\frac{\left(i-i_0\right)^2}{2}$$

The coefficient of $C_{\text{mod}}(i_0)$ is always positive $\left(\text{that is, } PV(i_0)\cdot\dfrac{(i-i_0)^2}{2}>0\right)$, regardless of whether i is greater or less than i_0. So a larger value of $C_{\text{mod}}(i_0)$ results in a larger present value (for values of i near i_0). Therefore, we want our assets to have a *larger* modified convexity than our liabilities. This is the third and final condition for immunization:

(7.60)

Immunization Condition 3

Modified Convexity is greater for Assets than for Liabilities:

$$C_{\text{mod}}^{A}(i_0) > C_{\text{mod}}^{L}(i_0)$$

When these three conditions are satisfied, the portfolio of assets and liabilities is said to be "immunized," which means that for values of i near i_0 the present value of the asset cash flows exceeds the present value of the liability cash flows. The concept of immunization based on these three conditions was developed by F. M. Redington, a British actuary, so it is referred to as **Redington Immunization**.

In summary, to achieve Redington immunization, the asset and liability cash flows must satisfy the following 3 conditions at i_0.

1. Assets and Liabilities have equal values:
$$PV^{A}(i_0) = PV^{L}(i_0)$$

2. Assets and Liabilities have equal modified durations
$$D_{\text{mod}}^{A}(i_0) = D_{\text{mod}}^{L}(i_0)$$

3. Assets have greater modified convexity than Liabilities:
$$C_{\text{mod}}^{A}(i_0) > C_{\text{mod}}^{L}(i_0)$$

When the three conditions for Redington immunization are satisfied, we know that assets currently equal liabilities (Condition 1.), that a small change in interest rates will cause about the same increase or decrease in assets and liabilities (Condition 2.), and that, in fact, the value of the assets after a small change in interest rates will exceed the value of the liabilities (Condition 3.).

The three conditions for Redington immunization can also be expressed in terms of the asset and liability cash flows, A_t and L_t:

(7.61)

Present Value:
$$\sum_t A_t v_{i_0}^t = \sum_t L_t v_{i_0}^t$$

(7.62)

Duration:
$$\sum_t t \cdot A_t v_{i_0}^t = \sum_t t \cdot L_t v_{i_0}^t$$

(7.63)

Convexity:
$$\sum_t t^2 \cdot A_t v_{i_0}^t > \sum_t t^2 \cdot L_t v_{i_0}^t$$

You will note that the second and third criteria (Equations **(7.62)** and **(7.63)**) have been expressed in terms of Macaulay duration and convexity, because these values are slightly easier to calculate than modified duration and convexity. If these criteria are satisfied (that is, if $D_{\text{mac}}^A(i_0) = D_{\text{mac}}^L(i_0)$ and $C_{\text{mac}}^A(i_0) > C_{\text{mac}}^L(i_0)$), then the requirements for modified duration and convexity (Equations **(7.59)** and **(7.60)**) will also be satisfied. You can prove this using the relationships between the Macaulay and modified measures, which are: $D_{\text{mod}} = \dfrac{D_{\text{mac}}}{1+i}$ and $C_{\text{mod}} = \dfrac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2}$.

To immunize a portfolio, the portfolio manager examines the known liabilities and selects assets such that Equations **(7.61)**, **(7.62)**, and **(7.63)** are satisfied. In real-world cases, where there can be thousands of liabilities and possible asset choices, computers are used to perform this task. In the above analysis, we have assumed that a single interest rate (i_0) is appropriate for present valuing all the asset and liability cash flows, and that all assets and liabilities will undergo the same change in interest rates. Real world situations are much more complex, and portfolio managers use more sophisticated analysis. However, we can illustrate the basic idea of immunization with a simple example in which there are only two assets and one liability, all of which are valued at the same interest rate. (This is a level of problem that might appear on Exam FM.)

Example (7.64)

You have a liability that requires a payment of 120,000 at time 6. You wish to create an immunized portfolio by buying two zero-coupon bonds with maturities at times 2 and 12. The valuation interest rate is $i_0 = 0.05$. Thus you know that $L_6 = 120,000$, and you need to select face amounts, A_2 and A_{12}, for the two bonds. We can create a system of equations to find A_2 and A_{12} using the first 2 conditions for Redington immunization:

Present Value: $$\frac{A_2}{1.05^2} + \frac{A_{12}}{1.05^{12}} = \frac{120,000}{1.05^6}$$

Duration: $$\frac{2 \cdot A_2}{1.05^2} + \frac{12 \cdot A_{12}}{1.05^{12}} = \frac{6 \cdot (120,000)}{1.05^6}$$

This reduces to a system of two equations in two unknowns:

$$0.90703A_2 + 0.55684A_{12} = 89,545.8476$$

$$1.81406A_2 + 6.68205A_{12} = 537,275.0856$$

The solution, rounded to dollars and cents, is:

$$A_2 = 59,234.58 \qquad A_{12} = 64,324.59$$

Now that we have found the face amounts, A_2 and A_{12}, needed to match the present values and durations of our assets and liabilities, we can check the convexity condition to see whether the portfolio is immunized:

$$\sum t^2 A_t v_{i_0}^t = 2^2\left(\frac{59,234.58}{1.05^2}\right) + 12^2\left(\frac{64,324.59}{1.05^{12}}\right) = 5,372,750.80$$

$$\sum t^2 L_t v_{i_0}^t = 6^2\left(\frac{120,000}{1.05^6}\right) = 3,223,650.51$$

We see that $\sum t^2 A_t v_{i_0}^t > \sum t^2 L_t v_{i_0}^t$, so the portfolio is immunized.

Exercise (7.65)

You have a single liability of 100,000 payable at time 5. You wish to create an immunized portfolio by buying two zero-coupon bonds that mature at times 3 and 10. The valuation interest rate is $i_0 = 0.06$. Find the face amounts of the two bonds and verify the portfolio is immunized.

Answer: $A_3 = 63,571.17$, $A_{10} = 38,235.02$ $\sum t^2 A_t v_{i_0}^t = 2,615,403.61 > \sum t^2 L_t v_{i_0}^t = 1,868,145.43$

Note that the conditions for Redington immunization protect us against a *small* change in the interest rate to a value i near i_0, but not necessarily against a larger change in interest rate. This is because D_{mod} and C_{mod} describe the slope and curvature of the *PV* functions at i_0, but they may not reflect the behavior of these functions at interest rates that are *not* near i_0. For this reason, portfolio managers often need to consider re-structuring their portfolios after a substantial shift in interest rates.

In some cases we get lucky and our portfolio is protected against *any* change in interest rates. When this happens, the portfolio is said to be **fully immunized**, meaning that $PV^A(i) > PV^L(i)$ for any positive interest rate $i \neq i_0$. The portfolios in Example **(7.64)** and Exercise **(7.65)** were fully immunized. It can be shown that when a single liability cash flow, L_s, is immunized by two asset cash flows, A_{t_1} and A_{t_2}, and all assets and liabilities are subject to the same interest rate, then if $t_1 < s < t_2$, the portfolio is *fully* immunized.

This principle of full immunization results from the fact that a set of cash flows will have greater convexity if they are more "spread out" over time. In the example of the previous paragraph, the liability cash flow(s) happened to be concentrated at time *s*, while the asset cash flows occurred before and after time *s*. Because they were more spread out, they had greater convexity, so as long as the present values and durations are matched, we can be certain that the portfolio is "fully immunized."

The above discussion of full immunization might imply that it is always best for asset cash flows to cover a very wide range of future times (i.e., that they should have a wide "dispersion"), since this increases the assets' convexity. And that would be true *if* we could be sure that future interest rate changes will always involve a parallel shift in the yield curve. But if the change is *not* a parallel shift (e.g., if short-term rates rise and nothing else changes, or if short-term rates fall and long-term rates rise), then there could be very adverse effects on surplus if the asset cash flows are much more widely dispersed than the liability cash flows.

The general wisdom is that the asset cash flows should be a bit more widely dispersed than the liability cash flows (in order for the assets to have greater convexity). However, they should not cover a significantly wider range of times, due to the risk of a non-parallel shift in interest rates.

Section 7.9

Stocks and Other Investments

Bonds are widely used as investment vehicles for insurance companies. However, there are many other financial instruments that insurers use to invest their assets. Examples include stocks, mutual funds, CDs, money market funds, and mortgage-backed securities. A basic understanding of these securities is required by the Exam FM syllabus, and we will review them here.

Stocks

If you own a share of a corporation's stock you are a partial owner of the corporation. Shareholders are typically paid a portion of the corporation's profits; these payments to shareholders are called **dividends**. A shareholder may benefit from owning a stock by receiving dividends, and also by selling the stock at a price higher than the purchase price, thus realizing a capital gain. However, a shareholder may also suffer a capital loss if the price of the stock declines and it is sold for less than the purchase price.

There are many ways to value a stock. We will discuss a relatively simple valuation method: the **dividend discount model**. This method assumes that the fair price of a stock is the present value of all the expected future dividends at a valuation interest rate i. If Div_k is the dividend expected to be paid at time k, we have:

(7.66)

$$P = \sum_k \frac{Div_k}{(1+i)^k}$$

A common technique for projecting the amounts of future dividends is to assume a constant percentage growth rate for the dividends. If the dividend expected at the end of the current period is Div and the constant percentage growth rate is g, then based on an effective interest rate of i per period, the price as of the beginning of the current period is given by:

$$\begin{aligned} P &= \frac{Div}{1+i} + \frac{Div\cdot(1+g)}{(1+i)^2} + \frac{Div\cdot(1+g)^2}{(1+i)^3} + \ldots \\ &= \frac{Div}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \ldots\right] \\ &= \frac{Div}{1+i}\left[\frac{1}{1-\left(\frac{1+g}{1+i}\right)}\right] = \frac{Div}{(i-g)} \end{aligned}$$

Thus the price of a stock can be estimated using the **dividend growth model**:

(7.67)

$$P = \frac{Div}{(i-g)}$$

This dividend growth model is widely used, and it appears in some of the sample exam problems at the end of this module.

Note: The "dividend discount model" is the technique of analyzing a stock's value as the present value of its expected dividends. The "dividend growth model" is a special case of the dividend discount model wherein dividends are assumed to increase geometrically (i.e., by a constant annual percentage).

Example (7.68)

A stock is expected to pay a dividend of 5 per share in one year. The valuation interest rate is $i = 0.05$. If each subsequent annual dividend is expected to be 3% larger than the preceding dividend, then based on the dividend growth model, the value of a share of the stock now is:

$$P = \frac{Div}{(i-g)} = \frac{5}{0.05-0.03} = 250$$

Stock valuation can also be performed using spot rates or forward rates instead of a single valuation rate. Using the notation s_n for the n-year spot rate and $i_{n-1,n}$ for the forward rate in year n (i.e., the interest rate for the period from time n-1 to time n), the valuation formulas are:

Spot rate: $$P = \sum_{k=1}^{\infty} \frac{Div_k}{(1+s_k)^k}$$

Forward rate: $$P = \frac{Div_1}{(1+i_{0,1})} + \frac{Div_2}{(1+i_{0,1})(1+i_{1,2})} + \ldots.$$

These two formulas are equivalent, provided that the spot rates and forward rates are consistent. Ideally, the single valuation rate i in Formula **(7.66)** should be chosen so as to give the same value as the above two formulas.

Mutual funds

A mutual fund provides a means to invest in a pool of stocks selected by professional investment managers. An investor buys unit shares of the fund by paying the current unit price per share to the fund managers. The managers use the total amount of cash received from the sale of unit shares to buy stocks that they select. An investor can redeem his shares by selling them back to the mutual fund and receiving the current **net asset value** (NAV) per share. The net asset value per share is equal to the total value of the fund's invested assets, divided by the total number of shares outstanding. The managers may sell stock to raise the cash needed to redeem shares for investors, or buy additional stock when investors purchase more shares of the mutual fund.

There are a wide variety of mutual funds with differing strategies. A fund might specialize in the stocks of one industry (e.g., an energy stock fund), a particular type of stock (e.g., a growth stock fund), or a geographical region (e.g., a Pacific Rim fund). Mutual funds may also specialize in other securities besides stock (e.g., a bond fund or a real estate fund).

Mutual funds give an investor the advantage of *diversification*, since the fund can invest in many more different stocks (or other securities) than most investors could afford to buy individually. The ultimate in diversification comes from an index fund, which buys shares in *all* of the stocks of a particular index such as the S&P 500, with purchases weighted so that the fund's share value will track the index.

Note: The above discussion describes <u>open-end mutual funds</u>. There are also <u>closed-end mutual funds</u> and <u>exchange-traded funds</u>. Closed-end funds represent a very small part of the mutual fund market. They have a fixed number of shares, and the fund does not redeem shares for shareholders. Instead, shares are traded between buyers and sellers on exchanges.

Exchange-traded funds (ETFs) are much more popular than closed-end funds. They are basically index funds that trade on stock exchanges. ETFs are available for a very large number of indexes. ETF shares generally trade at prices that are very close to their net asset values (NAVs). This is because shares can be created or retired, and this feature assures that supply and demand do not cause the price to vary significantly from the NAV.

CDs

CD is an abbreviation for **certificate of deposit.** CDs are offered by banks, credit unions, and savings and loan associations. To buy a CD, an investor makes a deposit that will earn a stated rate of interest for a specified period of time. A minimum deposit amount is required. As an example, a CD might earn a 1.80% annual effective rate for a term of 5 years and require a 2,500 minimum deposit.

The interest earned by a CD may be paid to the CD owner periodically (e.g., monthly, quarterly, or annually), or it may accumulate and be paid with the principal when the CD matures. If the owner redeems the CD before its maturity date, there is a charge deducted (an early termination penalty) that might, for example, equal 3 months or 6 months of interest.

CDs have the advantage of being insured by the Federal Deposit Insurance Corporation (FDIC), which protects deposits up to $250,000 in the event the issuing bank fails.

Money Market Funds

A money market fund is a mutual fund that invests in short-term secure investments like Treasury bills. It functions very much like a bank savings account, paying interest and possibly allowing the investor to write checks. The fund is managed in an effort to earn a higher interest rate than a bank account. Although the fund is managed with the goal of avoiding any loss of principal, it is not guaranteed against losses and is not insured by the FDIC.

Mortgage-Backed Securities

Interest rates on mortgages are typically higher than the rates earned on bank accounts or Treasury bonds, making mortgages an attractive investment for some investors. Mortgage lenders gather large numbers of mortgages into pools of loans and create **mortgage-backed securities (MBS).** For example, a lender who has just made 100 loans of $200,000 each can combine these into a $20,000,000 MBS. An insurance company with a large amount of money to invest can buy this security and receive mortgage rates of interest without the need to manage all of the individual loans.

A mortgage borrower may default. To provide security for MBS investors, mortgage loans can be insured by the Federal Housing Administration (FHA). In addition, many MBS pools are put into GNMA ("Ginnie Mae") securities, which are guaranteed by the Government National Mortgage Association.

Analysis of MBS securities is quite complex. Borrowers can prepay their mortgages, which makes the timing of payments from an MBS uncertain. Analysts use standard prepayment models and default assumptions to value MBS securities, but the assumptions in their calculations may not be accurate, leading to unexpected losses on mortgage-backed securities, as happened during the financial crisis of 2007-2008.

Section 7.10

Formula Sheet

If an investment consists of n equally spaced cash flows (not necessarily equal in amount) that occur at times 1 through n, we have:

Cash flows: $CF_1, \ldots, CF_n$

Investment price: $P = vCF_1 + v^2 CF_2 + \ldots + v^n CF_n = \sum_{k=1}^{n} CF_t (1+i)^{-t}$

Weights for D_{mac}: $w_t = \dfrac{CF_t \cdot (1+i)^{-t}}{P} = \dfrac{CF_t \cdot (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}$

Macaulay duration: $D_{\text{mac}} = 1 \cdot w_1 + 2 \cdot w_2 + \ldots + n \cdot w_n = \dfrac{\sum_{t=1}^{n} t \cdot CF_t (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}$

Modified duration: $D_{\text{mod}}(i) = -\dfrac{\left(\frac{dP}{di}\right)}{P} = \dfrac{\sum_{t=1}^{n} t \cdot CF_t (1+i)^{-(t+1)}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}} = \dfrac{D_{\text{mac}}}{1+i}$

Modified duration at a nominal rate $i^{(m)}$: $D_{\text{mod}}\left(i^{(m)}\right) = -\dfrac{\left(\frac{dP}{di^{(m)}}\right)}{P} = \dfrac{D_{\text{mac}}\left(i^{(m)}\right)}{1+\frac{i^{(m)}}{m}}$

Modified duration at a force of interest δ: $D_{\text{mod}}(\delta) = -\dfrac{\left(\frac{dP}{d\delta}\right)}{P} = D_{\text{mac}}(\delta)$

Macaulay duration of a level-payment annuity-immediate: $D_{\text{mac}} = \dfrac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}$

Macaulay duration (measured in coupon periods) for an n-period bond with face amount F, coupon Fr, and redemption value C:

$$D_{\text{mac}} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{Fr\left(a_{\overline{n}|}\right) + Cv^n} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{\text{Bond Price}}$$

For an annual-coupon "par" bond ($F=C$ and $r=i$): $D_{\text{mac}} = \ddot{a}_{\overline{n}|i}$

For a par bond with coupons payable m times per year: $D_{mac} = \ddot{a}_{\overline{n}|}^{(m)}$

For a zero-coupon bond maturing in n periods: $D_{mac} = n$

Convexity

Convexity is a measure of the curvature (second derivative) of the price function.

Modified convexity:

$$C_{\text{mod}} = \frac{P''(i)}{P(i)} = \frac{\sum_{t=1}^{n} t(t+1)\cdot CF_t(1+i)^{-(t+2)}}{P(i)} = \frac{\sum_{t=1}^{n} t(t+1)\cdot CF_t(1+i)^{-(t+2)}}{\sum_{t=1}^{n} CF_t(1+i)^{-t}}$$

Macaulay convexity: $$C_{\text{mac}} = \frac{\sum_{t=1}^{n} t^2\cdot CF_t(1+i)^{-t}}{P(i)} = \frac{\sum_{t=1}^{n} t^2\cdot CF_t(1+i)^{-t}}{\sum_{t=1}^{n} CF_t(1+i)^{-t}}$$

$$C_{\text{mod}} = \frac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2}$$

Formulas to approximate price using Duration and Convexity

First-order modified approximation:

$$P(i) \approx P(i_0) + P'(i_0)\cdot(i-i_0) = P(i_0) - D_{\text{mod}}(i_0)\cdot P(i_0)\cdot(i-i_0)$$

First-order Macaulay approximation:

$$P(i) \approx P(i_0)\cdot\left(\frac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}$$

Second-order modified approximation:

$$P(i) \approx P(i_0) + P'(i_0)\cdot(i-i_0) + P''(i_0)\cdot\frac{(i-i_0)^2}{2}$$

$$= P(i_0) - D_{\text{mod}}(i_0)\cdot P(i_0)\cdot(i-i_0) + C_{\text{mod}}(i_0)\cdot P(i_0)\cdot\frac{(i-i_0)^2}{2}$$

The **modified duration of a portfolio** is a weighted average of the modified durations of its component investments, with each investment having a weight equal to its fraction of the total portfolio value:

$$D_{\text{mod}}^{\text{portfolio}} = w_1\cdot D_{\text{mod}}^1 + w_2\cdot D_{\text{mod}}^2 + \ldots + w_m\cdot D_{\text{mod}}^m,$$

where $w_k = \left(\dfrac{X_k}{X_1 + X_2 + \ldots + X_m}\right)$ and X_k is the value of the k^{th} investment

Note: This weighted-average method for determining the modified duration of a portfolio does not require that all of the investments in the portfolio be valued at the same interest rate. However, if the modified duration of a portfolio is used to estimate how the portfolio's value will change when interest rates change, then one must assume a parallel shift in the interest rates that apply to the portfolio.

Immunization

If a portfolio satisfies the conditions for **Redington immunization**, then after a (small) parallel shift in interest rates, the value of the assets will exceed the value of the liabilities. The three conditions for Redington immunization are:

Equal Present Values: $PV^A(i_0) = PV^L(i_0)$

Equal Modified Durations: $D^A_{\text{mod}}(i_0) = D^L_{\text{mod}}(i_0)$

Greater Modified Convexity for Assets: $C^A_{\text{mod}}(i_0) > C^L_{\text{mod}}(i_0)$

In terms of the asset and liability cash flows (A_t and L_t) at each time t, the conditions for Redington immunization are:

Equal Present Values: $\sum A_t v_{i_0}^t = \sum L_t v_{i_0}^t$

Equal Macaulay Durations: $\sum t A_t v_{i_0}^t = \sum t L_t v_{i_0}^t$

Greater Macaulay Convexity for Assets: $\sum t^2 A_t v_{i_0}^t > \sum t^2 L_t v_{i_0}^t$

Note: These criteria are expressed in terms of Macaulay duration and Macaulay convexity. But if all three criteria are satisfied, then the values of modified duration and modified convexity will also satisfy the criteria for immunization.

Full immunization of a portfolio can be achieved when a portfolio consists of a single liability cash flow and two asset cash flows. If the first two conditions for Redington immunization are satisfied (equal present values and equal modified durations) and the asset cash flows occur one before and one after the liability cash flow, then the portfolio is fully immunized. Full immunization means that an interest rate change of *any* magnitude will cause the value of the assets to exceed the value of the liability.

Stock Valuation

Dividend Discount Model for valuing a stock:

$$P = \sum_{k=1}^{\infty} \frac{Div_k}{(1+i)^k}$$

Dividend growth model (for valuing a stock assuming that dividends will increase by a factor of $(1+g)$ each period):

$$P = \frac{Div}{(i-g)}$$

The above formulas for the dividend discount model and for the dividend growth model both use a valuation date that is exactly one period before the next dividend due date. To calculate P as of a different date, the formula value must be adjusted for the period between the formula's valuation date and the desired valuation date.

Section 7.11

Basic Review Problems

1. A company must make payments of 3,000 and 5,000 at the end of years 2 and 4, respectively. The only investments available to the company are the following two zero-coupon bonds:

Term (years)	Annual Effective Yield	Maturity Value
2	5.5%	1,000
4	6.8%	100

Find the cost of exactly matching the liabilities.

2. John has liabilities that require payments of 1,000 due 6 months from now and another 1,000 due one year from now. There are two available investments:

a. a 6-month bond with face amount 1,000, 4% nominal annual coupon rate payable semi-annually, and 3% nominal annual yield rate convertible semi-annually

b. a 1-year bond with face amount 1,000, 5% nominal annual coupon rate payable semi-annually, and 6% nominal annual yield rate convertible semi-annually

Find the amount of each bond to purchase and the total cost of the bonds.

3. An investment pays 1,000 in three years and 3,000 at the end of the fourth year. An investor has purchased it to yield an annual effective rate $i = 0.075$. Find the Macaulay duration and the modified duration.

4. An annual-coupon corporate bond is priced to yield 7% annually and has a price of 940.29 and a Macaulay duration of 6.5317. Estimate the change in price if the bond's yield to maturity increases by 0.10%, using:
(a) the first-order Macaulay approximation
(b) the first-order modified approximation

5. An investor has a portfolio containing 1,000 worth of a 3-year bond with a modified duration of 2.7; 4,000 worth of a 5-year bond with a modified duration of 4.6; and 5,000 worth of a 6-year bond with a modified duration of 5.50. Find the modified duration of the entire portfolio.

6. You have a single liability that requires a payment of 200,000 at time 7. The valuation interest rate for assets and liabilities is $i_0 = 0.06$. You wish to create an immunized portfolio by buying two zero-coupon bonds that will mature in 4 years and 10 years. Find the amounts of the two bonds you will buy, and verify that the portfolio is immunized.

Section 7.12

Basic Review Problem Solutions

1. For the year 2 cash flow, the company will purchase 5.5% two-year zero-coupon bonds with a maturity value of 3,000. The cost is:

$$\frac{3,000}{1.055^2} = 2,695.36$$

For the year 4 cash flow, the company will purchase 6.8% four-year zero-coupon bonds. The cost is:

$$\frac{5,000}{1.068^4} = 3,843.13$$

The total amount invested to match the liability cash flows is:

$$2,695.36 + 3,843.13 = 6,538.49$$

2. We look at the longest-term asset and liability first.

The total payments for the 12-month bond at month 12 consist of a coupon of 25 and the redemption value of 1,000 for a total of 1,025. To cover a liability of 1,000, the required amount of the 12-month bond is $\frac{1,000}{1,025} = 0.9756$

Once John purchases 0.9756 of the 12-month bond, it will provide a coupon payment of (0.9756)25 = 24.39 at month 6. To fund a total liability of 1,000 at month 6, the additional amount needed from the 6-month bond is:

1,000-24.39 = 975.61

The total payments for the 6-month bond at month 6 consist of a coupon of 20 and the redemption value of 1,000 for a total of 1,020. To cover a liability of 975.61, the amount of the 6-month bond required is $\frac{975.61}{1020} = 0.9565$.

John must purchase 0.9565 of the 6-month bond and 0.9756 of the 12-month bond. The prices of the separate bonds are as follows:

6-month: $\frac{1,020}{1.015} = 1,004.93$ 12-month: $\frac{25}{1.03} + \frac{1,025}{1.03^2} = 990.43$

The total cost of purchasing the required bonds is

0.9756(990.43) + 0.9565(1,004.93) = 1,927.48

3. The present value is

$$P = \frac{1{,}000}{1.075^3} + \frac{3{,}000}{1.075^4} = 804.96 + 2{,}246.40 = 3{,}051.36$$

The weights for the duration calculation are:

$$w_3 = \frac{804.96}{3{,}051.36} = 0.2638 \qquad w_4 = \frac{2{,}246.40}{3{,}051.36} = 0.7362$$

The Macaulay duration is the weighted average time of the payments:
D = 0.2638(3) + 0.7362(4) = 3.7362

The modified duration is: $D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{3.7362}{1.075} = 3.4755$

4. (a) $P(i) \approx P(i_0) \cdot \left(\frac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)} = 940.29 \cdot \left(\frac{1.071}{1.07}\right)^{6.5317} = 934.57$

$\Delta P = P(i) - P(i_0) \approx 934.57 - 940.29 = -5.72$

(b) $\Delta P = P(i) - P(i_0) \approx -\frac{D_{\text{mac}} \cdot P(i) \cdot (i - i_0)}{1+i_0} = -\frac{6.5317(940.29)(.001)}{1.07} = -5.74$

5. The modified duration of the entire portfolio is:
0.10(2.7) + 0.40(4.6) + 0.50(5.50) = 4.86

6. We know that $L_7 = 200{,}000$ and we need to find A_4 and A_{10}.

Present Value Matching: $\frac{A_4}{1.06^4} + \frac{A_{10}}{1.06^{10}} = \frac{200{,}000}{1.06^7}$

Duration Matching: $\frac{4A_4}{1.06^4} + \frac{10A_{10}}{1.06^{10}} = \frac{(7)200{,}000}{1.06^7}$

The solution to this system is $A_4 = 83{,}961.93$, $A_{10} = 119{,}101.60$.
$\sum t^2 L_t v_{i_0}^t = 6{,}517{,}559.71 < 7{,}714{,}662.52 = \sum t^2 A_t v_{i_0}^t$, so the portfolio is immunized.

Note: We can confirm that the portfolio is fully immunized, because it involves a single liability cash flow that is supported by two assets with longer and shorter durations, and the present values and durations of the assets and the liability are matched.

Section 7.13

Sample Exam Problems

1. (Fall 05 Sample Problems #35)
The current price of an annual coupon bond is 100. The derivative of the price of the bond with respect to the yield to maturity is -700. The yield to maturity is an annual effective rate of 8%.

Calculate the duration of the bond.

(A) 7.00 (B) 7.49 (C) 7.56 (D) 7.69 (E) 8.00

2. (Fall 05 Sample Problems #36)
Calculate the duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend is constant, and that the effective rate of interest is 10%.

(A) 7 (B) 9 (C) 11 (D) 19 (E) 27

3. (Fall 05 Sample Problems #37)
Calculate the duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 2% each year and that the effective rate of interest is 5%.

(A) 27 (B) 35 (C) 44 (D) 52 (E) 58

The following information applies to questions 4 thru 6.

Joe must pay liabilities of 1,000 due 6 months from now and another 1,000 due one year from now. There are two available investments:

1) a 6-month bond with face amount of 1,000, a 8% nominal annual coupon rate convertible semiannually and a 6% nominal annual yield rate convertible semiannually, and
2) a one-year bond with face amount of 1,000, a 5% nominal annual coupon rate convertible semiannually, and a 7% nominal annual yield rate convertible semiannually

4. (Fall 05 Sample Problems #51)
How much of each bond should Joe purchase in order to exactly (absolutely) match the liabilities?

	Bond I	Bond II
(A)	1	.97561
(B)	.93809	1
(C)	.97561	.94293
(D)	.93809	.97561
(E)	.98345	.97561

5. (Fall 05 Sample Problems #52)
What is Joe's total cost of purchasing the bonds required to exactly (absolutely) match the liabilities?

(A) 1894 (B) 1904 (C) 1914 (D) 1924 (E) 1934

6. (Fall 05 Sample Problems #53)
What is the annual effective yield rate for investment in the bonds required to exactly (absolutely) match the liabilities?

(A) 6.5% (B) 6.6% (C) 6.7% (D) 6.8% (E) 6.9%

7. (May 05 #3)
A bond will pay a coupon of 100 at the end of each of the next three years and will pay the face value of 1000 at the end of the three-year period. The bond's duration (Macaulay duration) when valued using an annual effective interest rate of 20% is X. Calculate X.

(A) 2.61 (B) 2.70 C) 2.77 (D) 2.89 (E) 3.00

8. (May 05 #6)
John purchased three bonds to form a portfolio as follows:

Bond A has semi-annual coupons at 4%, a duration of 21.46 years, and was purchased for 980.

Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.

Bond C has a duration of 16.67 years and was purchased for 1000.

Calculate the duration of the portfolio at the time of purchase.

(A) 16.62 years (B) 16.67 years (C) 16.72 years
(D) 16.77 years (E) 16.82 years

9. (May 05 #15)
An insurance company accepts an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds at a total cost of X in order to exactly match its obligation:

1-year 4% annual coupon bond with a yield rate of 5%
2-year 6% annual coupon bond with a yield rate of 5%.

Calculate X.

(A) 18,564 (B) 18,574 (C) 18,584 D) 18,594 (E) 18,604

10. (Nov 05 #2)
Calculate the Macaulay duration of an eight-year 100 par value bond with 10% annual coupons and an effective rate of interest equal to 8%.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

11. (Nov 05 #10)
A company must pay liabilities of 1000 and 2000 at the end of years 1 and 2, respectively. The only investments available to the company are the following two zero-coupon bonds:

Maturity (years)	Effective annual yield	Par
1	10%	1000
2	12%	1000

Determine the cost to the company today to match its liabilities exactly.
(A) 2007 (B) 2259 (C) 2503 (D) 2756 (E) 3001

12. (Nov 05 #21)
Which of the following statements about immunization strategies are true?

I. To achieve immunization, the convexity of the assets must equal the convexity of the liabilities.

II. The full immunization technique is designed to work for any change in the interest rate.

III. The theory of immunization was developed to protect against adverse effects created by changes in interest rates.

(A) None (B) I and II only
(C) I and III only (D) II and III only
(E) The correct answer is not given by (A), (B), (C), and (D).

13. (May 2005 #23)
The stock of Company X sells for 75 per share assuming an annual effective interest rate of i. Annual dividends will be paid at the end of each year forever. The first dividend is 6, with each subsequent dividend 3% greater than the previous year's dividend. Calculate i.

(A) 8% (B) 9% (C) 10% (D) 11% (E) 12%

14. (November 2005 #20)
The dividends of a common stock are expected to be 1 at the end of each of the next 5 years and 2 for each of the following 5 years. The dividends are expected to grow at a fixed rate of 2% per year thereafter. Assume an annual effective interest rate of 6%.Calculate the price of this stock using the dividend discount model.

(A) 29 (B) 33 (C) 37 (D) 39 (E) 41

Section 7.14

Sample Exam Problem Solutions

1.
This problem asks us to calculate the "duration" for an investment. Because it doesn't specify Macaulay or modified duration, it means Macaulay duration. We will use the relationship between Macaulay duration and modified duration.

$$\frac{D_{\text{mac}}}{1+i} = D_{\text{mod}}$$

We are given $P'(i) = -700$, $P(i) = 100$ and $i = .08$. Thus we have:

$$D_{\text{mod}} = \frac{-P'(i)}{P(i)} = \frac{700}{100} = 7 \qquad \frac{D_{\text{mac}}}{1.08} = 7$$

$$D_{\text{mac}} = 7.56$$

Answer C

2.
The series of dividends is a level perpetuity of the dividend *Div*.
Thus the price of the stock by the dividend discount model at rate i is:

$$P(i) = \frac{Div}{i}$$

Thus

$$P'(i) = \frac{-Div}{i^2}$$

From this we can get the modified duration and the Macaulay duration:

$$D_{\text{mod}} = \frac{-P'(i)}{P(i)} = \frac{1}{i} \text{ and } D_{\text{mac}} = (1+i)D_{\text{mod}} = \frac{(1+i)}{i}$$

When $i = .10$, we have:

$$D_{\text{mac}} = \frac{1.1}{0.1} = 11$$

Answer C

3.
We will use the relationship between (Macaulay) duration, D_{mac} and modified duration, D_{mod}:

$$D_{\text{mod}} = \frac{-P'(i)}{P(i)} = \frac{D_{\text{mac}}}{1+i}$$

First we denote the beginning dividend by Div and note that the price of the stock at interest rate i and growth rate 0.02 is given by:

$$P(i) = \frac{Div}{(i-0.02)}$$

Next we take the derivative of the expression above with respect to i and obtain:

$$P'(i) = \frac{-Div}{(i-0.02)^2}$$

It follows that:

$$D_{\text{mod}} = \frac{-P'(i)}{P(i)} = \frac{1}{i-0.02} = \frac{D_{\text{mac}}}{1+i}$$

For $i = 0.05$, we have:

$$\frac{1}{i-0.02} = \frac{1}{0.05-0.02} = \frac{1}{0.03} = \frac{D_{\text{mac}}}{1.05} \qquad D_{\text{mac}} = 35$$

Answer B

Note: In problems 2. and 3. we found D_{mac} *for an infinite series of cash flows.*

4.
In 12 months, the 6-month bond will be gone and only the 12-month bond will be available to pay the liability of 1,000 on that date. The total payments for the 12-month bond at month 12 consist of a coupon of 25 and the redemption value of 1,000 for a total of 1,025. To cover a liability of 1,000, the required fraction of a 12-month bond is:

$$\frac{1{,}000}{1{,}025} = 0.97561.$$

Once Joe purchases 0.97561 of the 12-month bond, it will provide a coupon payment of (0.97561)25 = 24.39 at month 6. To fund a total liability of 1,000 at month 6, the additional amount needed from the 6-month bond is:

$$1{,}000 - 24.39 = 975.61$$

The total payments from one 6-month bond at month 6 consist of a coupon of 40 and the redemption value of 1,000 for a total of 1,040. To cover a liability of 975.61, the required fraction of a 6-month bond is:

$$\frac{975.61}{1{,}040} = 0.93809$$

Answer D

5.
We have seen in the last problem that Joe must purchase 0.93809 of the 6-month bond and 0.97561 of the 12-month bond. The prices of the separate bonds are as follows:

<u>6-month</u> $\dfrac{1{,}040}{1.03} = 1{,}009.71$

<u>12-month</u> $\dfrac{25}{1.035} + \dfrac{1{,}025}{1.035^2} = 981.00$

The total cost of purchasing the required bonds is:

$$981.00(0.97561) + 1{,}009.71(0.93809) = 1{,}904.27$$

Answer B

6.
Joe will pay PV = -1,904.27 to receive N = 2 semi-annual payments of PMT= 1,000. The BA II Plus shows that the yield per semi-annual period for this sequence is $\text{I/Y} = 3.3332\%$. The annual effective yield is:

$$1.033332^2 - 1 = 0.067775$$

Answer D

7.
We use $D_{\text{mac}} = \dfrac{Fr(Ia)_{\overline{n|}} + nCv^n}{Fr(a_{\overline{n|}}) + Cv^n} = \dfrac{Fr(Ia)_{\overline{n|}} + nCv^n}{\text{Bond Price}}$. The price of the bond can be obtained using the BA II Plus with N=3, I/Y=20, PMT=100, and FV=1,000. It is PV=789.35. The coupon is $Fr = 100$ and $C = 1{,}000$. For an interest rate of 20%, $(Ia)_{\overline{3|}} = 3.9583$. Thus $D_{\text{mac}} = \dfrac{100(3.9583) + 3(1{,}000)/1.2^3}{789.35} = 2.70$

Answer B

Note: Since there are only 3 cash flows, this problem could easily have been solved directly from the definition of Macaulay duration:

$$D_{\text{mac}} = \frac{100(1)1.20^{-1} + 100(2)1.20^{-2} + 1{,}100(3)1.20^{-3}}{100 \cdot 1.20^{-1} + 100 \cdot 1.20^{-2} + 1{,}100 \cdot 1.20^{-3}} = 2.70$$

8.
The duration of the portfolio is the weighted average of the individual durations. (This is true regardless of whether we are talking about Macaulay or modified duration.) The total purchase price of the portfolio is:

$$980 + 1{,}015 + 1{,}000 = 2{,}995$$

Thus the duration is:

$$D = \left(\frac{980}{2{,}995}\right)21.46 + \left(\frac{1015}{2{,}995}\right)12.35 + \left(\frac{1000}{2{,}995}\right)16.67 = 16.77$$

Answer D

9.

To solve this problem, we must assume that the face amounts of the bonds are also their redemption values. Let x and y represent the face amounts of the 1-year and 2-year bonds, respectively. The total amount paid by the bonds should be 10,000 at times 1 and 2. In terms of x and y, we have:

Time 2 $\quad 1.06y = 10,000$

Time 1 $\quad 1.04x + 0.06y = 10,000$

Solving this system of equations yields: $x = 9,071.12$ and $y = 9,433.96$.

To calculate the prices of the bonds, we have:

1 year bond: $\dfrac{9,071.12(1+0.04)}{1.05} = 8,984.73$

2 year bond: $\dfrac{9,433.96(0.06)}{1.05} + \dfrac{9,433.96(1+0.06)}{1.05^2} = 9,609.38$

The total cost X is $8{,}984.73 + 9{,}609.38 = 18,594.11$

Answer D

10.

The Macaulay duration of a bond with face amount F, coupon Fr for n periods, and redemption value C is $\dfrac{Fr(Ia)_{\overline{n|}} + nCv^n}{Fr(a_n) + Cv^n}$.

In this case, $Fr = 10$, $F = C = 100$, and $n = 8$.

The price of the bond can be obtained using the BA II Plus with N = 8, I/Y = 8, PMT = 10, and FV = 100. Then CPT PV = -111.49, so the price is 111.49.

Thus the Macaulay duration is:

$$\frac{10(Ia)_{\overline{8|}} + 8(100)v^8}{111.49} = \frac{10(23.553) + 8(100)(0.5403)}{111.49} = 5.99$$

Answer C

11.

Since there are separate zero-coupon bonds for each year, the company can buy the appropriate amount of each bond to cover the liability for that year only.

For year 1, the company buys a 1-year zero-coupon bond with a 1,000 face amount at a cost of:

$$\frac{1{,}000}{1.1} = 909.09$$

For year 2, the company buys a 2-year zero-coupon bond with a 2,000 face amount at a cost of:

$$\frac{2{,}000}{1.12^2} = 1,594.39$$

The total amount invested to match the liabilities is $909.09 + 1,594.39 = 2,503.48$.

Answer C

12.
Statement I is false. The assets must have <u>greater</u> convexity for immunization.
Statement II is true.
Statement III is true.

Answer D

13.
This can be done directly with the dividend growth model

$$P(i) = \frac{Div}{(i-g)}.$$

Using $P = 75$, $Div = 6$ and $g = 0.03$, we have

$$75 = \frac{6}{(i-0.03)} \qquad\qquad i = 0.11$$

Answer D

14.
This stock will pay dividends in three different phases.

<u>Phase 1</u>. A level annuity of 1 for 5 years.
<u>Phase 2</u>. After a deferral period of 5 years, a level annuity of 2 for 5 years.
<u>Phase 3</u>. After a deferral period of 10 years, a geometric perpetuity for which we can use the dividend growth model:

$$\text{value of future dividends } \underline{\text{as of time 10}} = \frac{Div}{i-g}$$

Care must be taken with the final piece. It is clear that the growth rate is 2%. The value of *Div* to use is 2(1.02) = 2.04, since the next expected dividend (at the end of year 11) has already experienced one year of growth. (A common mistake is to use *Div*=2.)

Thus the price is

$$1a_{\overline{5|}} + 2v^5 a_{\overline{5|}} + v^{10}\left(\frac{2.04}{0.06-0.02}\right)$$

$$= 4.212 + \frac{2(4.212)}{1.06^5} + \frac{51}{1.06^{10}}$$

$$= 38.985$$

Answer D

Section 7.15

Supplemental Exercises

1. A company has liability payments of 3,000 and 6,000 due at the end of years 1 and 3, respectively. The investments available to the company are the following two zero-coupon bonds:

Maturity (years)	Annual Effective Rate	Par Value
1	4.8%	1,000
3	5.6%	1,000

   Find the cost of exactly matching these liabilities.

2. A company has liability payments of 1,000 and 2,000 due at the end of years 2 and 3, respectively. It can purchase two zero-coupon bonds to match these liabilities. The first has a par value of 1,000 and matures in 2 years. The second has a par value of 1,000 and matures in 3 years with an annual effective yield of 6%. If the cost of matching these liabilities is 2,586.27, what is the annual effective yield on the first bond?

3. An investment pays 1,000 at the end of year two; 2,000 at the end of year three; and 4,000 at the end of year four. It was purchased to yield an annual effective rate of 6.5%. Find the Macaulay duration for this investment.

4. An investor has a portfolio containing a 3-year bond with a value of 2,000 and a modified duration of 2.85; a 6-year bond with a value of 5,000 and a modified duration of 5.24; and a 10-year bond with a value of 8,000 and a modified duration of 9.13. Find the modified duration of the entire portfolio.

5. An annual-coupon corporate bond is trading at 972.18, resulting in an annual effective yield of 7.5%. Its Macaulay duration is 5.8215. Estimate the change in price if the yield rate decreases to 7.4% using (a) the 1^{st} order Macaulay approximation and (b) the 1^{st} order modified approximation.

6. An annual-coupon bond with face value 1,000 and a coupon rate of 4.5% matures in 3 years.
   Find the Macaulay duration of this bond at an annual effective yield of 4%.

Problems 7 and 8 use the following information:

A company has liability payments of 2,000 due in 6 months and another 2,000 due in 1 year.
It has two available investments:

1) a 6-month bond with face amount 1,000, 4% nominal annual coupon rate payable semi-annually, and 5% nominal annual yield convertible semi-annually
2) a 1-year bond with face amount 1,000, 7% nominal annual coupon rate payable semi-annually, and 6% nominal annual yield convertible semi-annually

7. How much of each bond must the company purchase in order to match its liabilities exactly?

8. What is the total cost of these bonds?

Section 7.16

Supplemental Exercise Solutions

1. For year 1, the company buys 1-year 4.8% zero-coupon bonds with a total face amount of 3,000. The cost is 3,000/1.048 = 2,826.60.

   For year 3, the company buys 3-year 5.6% zero-coupon bonds with a total face amount of 6,000. The cost is $6{,}000/1.056^3 = 5{,}095.18$.

   The total cost of buying assets to match the liabilities is:
   2,862.60 + 5,095.18 = 7,957.78

2. Let i be the annual effective yield of the first bond.
   The total cost of matching liabilities by purchasing a 2-year bond with a face amount of 1,000 and a 3-year bond with a face amount of 2,000 is:
   $$\frac{1{,}000}{(1+i)^2}+\frac{2{,}000}{1.06^3}=2{,}586.27$$
   Solving for i:
   $$(1+\ i)^2 \quad =\frac{1{,}000}{2{,}586.27-\dfrac{2{,}000}{1.06^3}}= 1.1025 \qquad i=0.05$$

3. The present value of the payments is
   $P = 1{,}000/1.065^2 + 2{,}000/1.065^3 + 4{,}000/1.065^4$
   = 881.66 + 1,655.70 + 3,109.29
   = 5646.65

   The weights for the duration calculation are:
   w_1 = 881.66/5,646.65 = 0.1561, w_2 = 1655.70/5,646.65 = 0.2932, and
   w_3 = 3109.29/5,646.65 = 0.5507.

   D_{mac} = 0.1561(2) + 0.2932(3) + 0.5507(4) = 3.3946

4. The price of the entire portfolio is 15,000. The weights for the modified durations are w_3 = 2/15, w_6 = 5/15, and w_{10} = 8/15.
   The modified duration for the entire portfolio is:
   D_{mod} = (2/15)(2.85) + (5/15)(5.24) + (8/15)(9.13) = 6.996

5. (a) $P(i)\approx P(i_0)\cdot\left(\dfrac{1+i_0}{1+i}\right)^{D_{mac}(i_0)}=972.18\cdot\left(\dfrac{1.075}{1.074}\right)^{5.8215}=977.46$
   $\Delta P\approx 977.46-972.18=5.28$

   (b) $\Delta P\approx -\dfrac{D_{mac}}{1+i}\cdot P(i)\cdot\Delta i=-\dfrac{(5.8215)\cdot(972.18)\cdot(-0.001)}{1.075}=5.26$

6. The Macaulay duration of the bond is:

$$D_{\text{mac}} = \frac{Fr(Ia)_{\overline{n|}} + n \cdot Cv^n}{\text{Bond Price}}$$

$F = C = 1{,}000$, $r = 0.045$, $n = 3$ and $i = 0.04$.

$v^3 = 0.8890$, $\ddot{a}_{\overline{3|}} = 2.8861$, $(Ia)_{\overline{3|}} = 5.4775$

Bond Price = 1,013.88. (N = 3, I/Y = 4, PMT = 45, FV = 1,000; CPT PV = −1,013.88)

$$D_{\text{mac}} = \frac{45(5.4775) + 3{,}000(0.889)}{1{,}013.88} = 2.8736$$

7. The total of the payments from the 1-year bond at redemption is 1,035. The required liability payment is 2,000, so the company needs 2,000/1,035 = 1.93237 of this bond (a face amount of 1,932.37).

At 6 months, the coupon on this amount of the 1-year bond is 35(1.93237) = 67.73. The additional amount to cover the payment due at 6 months is:

2,000–67.73 = 1,932.37

The total payment from one 6-month bond at redemption is 1,020.
The company needs 1,932.37/1,020 = 1.8945 of this bond to cover the required payment at 6 months.

8. The price of the 6-month bond is 1,020/1.025 = 995.12.
(N = 1, I/Y = 2.5, PMT = 20, FV = 1,000; CPT PV = -995.12)

The price of the 1-year bond is 1,009.57.
(N = 2, I/Y = 3, PMT = 35, FV = 1,000. CPT PV = -1,009.57)

The total cost of the bonds is 1.9324(1,009.57) + 1.8945(995.12) = 3,836.15

Module 8

Determinants of Interest Rates

The first 7 modules of this manual have focused on performing calculations using interest rates. Either we were given one or more interest rates to use in solving a problem, or we were given a set of facts and had to find the interest rate. In this module we will examine the factors that determine the level of interest rates in the real world. These factors include inflation, economic conditions, and the likelihood a borrower will default on the loan. As we learned in Module 6, the term of the loan also affects the interest rate. We will see that the interest rate for a loan can vary greatly depending on the characteristics of the loan and the borrower, and also that interest rates can fluctuate significantly over time. Finally, we will consider the influence that central banks can have on the level of interest rates in the economy.

Section 8.1

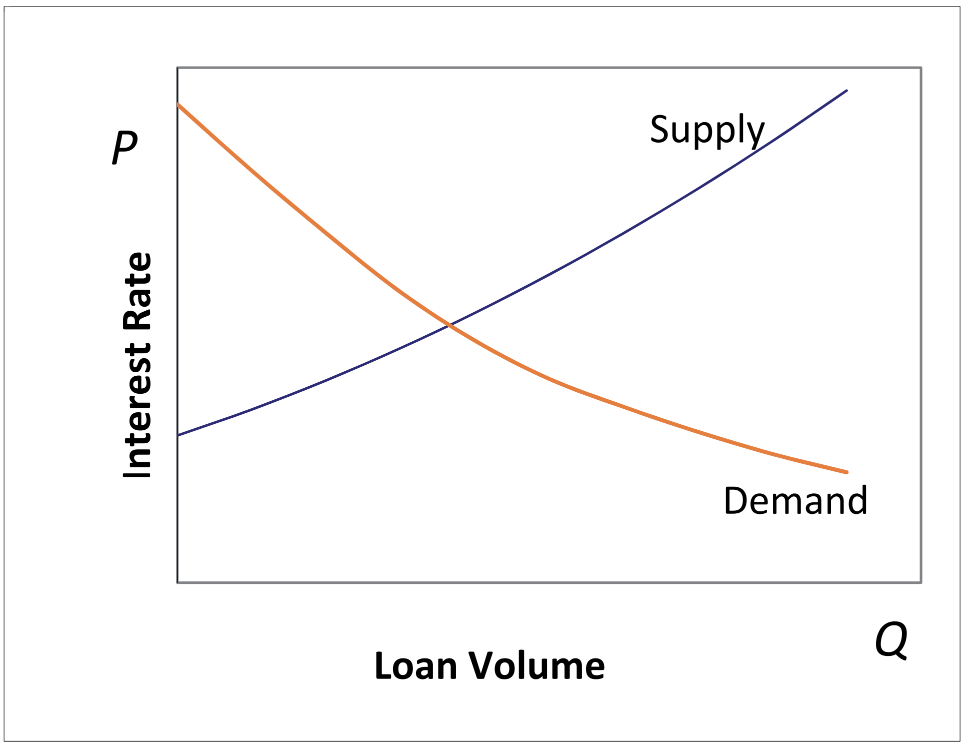
Background

From a lender's perspective, interest is the compensation that is received for delaying consumption. That is, the lender is rewarded for deferring purchases and allowing a borrower to use the lender's money for a period of time. From the borrower's perspective, interest is the cost incurred to make a purchase when the money for that purchase is not available. That is, the borrower pays a penalty in return for being able to make a purchase sooner than would have been possible without the loan.

A basic rule of economics is that incentives affect people's behavior. And high or low interest rates create incentives that affect people's decisions about whether to lend or borrow money. A higher interest rate increases the incentive for potential lenders to lend money rather than spending it. For potential borrowers, a higher interest rate is an incentive to defer purchases, whereas a lower interest rate increases the likelihood they will borrow money to make purchases.

Supply and Demand for Loans

Although it is an oversimplification of a complex financial structure, we can conceive of lending and borrowing in terms of the supply of and the demand for money. The price of money is the interest rate, and the quantity of money is the volume of loans. Lenders are the suppliers of money, and borrowers represent the demand for money. As with any other commodity, a higher price (a higher interest rate) results in greater supply (more lenders willing to lend), but it also results in less demand (fewer borrowers seeking loans). This can be illustrated by a graph of the **supply and demand** curves for loans. In the following graph, "Price" (P) represents the level of interest rates and "Quantity" (Q) represents loan volume (the amount of money lent).



No numerical interest rates or loan volumes are shown on the axes of the graph, as it is only a conceptual representation of a very complex system in which a huge number of borrowers and lenders are simultaneously creating a wide variety of loans with different amounts, terms, and interest rates. However, the graph reflects the fact that if interest rates rise, the supply of loans will increase and the demand for loans will decrease. The interest rate at the point where the supply and demand curves intersect represents an **equilibrium point**. When the interest rate is at that level, the supply and the demand for loans are in balance: all of the borrowers who are willing to borrow at that rate (or lower) can receive loans, and all of the lenders who are willing lend at that rate (or higher) can find borrowers to lend to.

If interest rates rise above the equilibrium value, not all of the willing lenders will find borrowers to lend to, so interest rates will tend to fall back toward the equilibrium point. Similarly, if interest rates decline from the equilibrium value, not all potential borrowers will be able to find willing lenders to borrow from, so interest rates will tend to increase toward the equilibrium value.

Based on this concept of a supply and demand curve for the lending markets, we can think of a market interest rate as being the equilibrium price for money. Conceptually, this equilibrium price is defined by the intersection of the supply and demand curves for money. Note, however, that these curves shift over time based on economic conditions, availability of goods and services, changes in the population, etc. As supply and demand shift, the equilibrium price of money and the volume of loans will also shift, resulting in rising or falling interest rates and changes in the volume of lending activity.

Quotation Bases for Interest Rates

In this manual we have introduced many different ways to express (or "quote") interest rates. We have used **annual effective rates**, as well as effective rates for other time periods (e.g., **quarterly effective rates**). We have used **nominal rates** as a means of *describing* an interest rate (but have noted that nominal rates must be converted to effective rates for calculation purposes). In some cases, we have applied a **continuously compounded interest rates** (i.e., a **force of interest)**. We have also seen that **rates of discount** can be used to measure an investment's rate of growth. And in Module 1 we examined the special methods used to quote **rates for Treasury bills** issued by the U.S. Treasury and by the Government of Canada.

When we compare interest rates, the rates being compared must be expressed on the same basis. Because rates can be quoted in many different forms, it is important to be able to convert them to a consistent basis. We have used annual effective interest rates most frequently, but rates expressed in other forms can be compared, as long as all of the rates being compared have been converted to the same basis.

In this module, we will often use continuously compounded rates, since they offer some calculation advantages over annual effective rates. In particular, continuously compounded rates can be added and averaged, making some calculations simpler than if we used annual effective rates.

For example, let $r_1, r_2, \ldots r_n$ be the continuously compounded interest rates for years 1, 2, …, n, and let $i_1, i_2, \ldots i_n$ be the corresponding annual effective rates. We can use the continuously compounded rates to calculate an n-year accumulation factor by simply adding the r's and taking e to that power. This is less complicated than multiplying the (1+i)'s:

$$a(n) = e^{r_1 + r_2 + \ldots + r_n} \qquad \text{vs.} \qquad a(n) = (1+i_1)\cdot(1+i_2)\cdot\ldots\cdot(1+i_n)$$

And to calculate an <u>average</u> rate of return over the n years, we can simply average the r's:

$$\bar{r} = \frac{(r_1 + r_2 + \ldots + r_n)}{n} \qquad \text{vs.} \qquad \bar{i} = \left[(1+i_1)(1+i_2)\ldots(1+i_n)\right]^{1/n} - 1$$

Clearly, the use of continuously compounded rates makes these calculations considerably simpler. For this reason, we will use continuously compounded rates in this module when we analyze the components of interest rates.

Note: The study note on Determinants of Interest Rates uses the variable "r" to represent the continuously compounded rate (i.e., the force of interest), and we will adopt that notation for this module only. In all other modules, we have used δ*, which is standard actuarial notation for the force of interest.*

Section 8.2

Components of the Interest Rate

In the previous section, we observed that interest rates are determined by the supply of and demand for money, and that the interest rate for a particular type of loan represents an equilibrium point where the supply and demand for that type of loan are in balance. In this section, we will examine the factors influencing supply and demand, and consider how these factors affect the interest rate charged for a loan. Relevant factors include the term of a loan, the likelihood that the loan will not be fully repaid, and the effect of inflation. We will use continuously compounded interest rates for this analysis, because that allows us to consider the interest rate as the *sum* of its components: compensation for **deferred consumption**, compensation for the **risk of default**, and compensation for the **loss of purchasing power**.

Loan Term

In Module 6, we studied yield curves and noted that longer-term loans usually have higher interest rates than shorter-term loans. There are a number of theories for explaining why long-term and short-term interest rates are different. These include the **market segmentation theory**, the **liquidity preference theory**, the **expectations theory**, and the **preferred habitat theory**.

The **market segmentation theory** is based on the idea that lenders who make short-term loans are a separate group from the lenders who make longer-term loans. Similarly, short-term borrowers are not the same people as long-term borrowers. Consequently, it is reasonable to expect that the supply of and demand for loans will be different in these two markets, and therefore the equilibrium interest rates for short-term and long-term loans will not be the same. This theory does not predict whether long-term rates will be higher or lower than short-term rates.

The **liquidity preference theory** (or **opportunity cost theory**) argues that lenders have a natural preference for shorter-term loans, since longer-term loans tie up their money for a longer period and prevent them from taking advantage of other investment (or buying) opportunities that may occur during the term of the loan. Therefore, lenders demand a higher interest rate on longer-term loans to compensate them for this loss of flexibility.

The **expectations theory** states that the rate charged for a longer-term loan provides information about the expected interest rates for future short-term loans. We have seen that forward rates can be calculated from the spot rates for various terms. (For example, the 3-year-forward, 1-year rate, $i_{3,4}$, can be calculated from the 3-year and 4-year spot rates, s_3 and s_4.) If we view forward rates as an indication of the expected level of future rates, then the difference between today's short-term and long-term rates is an indication of borrowers' and lenders' expectations for those future rates.

The **preferred habitat theory** holds that although borrowers and lenders have a *preferred* term for which they would like to borrow or lend (as described by the market segmentation theory), they can be persuaded to borrow or lend for a different term if the interest rate for that term is sufficiently attractive. Under normal conditions (where longer-term loans have higher interest rates), this means that a short-term lender might be persuaded to give up some flexibility (or liquidity) by lending for a longer term if the interest rate is high enough. Conversely, a borrower might be persuaded to give up the security of a long-term loan and accept a loan for a shorter term if the interest rate is low enough.

Yield Curves

We presented yield curves in Module 6, and identified various shapes that the yield curve can take. These include **normal** (increasing by term), **inverted** (decreasing by term), and **flat** (level at all terms). There is also the **bow-shaped** yield curve, where interest rates increase and then decrease with term (or decrease and then increase). The 4 theories discussed above generally explain why interest rates usually follow the normal pattern (increasing by term). Now we will discuss situations that can cause the other shapes to occur.

An **inverted yield curve** (where interest rates are higher for short-term loans than for long-term loans) can occur if fewer lenders are willing or able to make short-term loans. In that case, the supply curve for short-term loans shifts upward and to the left, increasing the equilibrium interest rate. An inverted yield curve could also be caused by an increase in the demand for short-term loans, causing the demand curve to shift upward and to the right, again increasing the equilibrium interest rate for short-term loans.

Because short-term interest rates are more variable than long-term rates, the inverted yield curve is most likely to be caused by an increase in short-term rates. However, a decrease in longer-term interest rates can also contribute to an inversion of the yield curve. This can happen if there is a reduced demand for long-term loans, as might happen when borrowers are pessimistic about the future and are unwilling to take on long-term debt obligations (e.g., home mortgages or business loans). In that case, the demand curve for long-term loans shifts downward and to the left, resulting in a lower equilibrium interest rate for long-term loans.

A **bow-shaped yield curve** has higher rates for medium-term loans than for either short-term or long-term loans. Or if medium-term rates are *lower* than both short-term and long-term rates, the yield curve would again be described as "bowed." Occurrences of a bow-shaped yield curve are very infrequent and tend not to last long, as the resulting incentives change borrowers' and lenders' preferences, which in turn alters the shape of the yield curve.

A **flat yield curve** has the same (or nearly the same) interest rate for loans of all terms. It can be viewed as an intermediate condition between a normal and an inverted yield curve, where the supply and demand for loans of all terms are such that there is little variation in rates by term. Flat yield curves rarely occur in the real world, but for simplicity, a single interest rate for all terms is often used in calculations. For example, calculations of duration and convexity often apply a single interest rate to all of the cash flows. This approach simplifies our calculations, but it is important to remember that the use of a single interest rate for all terms is only an approximate representation of reality.

Default

Whenever money is lent, there is a chance that it will not be repaid. The possibility that the borrower will default on the loan is a risk the lender takes on, and lenders demand compensation for assuming this risk, as shown in the following example.

Example (8.1)

A lender is considering lending 1,000 for 5 years. To compensate for the loss of liquidity (or for "deferred consumption"), the lender requires a continuously compounded yield of 0.04. If the loan is to be repaid in a lump sum at the end of its 5-year term, the repayment amount is $1{,}000 \cdot e^{5 \cdot (0.04)} = 1{,}221.40$. If there is a 2% chance that the loan will *not* be repaid (and that the lender will not receive even a *partial* repayment), then the expected repayment amount is only $(0.98) \cdot 1{,}221.40 = 1{,}196.97$, which corresponds to a continuously compounded yield of only 0.03596.

In order to have an expected cash flow of 1,221.40, the lender would have to set the repayment amount at $1{,}221.40 / 0.98 = 1{,}246.33$. This corresponds to a continuously compounded interest rate of:

$$\frac{\ln(1.24633)}{5} = 0.04404$$

To have an *expected* continuous return of 4%, the lender should charge a continuously compounded interest rate of 4.404%. But there is *uncertainty* associated with such a loan, since the lender could either realize a 4.404% return (if the borrower repays the loan) or suffer a total loss (if the borrower defaults). Consequently, the lender will demand a continuously compounded interest rate of *at least* 4.404%.

Exercise (8.2)

A lender requires a continuously compounded yield of 5% for a 10-year loan. There is a 6% chance that the loan will not be repaid (and that the lender will not receive even a partial repayment). What continuously compounded rate must the lender charge in order to have an *expected* continuous return of 5%?

Answer: 5.62%

When a borrower defaults on a loan, the lender may be able to recover a portion of the outstanding loan balance. This can occur when the lender takes title to the collateral for a loan (e.g., by foreclosing on a mortgage or repossessing a car). It can also happen when the lender agrees to accept a payment of less than the full outstanding balance of the loan after the borrower defaults.

Example (8.3)

Suppose that the loan of Example **(8.1)** has a 2% chance of default, but in the event of default the lender will be able to recover 60% of the amount owed. If the lender again seeks an expected (continuously compounded) yield of 0.04, the required repayment amount, x, can be calculated as follows:

$$0.98x + 0.02 \cdot (0.60 \cdot x) = 1{,}221.40$$

$$x = \frac{1{,}221.40}{0.98 + 0.02 \cdot (0.60)} = 1{,}231.25$$

This required repayment amount corresponds to a continuously compounded interest rate of $\frac{\ln(1.23125)}{5} = 0.04161$. Because there is uncertainty (about the probability of default and the possible recovery), the lender would require a continuously compounded interest rate of *at least* 4.161%. The expected 60% recovery results in a rate lower than the 4.404% that we calculated in Example **(8.1)** when we assumed there would be *no* recovery in the event of a default.

Exercise (8.4)

Suppose that the loan of Exercise **(8.2)** has a 6% chance of default, but in the event of default the lender will be able to recover 40% of the amount owed. What continuously compounded rate must the lender charge in order to have an *expected* continuous compounded return of 5%?

Answer: 5.37%

In practice, of course, there is considerable uncertainty about the probability of default and the likely recovery in the event of a default. Also, lenders incur some expense in assessing the risk of default and collecting the recovery amount after a default. Different lenders will have greater or lesser skill in measuring the default risk, and will have different cost structures for analyzing loan applicants and for seeking recovery in the event of default. As a result, different lenders will have different levels of expenses and different degrees of uncertainty about repayment. This affects the interest rates that different lenders demand, which in turn determines the supply curve for loans.

Similarly, different borrowers will have different degrees of willingness to pay various levels of interest rates, and their behaviors will define the demand curve. Each loan type, loan term, and borrower category will have its own set of supply and demand curves, and its own equilibrium interest rate. These equilibrium rates cannot be calculated; they can only be observed as borrowers and lenders make loan agreements with various terms and in various market conditions.

Compensation for Default Risk

In the preceding example, we calculated (minimum) interest rates that reflected the risk of default. Now we will consider the compensation that a lender actually demands for assuming the default risk. If r is the (continuously compounded) interest rate that would be charged for a loan with no default risk, and R is the rate actually charged, then we can write:

(8.5) $$\boxed{R = r + s}$$

where s, the difference between R and r, represents compensation to the lender for the risk of default.

If we instead use annual effective rates, then (using the same variable names) we would have:

(8.6) $$\boxed{R = (1+r)\cdot(1+s) - 1}$$

In this case, R is not simply the sum of r and s. Instead, $R = r + s + r \cdot s$. Since the $r \cdot s$ term in this formula tends to be small, sometimes the approximate relation $R \approx r + s$ is used. But if either r or s is relatively large, this approximation becomes much less accurate.

In Examples **(8.1)** and **(8.3)**, we calculated the minimum continuously compounded rate that a lender would need to charge in order to compensate for default risk (with no recovery, and with a 60% recovery). The lender targeted a continuously compounded yield of 4%. In Example **(8.1)**, we determined that the continuously compounded loan interest rate would have to be at least 4.404% to compensate for a 2% probability of default. Assuming this is the actual rate charged, the value of s for that example is 0.404%, and we have:

$$s = R - r = 4.404\% - 4\% = 0.404\%$$

Given a 2% probability of default over a 5-year period, the value of s could have been calculated as follows, independent of the lender's target yield:

$$s = -\ln(1-0.02)/5 = 0.00404$$

Similarly, if we assume 60% recovery in the event of default, then the default component is:

$$s = -\ln\left(1-\left[0.02\cdot(1-0.60)\right]\right)/5 = -\ln(1-0.008)/5 = 0.00161$$

This is consistent with the continuously compounded loan interest rate of 4.161% that was calculated in Example **(8.3)**.

Inflation

Inflation (or **monetary inflation** or **price inflation**) refers to the increase in the prices of goods and services over time. The rate at which prices increase is called the **inflation rate**. In the United States, the inflation rate is measured by the Bureau of Labor Statistics by means of two indexes: the **Consumer Price Index** (**CPI**) and the **Producer Price Index** (**PPI**).

The CPI measures the change over time in the prices paid by urban consumers for a "market basket" of consumer goods and services, including such things as food, clothing, medical services, and transportation. It is similar to, but not the same as, a "cost-of-living" index, which is a conceptual measurement that would reflect the expenditures required to maintain a certain standard of living. A cost-of-living index would be affected by environmental and governmental factors that affect consumers' standard of living. The CPI instead measures changes in the prices of things that consumers purchase, making appropriate adjustments to reflect changes in the quality of those items.

The PPI measures the change over time in the prices of goods and services provided by U.S. producers. This includes goods and services sold to other producers, as well as those sold to consumers, either directly or through retailers. The PPI is actually a family of about 10,000 indexes that measure inflation for a very wide range of goods-producing sectors of the U.S. economy.

The inflation rate, as measured by the CPI or PPI, represents the *average* rate of increase in the prices of the goods and services included in the index. Some items will have increased in price more rapidly than others, and some prices may decrease from one period to the next. The reported rate of inflation reflects a weighted average of the rates at which prices increased over the period. These increases are typically reported as annual rates, and may be "seasonally adjusted" to compensate for known patterns of seasonal price fluctuation in certain items.

Inflation is an important consideration in investment decisions. Expecting that inflation will occur during the term of a loan, the lender realizes that the loan will be repaid in dollars that have less purchasing power than the dollars being lent. The lender therefore demands a higher interest rate than would be required in the absence of inflation, so that the repayment will consist of a larger number of the less-valuable future dollars.

By contrast, the borrower is, in a sense, aided by inflation, as he or she will repay the loan using inflated dollars. This is particularly important where the borrower is employed and can anticipate an increasing income level in future periods, as wages also tend to rise due to inflation. The expectation of a higher future income tends to increase the borrower's ability and willingness to borrow at a given interest rate.

Due to inflation, lenders will demand, and borrowers will be willing to pay, a higher rate of interest than if there were no inflation. However, since the interest rate for a loan is determined in advance and inflation can be measured only in retrospect (after we see how much prices have risen), interest rates must be set based on *expectations* as to the amount of inflation that will occur during the term of the loan. In our analysis, we will initially assume that the rate of future inflation is known, and we will examine how a *known* rate of inflation affects the interest rate for a loan. Then we will consider how borrowers and lenders might deal with an *uncertain* rate of inflation.

We will again use R to represent a continuously compounded loan interest rate that reflects compensation for deferred consumption (r) and an adjustment for the risk of default (s). In the absence of inflation, a lender would be willing to lend an amount P_0 for a period of t years at rate $R = r + s$, and be repaid an amount $P_0 \cdot e^{Rt}$ at the end of the period.

Now suppose that there will be inflation over the term of the loan at a known rate i, where i is a continuously compounded annual rate. In order for the loan repayment to have the same purchasing power at time t as $P_0 \cdot e^{Rt}$ would have had with no inflation, the lender will now need to receive $P_0 \cdot e^{Rt} \cdot e^{it} = P_0 \cdot e^{(R+i)\cdot t}$ as repayment for the loan. Therefore, when the rate of inflation during the term of the loan is known to be i, the lender will require a continuously compounded interest rate of $R + i$. If we use R^* to represent a rate that reflects deferred consumption, risk of default, and known inflation, then we have:

(8.7)

$$R^* = r + s + i$$

If these same symbols were used to represent annual effective rates, we would have the more complex formula:

(8.8)

$$R^* = (1+r)(1+s)(1+i) - 1$$

Example (8.9)

Consider a 5-year loan of 1,000 on which the lender would require a continuously compounded yield of 4% if there were no inflation. We will assume there is no chance of default and that the continuously compounded rate of inflation is known to be a constant 2%.

Without inflation, the lender would require a repayment amount of:

$$1{,}000 \cdot e^{5\cdot(0.04)} = 1{,}221.40$$

However, 5 years of inflation will reduce the value of the repayment. To realize the same purchasing power as 1,221.40 in a scenario of no inflation, the lender must receive a repayment of:

$$1{,}221.40 \cdot e^{5\cdot(0.02)} = 1{,}349.86$$

This corresponds to a continuously compounded loan interest rate of:

$$\frac{\ln(1.34986)}{5} = 6\%$$

We can also find this interest rate using Formula **(8.7)**. Since s (the compensation for default risk) is 0, we have:

$$R^* = r + i = 4\% + 2\% = 6\%$$

Exercise (8.10)

An investor plans to purchase a 10-year zero-coupon bond with a face amount of 1,000. The continuously compounded rate of inflation is known to be a constant 1.5%. Assuming there is no risk of default, what continuously compounded rate must the lender earn on this investment in order to realize 5% continuously compounded growth in purchasing power?

Answer: 6.5%

In reality, of course, the level of future inflation over the term of a loan is not known. This uncertainty about the rate of inflation causes lenders to demand greater compensation (higher interest rates), so that the supply curve shifts upward. Borrowers, however, may not be willing to pay higher interest rates, as they can't be certain that inflation will raise their wages sufficiently to allow them to make the larger repayments. This makes it difficult to predict the equilibrium level for interest rates and to decompose the interest rate into its component parts.

In order to address the uncertainty about future inflation rates, a loan can be arranged so that its repayment amounts are based on *actual* inflation during the loan term. This eliminates the lender's uncertainty as to the purchasing power of the loan payments that will be received. Also, assuming the borrower's income will rise with inflation, an inflation-protected loan allows the borrower to be comfortable that the repayment amounts will be affordable. The following example demonstrates how such an **inflation-protected loan** might be structured.

Example (8.11)

Consider a 5-year loan of 1,000 with inflation protection. The loan agreement specifies a continuously compounded interest rate of 3%, and that the repayment amount will be adjusted by a factor equal to the value of a particular price index on the repayment date, divided by the value of that index on the date of the loan.

The unadjusted repayment amount is 1,161.83 $\left(=1{,}000\cdot e^{5\cdot(0.03)}\right)$.

Suppose that the price index specified in the agreement increases by a factor of 1.0868 during the term of the loan. (This corresponds to a continuously compounded inflation rate of $1.665\% = \left(\ln 1.0868\right)/5$. Of course, the actual inflation rates would vary from year to year.) The repayment amount equals the unadjusted amount stated in the contract, times the inflation adjustment factor:

$$1{,}161.83 \cdot 1.0868 = 1{,}262.68$$

This repayment amount is consistent with a continuously compounded interest rate of 4.665% (which is equal to the equivalent continuously compounded inflation rate of 1.665%, plus the contractual rate of 3%).

Exercise (8.12)

Consider a 10-year loan of 1,000 with inflation protection. The loan agreement specifies a continuously compounded interest rate of 4%, and that the repayment amount will be adjusted by a factor equal to the value of a particular price index on the repayment date, divided by the value of that index on the date of the loan. Suppose that the value of the price index specified in the agreement is 201.9 on the date of the loan and 241.8 at the end of the loan's 10-year term.
What is the repayment amount the lender receives? What was the real rate of return for this loan, and what was the nominal rate of return? (Express your answers as continuously compounded rates.)

Answers: 1,786.64 Real: 4% Nominal: 5.80%

It is important to understand that the inflation protection in a loan of this type provides a benefit to the lender, and it is reasonable to expect that the lender pays a price for this benefit. The price of inflation protection is reflected as a reduction in the interest rate charged for the loan. We will use c to represent this reduction in the interest rate, with c being expressed as a continuously compounded rate. If R_1 is the continuously compounded interest rate for an inflation-protected loan with no risk of default, we have:

(8.13) $$R_1 = r - c$$

The actual rate earned on the loan is:

(8.14) $$R_1^{(a)} = r - c + i_a$$

where i_a is the *actual* rate of inflation during the loan period, expressed as a continuously compounded rate.

In practice, it is not possible to measure the values of r and c separately. Only their difference ($r - c$) can be observed, as that is the value that appears in the loan contract. Significantly, this value can be negative in some cases. This is more likely to occur with short-term loans, where the value of r tends to be small. When an inflation-protected loan specifies a negative interest rate, we do not know the values of r and c. We can, however, conclude that c exceeds r; that is, the **cost of inflation protection** (c) is greater than the compensation for deferred consumption (r).

Most loans are *not* inflation-protected. Instead, they specify the interest rate that will be charged, and no adjustment is made for actual inflation during the loan term. (Interest may be charged at a fixed rate, or the interest rate may vary during the term of the loan based on an index of market interest rates. In either case, the interest rate is not adjusted for inflation, so the lender does not have inflation protection.) In this situation, the lender assumes the risk that inflation will diminish the value of the money that will be repaid at the end of the loan term. As compensation for assuming this risk, lenders require a higher interest rate on loans that are not inflation-protected.

We will consider the compensation to the lender for inflation risk in two parts:

- i_e, compensation for expected inflation, and
- i_u, compensation for unexpected inflation (i.e., compensation for the risk that inflation will be greater than expected)

If there is no risk of default, we can express the (continuously compounded) interest rate for a loan without inflation protection as:

(8.15) $$R_2 = r + i_e + i_u$$

The value of i_e (the compensation for expected inflation) is, of course, simply the expected rate of inflation. The value of i_u (the compensation for unexpected inflation) is likely to be small when the rate of inflation is stable and predictable, but large when it is volatile and unpredictable.

The **"real" rate of interest** is defined as the rate at which the purchasing power of an investment grows. For an inflation-protected loan, this is the interest rate specified in the loan agreement (the rate that will be adjusted for inflation when the borrower repays the loan). That is, the real interest rate is $R_1 = r - c$, and the actual rate of interest earned on an inflation-protected loan is $R_1^{(a)} = r - c + i_a$. The **"nominal" rate of interest** is defined as the interest rate on a non-inflation-protected loan. In terms of the variables we are using, the nominal rate is $R_2 = r + i_e + i_u$. (Note that here we are using nominal in a different sense than when we speak of a "nominal" annual rate of interest convertible *m* times per year.)

The difference between the nominal and real rates of interest is:

(8.16) $$R_2 - R_1 = i_e + i_u + c$$

Although we cannot observe any of the individual components of the interest rates (r, c, i_e, or i_u), we can observe the *difference* between R_2 and R_1, which represents the combined value of i_e, i_u, and c. Commentators sometimes say that the difference between the nominal and real interest rates $(R_2 - R_1)$ is the "expected" rate of inflation over the term of the loan. We can see that this is not accurate, since it ignores the effect of i_u and c on that difference, so $(R_2 - R_1)$ *overstates* the expected rate of inflation. However, when the values of i_u and c are small relative to i_e, the statement is relatively accurate:

(8.17)

When i_u and c are small relative to i_e:

$$i_e \approx R_2 - R_1$$

We noted earlier that the real rate of interest can be negative (when the compensation for inflation protection (c) is greater than the compensation for deferred consumption (r). The nominal rate of interest can also be negative under certain conditions. This can happen when $r + i_e + i_u$ is less than 0. Since i_u, the compensation for unexpected inflation, cannot be less than 0, either r or i_e must be negative in order for the nominal rate to be negative.

It is possible for expected inflation, i_e, to be negative in a time of decreasing prices (negative inflation, known as **deflation**). When that happens, a loan will be repaid with dollars that are *more valuable* than the dollars that were borrowed. Logically, this could lead to a negative nominal interest rate. But a lender who faces the prospect of lending at a negative interest rate could simply store the money instead of lending it, and thus realize an increase in purchasing power without suffering the cost of a negative interest rate. In effect, a borrower who pays a negative interest rate is charging a fee for storing the lender's money. If the lender has a less expensive way to store the money safely, then that is a better alternative than lending at a negative rate.

Note: Under current conditions at this writing (2017), negative interest rates are being used in certain situations. Some central banks (including the Bank of Japan and the European Central Bank) pay negative interest on the excess reserves that their member banks maintain at the central bank. This is done in order to encourage the member banks to make loans (and thus reduce their excess reserves), with the goal of stimulating the economy. In addition, the German government's short-term zero-coupon bonds are trading at a premium, so their yield to maturity is negative. In effect, the bondholders are paying a price to have their money safely stored.

In the preceding analysis of inflation and interest rates, we have assumed that there is no risk of default. Considering that real-world loans do have a risk of default, we can express R^*, the interest rate for a non-inflation-protected loan as:

(8.18)

$$R^* = r + s + i_e + i_u$$

The quantity s, the compensation for default risk, is called the **credit spread** or **spread for credit risk** (or simply the "**spread**"). By comparing Formula **(8.18)** to Formula **(8.15)**, we can see that $s = R^* - R_2$. That is, the credit spread is the difference between the interest rate for a given loan and the interest rate for a comparable loan that is considered to have no credit risk (e.g., a Treasury security). A borrower with a poor credit rating will be offered a loan with a larger credit spread, and will thus pay a higher interest rate than a more creditworthy borrower.

The credit spread for a loan depends on the loan term, as well as the creditworthiness of the borrower. The set of spreads broken down by loan term is called the **spread curve**. Spread curves are similar to yield curves; they can take on various shapes, but they are usually upward-sloping (like the normal yield curve). The upward slope of the spread curve reflects the fact that longer-term loans present a greater risk of default. This makes the yield curve for loans with default risk steeper than the yield curve for risk-free loans. It is important to note that a higher rate of inflation could lead to lower credit spreads, since higher inflation can make it easier for borrowers to repay their loans and avoid default.

To make it easier to remember the various formulas, the following table shows the components of the loan interest rate for each situation we have described. For those situations where the study note has defined a variable (R, R^*, etc.), that variable is shown; otherwise, the table contains just an expression using the components of the interest rate. (All rates are continuously compounded.)

	with no default risk	with default risk
with no inflation	r	$R = r + s$
with a known rate of inflation	$r + i$	$R^* = r + s + i$
with an unknown rate of inflation ("nominal" int. rate)	$R_2 = r + i_e + i_u$	$R^* = r + s + i_e + i_u$
inflation-protected loan: contractual rate ("real" interest rate)	$R_1 = r - c$	$r + s - c$
inflation-protected loan: actual rate paid ("nominal" int. rate)	$R_1^{(a)} = r - c + i_a$	$r + s - c + i_a$

Section 8.3

Retail Savings and Lending Interest Rates

In the previous section, we used a conceptual approach to examine the "components" of an interest rate from a theoretical perspective. In this section we will take a practical approach and identify additional factors that affect the level of interest rates in the world of retail savings and lending. These factors include overhead costs, the options or guarantees included in a product, and the liquidity of financial markets.

One issue that was not addressed in the previous section is how lenders and borrowers find each other in order to arrange loans. In the real world, lenders and borrowers usually need assistance identifying each other. This assistance is provided by **financial intermediaries**, which are third parties who identify potential lenders (i.e., savers and investors) and potential borrowers, and provide a means for loan money to flow from lenders to borrowers, and for repayments to flow back to the lenders.

The most significant financial intermediaries in the retail market are **banks** and **savings and loan companies**. These institutions accept deposits from savers and make loans to individuals, businesses, and corporations. Their business model involves lending money at interest rates that are higher than the rates they pay on deposits. The margin between these interest rates must be adequate to pay overhead expenses, cover default losses, and provide a profit for the owners.

Note: Credit unions are another category of financial intermediaries, and their total assets are comparable to those of savings and loans. However, the study note does not mention credit unions, so they will not be discussed further.

Banks and savings and loans are heavily regulated in order to ensure the safety of their customers' money and to assure that the nation's payment system is safe and secure. The payment system is a network of banks and other financial organizations that process payments and arrange for funds to flow from the payer's account to the payee's account. Most transactions in developed economies are made by check or by debit or credit card, and these payments must pass through the banking system. As a result, the failure of a bank (especially a large bank) would damage or inconvenience many people beyond that bank's depositors.

Today there are other financial intermediaries besides banks and savings and loans. **Alternative lenders**, including many that operate only online, raise funds directly from investors, rather than accepting deposits from the general public. Because they do not accept deposits from the general public, these companies are not subject to banking regulations. But their business is similar to banking, so it has sometimes been described as "shadow banking." A number of these alternative lenders failed during the 2008-09 financial crisis, which added to the negative economic effects of that crisis. Although these alternative lenders are not banks, their underlying business model is the same: they charge more for loans than the cost of funding those loans.

In addition to alternative lenders, there are **alternative payment providers** that are not part of the traditional banking system. These include PayPal, Apple Pay, and (in some foreign countries) M-PESA. These organizations process payments for their users, usually via the Internet or smartphones, using encryption to keep the users' identities and transaction information secure. They are part of the growing industry known as **financial technology**, or **fintech**, which use new technologies to offer banking services in non-traditional ways. Other examples of fintech include crowdfunding and peer-to-peer lending.

Note: The study note mentions <u>Bitcoin</u> as an alternative payment provider. Technically, however, Bitcoin is a currency (or "cryptocurrency"), rather than a payment provider. The payment provider in this case (i.e., the means of transferring ownership of the Bitcoin currency) is the <u>Bitcoin blockchain</u>, which is a "distributed ledger" that is maintained by a large number of Bitcoin users, rather than by a centralized banking system.

Banks and savings and loans offer <u>two types of savings vehicles</u>: **savings accounts** and **certificates of deposit** (**CDs**). A savings account earns interest at a rate that may be adjusted by the institution from time to time, and the depositor is able to make withdrawals (and deposits) without incurring a penalty. By contrast, a certificate of deposit has a contractual interest rate and a fixed term; the depositor may not withdraw the funds before the end of the CD's term without incurring a "**penalty for early withdrawal**." The penalty typically involves the loss of part of the interest earned as of the withdrawal date (e.g., 3 months or 6 months of interest).

Note: Financial institutions also offer <u>checking accounts</u> (sometimes called demand deposits, current accounts, or transaction accounts). Since checking accounts pay no interest (or very little interest), and are used for transactional (rather than investment) purposes, they will not be included in this discussion of interest rates.

A very important factor affecting the interest rate paid on savings deposits is the financial institution's overhead expenses. A **full-service bank** that has many lines of business and a large number of branches has higher overhead expenses than an **online bank** with a narrower range of products and services. As a result, a full-service bank will frequently pay lower interest rates for savings, or charge higher interest rates for loans, or both, and the difference in interest rates (for traditional vs. online banking) can be significant. On the other hand, a large full-service bank may have economies of scale that some online competitors have not achieved, and this could allow the full-service bank to set its interest rates at competitive levels. The rate differences between traditional and online banks have decreased in recent years.

Another factor affecting interest rates is the business environment where the bank or savings and loan is located. If demand for loans is high in a particular area, banks that operate in that area may pay higher interest rates on savings in order to attract more deposits to fund their lending operations, and may also be able to charge higher rates for loans. This can lead to significant differences in interest rates between different regions of the country. Similarly, a bank that intends to increase the volume of loans on its books might pay higher savings rates to attract more deposits and/or charge lower rates for loans.

In the previous section, we noted that borrowers with poor credit ratings generally have to pay higher interest rates due to the higher risk of default. Similarly, a bank that has a weak credit rating or is perceived as less creditworthy may have to offer a higher interest rate in order to attract deposits. Even with their balances protected by the **FDIC** (**Federal Deposit Insurance Corporation**), depositors experience inconvenience and unexpected costs when a financial institution fails (e.g., lack of access to their funds for a period of time, and the need to transfer their accounts to a different bank). Lower-rated banks may need to pay higher interest rates in order to attract deposits. CD interest rates can be particularly affected, because the depositor must make a commitment to keep the money at that bank for the term of the CD. There are other ways that a bank can compensate for perceived weakness, such as by providing additional or discounted services, or offering free checking.

There are <u>three broad categories of lending products</u>:

- **Secured loans**: These are loans that are secured by property that can be repossessed or foreclosed on in the event of default; they include mortgages, home equity loans, and auto loans. The property that is pledged as security for the loan is called **collateral**.
- **Unsecured loans**: These are loans that are not secured by property, such as credit card debt.
- **Guaranteed loans** (or **loans with guaranteed repayment**): These are loans that are guaranteed by a third party; they include student loans (which may be guaranteed by the federal government) and high-ratio mortgages, where the loan amount is nearly 100% of the property value (which may be guaranteed by an individual, a corporation, or a federal government agency).

When considering the default risk of a prospective loan, the bank's loan officer will consider the borrower's credit rating, and also whether the loan is secured, unsecured, or guaranteed, as well as the characteristics of the security or guarantee. Just as a higher-risk borrower is likely to pay a higher interest rate, an unsecured loan generally carries a higher interest rate than a secured loan, and a non-guaranteed loan will have a higher interest rate than a guaranteed loan.

In the case of savings products, all customers generally receive the same interest rate for the same product (although there may be higher rates paid on larger deposits, reflecting economies of scale). This is not true for loans, where the interest rate will be affected by the borrower's creditworthiness, which depends on such things as the borrower's income, assets, and credit history. In analyzing a borrower's income, a record of paying bills on time is very important, and a steady paycheck (salary or wages) is viewed more favorably than an income dependent on sales commissions or the profits of a small business.

Interest rates for credit card debt have to comply with government regulations. Credit card companies generally cannot charge one customer a higher rate than another, unless that customer has missed required payments *on that credit card account*. In response to this restriction, credit card companies often segment their customers by designing different cards to attract different groups of customers. A card with characteristics preferred by high-risk borrowers will carry a higher interest rate than a card designed to attract low-risk borrowers.

A bank's **prime rate** is the interest rate that it charges to those borrowers that it judges to be most creditworthy. Few customers qualify for the prime rate, but it serves as a benchmark for setting the bank's other loan interest rates that are charged to the rest of its loan customers. The prime rates of different banks can also serve as a point of comparison among those banks, since a bank with a higher prime rate will likely have higher rates for all of its loan offerings.

The mortgage rates posted by banks are usually *higher* than the rates that are actually charged for an existing customer with a good credit record. Banks provide discounts from posted mortgage rates to reflect that a mortgage customer is likely to use the bank's other products. Profits from these additional products allow the bank to recover the value of the discount given on the mortgage interest rate.

Section 8.4

Bonds

Governments and large corporations, due to their massive scale, frequently borrow large sums of money (often billions of dollars). Even a large bank would be reluctant to lend such a large amount to a single borrower, because default could be catastrophic for the bank. Therefore, to borrow large amounts of money, governments and corporations access public financial markets by issuing bonds. We have studied bonds in Module 4 and have performed various calculations involving bonds. In this section, we will consider the factors that affect the interest rates (yields) that investors earn from bonds.

Bonds have some of the characteristics of CDs, since they have a stated term and interest rate (or coupon rate). But unlike CDs, bonds can be bought and sold after issue. In Module 4, we observed that bond prices are affected by market interest rates; an increase in interest rates causes bond prices to fall, and a decrease in rates causes bond prices to rise. Besides market interest rates, a number of other factors affect a bond's price (and therefore its yield), both at the time of issue and also when the bond is traded in the secondary market. These factors include:

- **creditworthiness** of the issuing corporation (or government)
- **liquidity** of the bond market
- **currency** in which the bond is denominated
- **time until maturity**
- anticipated rate of **inflation**
- **tax treatment** of the bond's coupons
- "**seniority**" of the bond issue in a corporation's capital structure and whether the bond issue is "secured" by specific corporate assets
- **aging** of the bond
  (i.e., the shortening of its term as the maturity date approaches)

The concept of seniority deserves explanation. A bond's seniority determines whether the bond's owner will be first (or last) in line to be paid off in the event of insolvency. Some bonds are "secured" by specific assets whose value will be used to redeem the bond issue after a default. Others are unsecured but "senior" to other unsecured debt. And some bonds may be "subordinated" to other more senior debt, but still senior to preferred and common stock.)

Zero-coupon bonds are useful in analyzing interest rates, because a zero-coupon bond's yield rate is a spot rate. (For example, the yield on a 3-year zero-coupon bond is the 3-year spot rate.) By contrast, the yield on a coupon bond represents a complex weighted average of the various spot rates for all of the bond's coupons and its maturity value. So a coupon bond's yield is not the spot rate for any particular term.

Zero-coupon bonds are created by "**stripping**" the coupon bonds of a particular issuer (most commonly the U.S. Treasury, but they could be the bonds of a different issuer). To "strip" bonds, a financial intermediary buys a large quantity of bonds and sells their coupons separately as zero-coupon bonds. (The coupons due in 6 months are combined and sold as 6-month zero-coupon bonds; the coupons due in 12 months are used to create 1-year zero-coupon bonds; etc.)

Based on the prices that investors are willing to pay for the zero-coupon bonds of an issuer, it is possible to compute the implied spot rates and create a yield curve, known as the **zero-coupon yield curve**, for that issuer. Using that yield curve, one can then estimate the price investors would pay for that issuer's coupon bonds with a particular coupon rate and term.

Example (8.19)

Prices for zero-coupon bonds are as follows:

Bond Term	6 months	1 year	18 months	2 years
Price per 100 Face	99.01	96.88	94.23	91.31

A bond that pays semi-annual coupons at a 6% (annual) rate has 2 years remaining until maturity. The bond's face amount (and maturity value) is 1,000. Calculate the bond's market price and its yield to maturity (as a nominal rate convertible semi-annually).

Solution.

The prices for zero-coupon bonds represent present value factors. For instance, the 1-year present value factor is $\frac{96.88}{100} = 0.9688$.

We are pricing a bond that pays a coupon of 30 every 6 months, and 1,000 at maturity. Its price is the present value of these payments:

$$\text{Price} = 0.9901 \cdot 30 + 0.9688 \cdot 30 + 0.9423 \cdot 30 + 0.9131 \cdot 1{,}030 = 1{,}027.53$$

To find the bond's yield to maturity, enter the following values into the BA II Plus's TVM worksheet:

N=4 PV=-1,027.53 PMT=30 FV=1,000 CPT I/Y→2.27

The effective yield *per period* is 2.27%. This is a 6-month effective rate, so the nominal yield convertible semi-annually is:

$$2.27\% \times 2 = 4.54\%$$

Exercise (8.20)

A bond that pays semi-annual coupons at a 2% (annual) rate has 2 years remaining until maturity. The bond's face amount (and maturity value) is 1,000. Using the zero-coupon bond prices in Example **(8.19)**, calculate the bond's market price and its yield to maturity (as a nominal rate convertible semi-annually).

Answers: 951.24 4.58%

In Example **(8.19)** and Exercise **(8.20)**, The zero-coupon yields for the issuer are higher for its longer-term zero-coupon bonds. (This is to be expected when we have a normal yield curve.) As a result, the yield for the bond with a 2% coupon rate is higher than the yield for the bond with a 6% coupon rate. This is because a bond with smaller coupons has a yield that puts more emphasis on the spot rate for its maturity value (2 years, in this case), and less emphasis on the shorter-term spot rates (the ones that apply to the coupons). In general, bonds with smaller coupons will have higher yields than bonds from the same issuer that have larger coupons (and have the same term).

If we want to create a yield curve for *coupon* bonds (rather than zero-coupon bonds), we have to recognize that the yield will vary by coupon rate as well as by term. The usual approach for creating a **coupon bond yield curve** is to use the yields for *par* bonds, thus creating a **par bond yield curve**. (A par bond is a bond whose price is equal to its face amount (i.e., its "par value").)The following example demonstrate how spot rates can be used to find the yield for a par bond.

Example (8.21)

The following are annual effective rates from a spot-rate yield curve:

Term	1 year	2 years	3 years
Spot Rate	3.3%	4.8%	5.8%

Find the annual effective yield for a 3-year annual-coupon par bond.

Solution.
A par bond sells at par because its coupon rate equals its yield. In this case, we don't know the coupon rate or the yield. However, if we solve for a coupon rate that will make the bond's price equal to its face amount, then that yield will also be the coupon rate for a par bond.

Let x be the annual coupon for this bond. Then its price is:

$$\text{Price} = x \cdot \left(1.033^{-1} + 1.048^{-2} + 1.058^{-3}\right) + 1{,}000 \cdot 1.058^{-3} = 2.7229x + 844.39$$

Setting the price equal to 1,000, we have:

$$2.7229x + 844.39 = 1{,}000$$

$$x = \frac{1{,}000 - 844.39}{2.7229} = 57.15$$

The coupon rate (and yield) for a 3-year par bond is: $\dfrac{57.15}{1{,}000} = 5.715\%$.

Exercise (8.22)

Using the same spot rates as in Example **(8.21)**, find the annual effective yield (and coupon rate) for a 2-year annual-coupon par bond.

Answers: 4.765%

In **(8.21)** and **(8.22)**, we have found the par coupon bond rates for 2-year and 3-year bonds. In combination with the 1-year spot rate (which is also the coupon rate for a 1-year par bond), we have values for a par bond yield curve:

Bond Term	1 year	2 years	3 years
Par Bond Yield	3.300%	4.765%	5.715%

As expected, these yields are slightly lower than the spot rates on which they are based.

U.S. Treasury Securities

U.S. Treasury bonds have for long been considered the most secure bonds available anywhere, and they are also the most widely held and actively traded of all bonds. Treasury securities go by three names: bills, notes, and bonds. **Treasury bills** (**T-bills**) are short-term (52 weeks or less) zero-coupon bonds. T-bills mature for their face amount, and are offered to the public at a discount to the face amount. **Treasury notes** pay semi-annual coupons and have terms of 2 to 10 years. **Treasury bonds** also pay semi-annual coupons, and they have terms greater than 10 years (up to 30 years).

Note: The Exam FM study note on Determination of Interest Rates uses the word "note" to describe all Treasury securities with a term of more than 1 year (i.e., it uses "note" to include both notes and bonds).

Treasury notes and bonds are issued with or without inflation protection. Those with fixed payments (i.e., without inflation protection) are called **nominal return bonds**. Treasury securities with inflation protection have been issued since the 1990s, and are less common than those without inflation protection. The inflation-protected Treasury securities are called **real-return bonds**, or **inflation-indexed** (or **inflation-linked**) **bonds**, or **Treasury Inflation-Protected Securities** (**TIPS**).

Real-return bonds are used by managers of pension plans that have benefits indexed to inflation. This provides a hedge against unexpectedly high inflation. The federal government is in a position to assume this risk, because its revenue (from taxes) is closely linked to changes in incomes and prices and asset values. Higher inflation means that the government collects more tax money, some of which is used to pay the increased coupons and maturity values on real-return bonds. Pension plans that hold inflation-indexed Treasury securities are transferring the risk of inflation to the federal government. Individual investors, particularly retirees, may also be attracted to real-return bonds to protect their savings from the effects of inflation.

Current yields for U.S. Treasury securities are published daily by the U.S. Federal Reserve. Yields are published for Treasury bills that are 1, 3, 6, and 12 months from maturity; these are, of course yield rates for zero-coupon bonds, so they are spot rates. And yields are published for Treasury notes and bonds that are 1, 3, and 6 months from maturity, and also 1, 2, 3, 5, 7, 10, 20, and 30 years from maturity. These are referred to as **"constant maturity" Treasury yields**.

These published rates are based on the daily closing prices for actively-traded Treasury securities. Since there may not be a Treasury security available with a particular term to maturity (or there may have been no trades in that security during the day), the Federal Reserve interpolates data from other securities that were traded. The yield rates for Treasury notes and bonds are based on securities that are trading close to par, and the published yields form a par bond yield curve.

The Treasury issues new securities in regularly scheduled auctions. These are usually held weekly for T-bills and less frequently for notes and bonds (e.g., 30-year bond auctions are typically held once every 3 months). At any time, the most recently issued bond of a particular type is called the **"on the run" bond**. Older bonds are described as **"off the run."** There is generally more active trading of on-the-run bonds than older bonds. As a result, on-the-run bonds are more liquid and trade at a higher price than off-the-run bonds that will mature on the same date. This higher price (and lower yield) is referred to as the **"on-the-run premium."**

The Federal Reserve also makes *historical* interest rates available. Page 31 of the study note contains a graph of Treasury yields from 1955 to 2015 (for 1-, 5-, and 10-year securities). It shows that interest rates had a long-term uptrend, from 2-4% in the 1950s to 15-17% in the early 1980s, followed by a long-term downtrend from that peak to rates of 0-2% in 2015, a level that had not been seen since the Great Depression years of the 1930s. Within those long-term trends there have been a series of short-term fluctuations, with rates rising and then falling in cycles of perhaps 4-5 years. Short-term rates have been the most volatile, falling farther than long-term rates during downtrends, and rising more than long-term rates during uptrends.

Treasury securities had a "normal" yield curve during most of the period shown in the graph (i.e., long-term rates were usually higher than short-term rates). However, at the peaks of **business cycles**, when the demand for money had driven up interest rates, short term rates sometimes rose above long-term rates, resulting in an "inverted" yield curve. An inverted yield curve tends to occur at the peak of an upward fluctuation in rates, and this is widely viewed as a sign that a **recession** is about to begin. A recession (a period of negative growth), or even a slowdown in an economy's growth, will generally reduce the demand for loans, as businesses and individuals are less willing to take on new debt. This results in a decrease in interest rates, reversing the short-term upward trend.

Municipal Bonds

Many state and local governments in the United States issue bonds. These are referred to as **municipal bonds**. (The word "municipal implies that these bonds are issued by cities or towns, but bonds issued by states, counties, and other governmental entities are also categorized as municipal bonds.) These bonds are usually issued to fund the construction or maintenance of infrastructure (e.g., roads and bridges) or buildings (e.g., schools and hospitals). Many of these governmental bodies have "balanced budget" requirements that prevent them from borrowing money to fund routine expenditures.

Municipal bonds fall into two categories:

- **Revenue bonds**, whose coupons and maturity values are paid from the revenues collected from a particular project (e.g., a toll bridge or a recycling facility)
- **General obligation bonds**, which are backed by the issuing entity's ability to levy taxes

While U.S. Treasury securities are generally regarded as being risk-free, municipal bonds carry some amount of default risk. There have been defaults on municipal bonds when the issuing governments went bankrupt. This suggests that the yields for municipal bonds should be higher than those for comparable Treasury securities, with the credit spread varying depending on the issuing body's creditworthiness. However, an investor's income from bonds issued by state and local governments in the U.S. is generally not subject to federal income tax, and also is usually not subject to state income tax in the state where the bonds are issued. Investors (particularly investors in high tax brackets) are therefore willing to accept a lower rate of return on a municipal bond than they would demand on a taxable bond (including Treasury securities, which are taxable at the federal level, but not at the state level). As a result, yields for municipal bonds are often lower than yields for similar bonds issued by the U.S. Treasury or by corporations. The issuers of municipal bonds typically set the coupon rates lower than comparable taxable bonds, knowing that this will allow the bonds to sell at or near par (instead of selling at a premium).

Government of Canada Bonds

Like the U.S. federal government, the **Canadian government** issues bonds, with terms ranging from 91 days to 30 years. Most of these are issued in Canadian dollars, but some are denominated in U.S. dollars. The reason Canada issues bonds in U.S. dollars is to meet the needs of foreign (particularly U.S.) investors who do not want their investment income to be affected by fluctuations in the value of Canadian dollars. The value of the Canadian dollar relative to other currencies varies over time, depending on the demand for Canadian dollars in international currency markets, which is largely driven by demand for Canadian assets or for goods and services provided by Canada. A foreign buyer of a Government of Canada bond will earn a return that depends on not only the return measured in Canadian dollars, but also on variations in the value of Canadian dollars relative to the currency of the investor's home country.

The Government of Canada has never defaulted on any of its bonds, and the safety of its bonds is regarded as comparable to that of U.S. Treasury securities. Thus the yields on Canadian government bonds denominated in U.S. dollars are nearly the same as the yields on comparable U.S. Treasury securities. Because the Canadian issues trade less frequently than U.S. securities, there can be some variation in yields between U.S. and Canadian government securities. By contrast, the yields for bonds issued in Canadian currency can vary significantly from yields on U.S. Treasury securities, since they are affected by the level of activity in the Canadian economy and the resultant supply and demand for Canadian dollars. However, for a Canadian investor, Government of Canada bonds can be viewed as risk-free, in the same way that a U.S. investor regards U.S. Treasury securities as risk-free. This highlights the fact that "risk-free rate" depends not only on the creditworthiness of the issuing body, but also on the currency in which the bond is denominated and whether that is the home currency of the investor.

Just as in the United States, **Canadian provincial and local governments** issue bonds, and these bonds are subject to default risk. In general, the yields on these bonds will be higher than the yields on comparable bonds issued by the Government of Canada.

Corporate Bonds

Corporations issue bonds for a variety of purposes that range from providing short-term cash to funding major construction or expansion projects. After issue, however, **corporate bonds** trade much less frequently than government bonds, with some issues experiencing little or no trading as the original purchasers collect their payments over the term of the bond.

Investment brokers maintain an inventory of the more frequently traded bonds and make a market for these bonds by being willing to trade them with the public as either buyer or seller. Naturally, the broker seeks to earn some margin on these purchases and sales. To accomplish that, the broker maintains a **bid price** at which it is willing to buy a particular bond, and a higher **ask price** (or offer price) at which it is willing to sell the bond. The difference between these two prices (ask price minus bid price) is called the **bid-ask spread** (also known as the bid-offer spread or bid-sell spread). A yield for the bond can be calculated using either the bid price or the ask price. Because the bid price is *lower* than the ask price, the **bid yield** will be *higher* than the **ask yield**.

If a bond is traded frequently, its bid and ask prices are more readily measured than if it is traded very infrequently. It is easier for an investor to buy or sell an actively traded bond, as there are many buyers and sellers willing to participate in the trade. Such a bond is said to have greater **liquidity** than a **thinly traded** bond (i.e., it is easier to buy or sell the heavily traded bond). If a bond is thinly traded, a broker who maintains an inventory of that bond will set a larger bid-ask spread as compensation for holding bonds that are not easily traded, and to protect itself against a sudden drop in price (for that bond, or for the entire bond market).

Rating agencies (such as Moody's Investors Service, Standard and Poor's, and Fitch Ratings Inc.) rate bonds by evaluating the financial strength of the issuing entity and its ability to pay the coupons and maturity value on a timely basis. For a given publicly traded bond, each of these rating agencies may publish a **rating** that reflects the likelihood that the issuer will default on the bond (and the potential recovery in the event of default). Naturally, a bond with a higher rating will tend to have a lower yield. These ratings use the letters of the alphabet, with A ranking higher than B, and AA ranking higher than A. The highest-rated bonds get a rating of AAA (or Aaa, in the case of Moody's). The ratings assigned to a given bond issue by the different agencies tend to be quite similar, although the lettering schemes they follow are somewhat different, as shown in the table on the following page.

A bond is in default when the issuing corporation (or government) is bankrupt and fails to pay the bond's scheduled coupons (or maturity value). The amount that investors can then recover on their investment will depend on the bond's **seniority** in the issuing corporation's capital structure. Seniority determines the order in which the various creditors of a bankrupt corporation will have their claims met from the corporation's remaining assets. Common stock has the lowest seniority, and common stockholders can expect to receive little or nothing from a bankrupt corporation. Preferred stockholders rank slightly higher, but not as high as holders of the corporation's debt, including its bondholders. A corporation will often issue different bond series with different seniorities. Some bonds may be "**secured debt**" (i.e., they are secured by specified corporate assets); some may be unsecured but "senior" to other bond series; and others will be **"subordinated"** (i.e., ranking lower than the corporation's other debt, but higher than its preferred and common stock).

<table>
<tr><th colspan="2">Moody's</th><th colspan="2">S&P</th><th colspan="2">Fitch</th><th colspan="2" rowspan="2">Rating description</th></tr>
<tr><th>Long-term</th><th>Short-term</th><th>Long-term</th><th>Short-term</th><th>Long-term</th><th>Short-term</th></tr>
<tr><td>Aaa</td><td rowspan="6">P-1</td><td>AAA</td><td rowspan="4">A-1+</td><td>AAA</td><td rowspan="4">F1+</td><td>Prime</td><td rowspan="10">Investment-grade</td></tr>
<tr><td>Aa1</td><td>AA+</td><td>AA+</td><td rowspan="3">High grade</td></tr>
<tr><td>Aa2</td><td>AA</td><td>AA</td></tr>
<tr><td>Aa3</td><td>AA−</td><td>AA−</td></tr>
<tr><td>A1</td><td>A+</td><td rowspan="2">A-1</td><td>A+</td><td rowspan="2">F1</td><td rowspan="3">Upper medium grade</td></tr>
<tr><td>A2</td><td>A</td><td>A</td></tr>
<tr><td>A3</td><td rowspan="2">P-2</td><td>A−</td><td rowspan="2">A-2</td><td>A−</td><td rowspan="2">F2</td></tr>
<tr><td>Baa1</td><td>BBB+</td><td>BBB+</td><td rowspan="3">Lower medium grade</td></tr>
<tr><td>Baa2</td><td rowspan="2">P-3</td><td>BBB</td><td rowspan="2">A-3</td><td>BBB</td><td rowspan="2">F3</td></tr>
<tr><td>Baa3</td><td>BBB−</td><td>BBB−</td></tr>
<tr><td>Ba1</td><td rowspan="14">Not prime</td><td>BB+</td><td rowspan="6">B</td><td>BB+</td><td rowspan="6">B</td><td rowspan="3">Non-investment grade speculative</td><td rowspan="14">Non-investment grade
aka high-yield bonds
aka junk bonds</td></tr>
<tr><td>Ba2</td><td>BB</td><td>BB</td></tr>
<tr><td>Ba3</td><td>BB−</td><td>BB−</td></tr>
<tr><td>B1</td><td>B+</td><td>B+</td><td rowspan="3">Highly speculative</td></tr>
<tr><td>B2</td><td>B</td><td>B</td></tr>
<tr><td>B3</td><td>B−</td><td>B−</td></tr>
<tr><td>Caa1</td><td>CCC+</td><td rowspan="5">C</td><td rowspan="5">CCC</td><td rowspan="5">C</td><td>Substantial risks</td></tr>
<tr><td>Caa2</td><td>CCC</td><td>Extremely speculative</td></tr>
<tr><td>Caa3</td><td>CCC−</td><td rowspan="3">Default imminent with little prospect for recovery</td></tr>
<tr><td rowspan="2">Ca</td><td>CC</td></tr>
<tr><td>C</td></tr>
<tr><td>C</td><td rowspan="3">D</td><td rowspan="3">/</td><td>DDD</td><td rowspan="3">/</td><td rowspan="3">In default</td></tr>
<tr><td rowspan="2">/</td><td>DD</td></tr>
<tr><td>D</td></tr>
</table>

from https://en.wikipedia.org/wiki/Bond_credit_rating

For two similar bonds issued by the same corporation, the more senior bond should have a higher price (and a lower yield) than the less senior bond. Although seniority is a consideration in assigning bond ratings, it is less significant than the corporation's overall creditworthiness, since ratings are mainly an indicator of the likelihood of default, rather than the expected recovery after a default has occurred. (Obviously, seniority is more important in the case of lower-rated issues, where default is not an unlikely occurrence.)

Some bonds contain an option for the issuer or the owner to redeem the bond prior to maturity. An *issuer's* right to redeem a bond is referred to as a **call provision**. As was discussed in Module 4, a call provision usually provides that the issuer may "call" the bond on any coupon date after a certain period (e.g., after 5 years from the issue date), and it may also specify that the redemption value will exceed the maturity value by a particular amount (the "call premium"). A *bondholder's* right to redeem a bond is a **put provision**, and it allows the holder to "put" (sell) the bond back to the issuer before maturity at a specified price (usually below the bond's maturity value). Such provisions provide an **option** to either the issuer (a call) or the bondholder (a put). The value of that option will affect the bond's price (and therefore its yield).

Naturally, an issuer will have an incentive to call an outstanding bond if interest rates drop to a level where the bond's price exceeds its redemption value (including the call premium, if any). Actually, interest rates must be low enough to justify incurring the expense of issuing new bonds at the lower current interest rates. Buyers of callable bonds assume the risk that the bonds will be called away from them at a time when reinvestment opportunities offer lower yields. Consequently, the prices they are willing to pay for callable bonds are lower (and the yields are higher) than for similar noncallable bonds.

The characteristics of put provisions are the mirror image of those for call provisions. The option of "putting" a bond when interest rates rise (and bond prices fall) allows a bondholder to redeem a bond and then reinvest the proceeds in a higher-yielding bond. The issuer is required to pay the redemption value, and will then likely have to borrow funds at the higher interest rates in effect at that time. Because of the value of the put provision, a puttable bond will sell at a higher price and provide a lower yield than an otherwise similar bond without the put provision.

Corporations that do business in more than one country may issue different series of bonds denominated in different currencies. This can provide the issuer with some protection against the risk of currency fluctuations in the various countries where it operates. As with government bonds denominated in different currencies, the particular currency used and the economic conditions (including inflation) in the country that issues that currency will affect the prices and yields of such bonds.

It is important to understand that the spread between the yield of a corporate bond and the yield of a similar risk-free government bond may reflect additional considerations beyond the corporation's creditworthiness. In particular, the spread can be affected by the value of any embedded options (e.g., put or call provisions), and by any currency risk that the bondholder takes on (if the bond is not denominated in the bondholder's home currency).

Aging of a bond refers to the gradual shortening of its term as time passes. For example, a 10-year Treasury bond eventually becomes a 1-year bond, and at that point its yield to maturity will be typical of a 1-year bond, rather than a 10-year bond. The fact that a bond's yield tends to decrease as it ages leads to an investment strategy known as "**rolling down the yield curve**." This refers to a bondholder's selling a bond prior to its maturity date in order to realize a higher yield. This is illustrated by the following example.

Example (8.23)

An investor purchases a 5-year bond that pays a 6% annual coupon. Based on the bond's purchase price, its yield to maturity is 5.4%. After 4 years (when the bond has only 1 year remaining), the investor sells the bond at a price such that the buyer will realize a 3.2% yield to maturity. What was the investor's annual effective yield over the 4-year period that the bond was owned?

Solution.
The purchase price can be found using the TVM worksheet:
Set N=5, I/Y=5.4%, PMT=60, FV=1,000. Then CPT PV→-1,025.69.

The sale price can be found by simply calculating the present value of the single remaining payment at the 3.2% yield to maturity:

$$\text{Price} = \frac{1{,}060}{1.032} = 1{,}027.13$$

Finally, the investor's yield over the 4 years can be found using the TVM worksheet:

Set N=4, PV=-1,025.69, PMT=60, FV=1,027.13. Then CPT I/Y→5.882

The investor has earned an annual effective yield of 5.882%.

In this example, the bond was purchased to yield 5.4% to maturity. But by selling it one year before its maturity date, when its yield to maturity was only 3.2% (the 1-year spot rate on the sale date), the investor has earned a 5.882% yield over the 4 years that the bond was owned. The investor has taken advantage of the yield curve by selling the bond after its yield decreased due to "aging." This is an example of "rolling down the yield curve."

It is important to understand that the strategy of rolling down the yield curve is not guaranteed to improve the investor's rate of return. If interest rates rise so that (in the above example) the 1-year rate is greater than 5.4%, the investor's yield will not be improved by selling the bond after 4 years. This could be caused by an inversion of the yield curve, or simply by a parallel shift upward in a normal yield curve.

Exercise (8.24)

An investor purchases a 10-year bond that pays a 6% annual coupon rate. Based on the bond's purchase price, its yield to maturity is 5.8%. After 8 years (when the bond has only 2 years remaining), the investor sells the bond at a price such that the buyer will realize a 4.1% yield to maturity. What was the investor's annual effective yield over the 8-year period that the bond was owned? What would that yield have been if the bond had been sold with a 2-year yield to maturity of 6%?

Answers: 6.120% 5.763%

Section 8.5

The Role of Central Banks

In addition to being affected by the factors mentioned in previous sections, a country's interest rates are also influenced by actions of the country's central bank. Nearly every developed country has a central bank, which serves as a banker for that country's commercial banks. The two key responsibilities of a central bank are the operation of the country's payment system and serving as "lender of last resort" for the country's banks. A central bank usually issues the country's currency, and may also have very broad responsibilities, such as supervising banks, controlling the currency, and even managing the country's economy.

Operation of the Payment System

When a purchase is made by credit card, debit card, check, or electronic funds transfer, the money must be moved from the buyer's account to the seller's account through the country's **payment system**. This is easy to accomplish in the case of a check on Bank A that is deposited into another account at Bank A; the bank simply records a transfer of funds between the two accounts. But when the check is deposited at a different bank, then the information (and the money) must pass between the two banks through a network of financial intermediaries. This could involve only the two banks if they maintain a relationship, or it could involve a local bank network that they both belong to, or the transaction might pass through a whole series of intermediaries, including commercial banks and the central bank.

Each of the institutions that participates in such a transaction must have confidence that the other participants will meet their obligations. The central bank plays a key role in this, because it will make good on these obligations if one of the intermediaries is not able to do so. The central bank can provide this assurance because all authorized banks are required to maintain a certain amount on deposit with the central bank. The required balance depends on that bank's customers' deposits, the volume of transactions the bank processes, and the relative importance of the bank to the overall payment system. The bank's balance with the central bank is called its **reserve**, and the balance must equal or exceed the bank's **reserve requirement**.

If the central bank is called on to make good on a bank's obligation, it will withdraw the required funds from that bank's reserve account to complete the transaction. Banks' reserve accounts also provide the means for settling transactions. Each day a bank's reserve increases or decreases, depending on whether all of its transactions that flowed through the Federal Reserve bank resulted in a net positive or negative change in its balance. Banks that do not have an account at the central bank maintain accounts at **correspondent banks** that do have central bank accounts, and those correspondent banks settle transactions for their customer banks in a similar manner.

Lender of Last Resort

A country's central bank also acts as **lender of last resort** for its member banks. If a bank experiences heavy withdrawals by its depositors, impairing its ability to maintain the required reserve, the central bank will lend the funds needed to meet the required reserve, using as collateral the bank's invested assets (e.g., the loans on its books and the securities that it owns). In this situation, the bank could borrow from other banks rather than from the central bank. (This is arranged by means of overnight borrowing of *excess* reserves that another bank holds with the central bank.) However, even if the other banks are unwilling to lend to the bank that is experiencing high withdrawal rates, the central bank is committed to providing such loans, which is why it is called the "lender of last resort."

The central bank has two sources of money to lend to banks in these situations. One source is the reserve accounts maintained by its member banks. The other is currency that the central bank is authorized to issue (assuming that it does have this authority). If a central bank does not have authority to issue currency, it could have difficulty satisfying the borrowing needs of its member banks when withdrawal rates are high throughout the economy. In that situation, it may be necessary to limit the amounts that bank customers can withdraw from their accounts.

United States Federal Reserve System

The central bank of the United States is the **Federal Reserve System**, which is also known as "the Fed." It consists of 12 regional **Federal Reserve Banks**, a 7-person **Board of Governors**, and the **Federal Open Market Committee**. A regional Federal Reserve bank holds the reserves of its member banks, provides payment services, and makes loans to member banks. The Board of Governors oversees the Federal Reserve banks and helps implement the **monetary policy** of the United States. Governors are appointed by the U.S. President and confirmed by the U.S. Senate; they serve staggered terms of 14 years (with one new Governor appointed every two years). The Chairman of the Board of Governors (one of its 7 members) is appointed by the U.S. President to serve a 4-year term as Chairman.

The 12-member Federal Open Market Committee (FOMC) consists of the 7 members of the Board of Governors, plus the presidents of 5 of the regional Federal Reserve banks. The FOMC sets policy for the Fed's **open market operations**. Its most visible task is to set a target level for the **federal funds rate**. This is the interest rate at which banks lend funds to each other; this lending occurs when a bank with excess reserve funds at its regional Federal Reserve bank lends part of that excess to another bank that has inadequate reserves. These are overnight loans. To influence the level of the federal funds rate, Federal Reserve banks buy and sell Treasury bills in the secondary market.

When Federal Reserve banks buy Treasury bills, that tends to decrease short-term interest rates, including the rates for overnight loans (because the purchase tends to raise the price of Treasury bills and lower their yields). Purchasing T-bills also injects cash into the economy, increasing banks' reserves and decreasing the demand for overnight loans, thus decreasing the equilibrium interest rate for those loans. When Federal Reserve banks *sell* Treasury bills, it has the opposite effect, and the interest rate for overnight loans tends to increase. By buying or selling Treasury securities, Federal Reserve banks influence the federal funds rate that banks charge each other, and seek to keep that rate close to the target rate set by the FOMC.

As previously mentioned, banks can borrow reserves from their Federal Reserve bank. The Board of Governors sets the interest rate for these loans, and it is called the **discount rate**. Banks usually borrow from other banks rather than "going to the discount window" (i.e., borrowing from the Federal Reserve). There are two reasons that banks prefer to borrow from other banks rather than from the Fed. First, the rate offered by other banks is usually lower than the Fed's discount rate. In addition, if a bank borrows from another bank, this demonstrates that its peer banks consider it creditworthy so that it doesn't need to turn to the lender of last resort. (As of December 22, 2016, the effective federal funds rate (the average rate charged on overnight interbank loans) was 0.66%, and the Fed's target range for this rate was 0.50%-0.75%. The discount rate for loans from the Fed was 1.25%.)

Although the federal funds rate is a short-term rate, when the FOMC adjusts its target for the federal funds rate, it affects both short-term and long-term interest rates in the economy. If the Fed increases its target rate, banks will seek to avoid borrowing reserves, which means they need to increase their deposits with the Fed. They accomplish this by making fewer loans to customers, which reduces the supply of loans to businesses and individuals, resulting in higher interest rates for both short-term and long-term loans. When the FOMC acts to *decrease* the federal funds rate, it has the opposite effect, so that both short-term and long-term interest rates tend to *decrease*.

The FOMC also influences longer-term market interest rates by providing **forward guidance** after each of its meetings. This provides information to individuals and businesses about the expected future course of the Fed's monetary policy. The forward guidance gives an indication of the direction of future changes in the federal funds rate. This assists borrowers and lenders in making decisions about longer-term loans.

The responsibilities of the Federal Open Market Committee are referred to as **monetary policy**. (This is in contrast to **fiscal policy**, which refers to the government's taxing and spending policies, which are *not* the responsibility of the FOMC or the Fed.) The FOMC's broad goals include **price stability**, **full employment**, **moderate long-term interest rates**, and **continuing economic growth**. These goals can be in conflict. For example, low interest rates promote full employment, but they also tend to result in higher inflation. Consequently, the FOMC must take a balanced approach to achieving its various goals.

Section 8.6

Formula Sheet

All functions are expressed as continuously compounded rates.

r loan interest rate without default risk or inflation
(reflects compensation for deferred consumption only)

s compensation for risk of default

i rate of inflation (used when we assume the rate of inflation is known)

i_e expected rate of inflation

i_u compensation for the risk of unexpected inflation

i_a actual rate of inflation (not known until the end of the loan period)

c cost (to lender) of inflation protection

R loan rate reflecting deferred consumption and default risk

$$R = r + s$$

R_1 "real" rate for an inflation-protected loan with no default risk
(reflects deferred consumption and cost of inflation protection)

$$R_1 = r - c$$

$R_1^{(a)}$ actual rate earned on an inflation-protected loan with no default risk

$$R_1^{(a)} = r - c + i_a$$

R_2 "nominal" rate for a non-inflation-protected loan with no default risk
(reflects deferred consumption and inflation)

$$R_2 = r + i_e + i_u$$

Nominal rate – Real rate: $R_2 - R_1 = i_e + i_u + c$
if i_u and c are small, $R_2 - R_1 \approx i_e$

R^* loan rate reflecting deferred consumption, default risk, and inflation (without inflation protection)

- if inflation rate is known: $R^* = r + s + i$
- if inflation rate is unknown: $R^* = r + s + i_e + i_u$

Credit spread: $R^* - R_2 = s$

Section 8.7

Basic Review Problems

1. If the number of people interested in lending money for short periods of time decreases, in which direction does this shift the supply curve for short-term loans, and how does that affect the equilibrium interest rate for short-term loans?

2. How does an increase in the rate of inflation affect interest rates? Explain the reason for this from the perspective of lenders (the supply), and also from the perspective of borrowers (the demand).

3. Who assumes the risk of inflation in an inflation-protected loan, the borrower or the lender? Who assumes the risk of inflation in a non-inflation-protected loan?

4. The interest rates that a bank charges for loans are higher than the rates it pays depositors. What are the items that must be provided for by the difference between these rates?

5. What is the term that describes a business that provides a conduit between those wishing to save or invest money and those wishing to borrow money? What are the two major categories of these institutions?

6. If there is an increased demand for loans in a particular region of the country, how might that affect the interest rates that banks in that region offer for savings accounts and certificates of deposit?

7. What is a bank's "prime rate"? Why is it significant to borrowers who are comparing banks?

8. Interest rates for Treasury securities have tended to move in cycles of 4-5 years, first rising and then falling. At what point in the cycle is the yield curve for Treasury securities most likely to be "inverted" (with higher rates for short-term securities than for long-term securities)?

9. State (or provincial) and local governments issue revenue bonds and general obligation bonds. What is the difference between these two types of bonds?

10. What is meant by the "bid price" and the "ask price" for a bond? Which price is higher? What does this mean for the "bid yield" and "ask yield" for the bond? How is the "spread" between the bid and ask prices affected by the bond's liquidity?

11. What is meant by the "seniority" of a bond? How does seniority affect the bond's yield? Does seniority have a larger or smaller effect on yield than the issuer's overall creditworthiness?

12. What are the two core functions of all central banks? What other functions might central banks be authorized to perform?

13. What are the three components of the U.S. Federal Reserve System?

14. What is the federal funds rate? What is the Federal Reserve's "discount" rate? Which rate is higher?

15. How does an increase in the Federal Reserve's target for the federal funds rate affect interest rates and the volume of loans in the economy? How is the supply curve for loans affected?

Section 8.8

Basic Review Problem Solutions

1. When there are fewer lenders in a given category (e.g., short-term loans), the supply curve for loans of that type shifts upward and to the left. As a result, the equilibrium interest rate (where the supply and demand curves intersect) is increased.

2. A higher inflation rate tends to cause interest rates to increase. Lenders require a higher rate of interest to compensate for the decreased purchasing power of the repayments they will receive. Borrowers tend to be willing to pay a higher interest rate, because inflation will likely raise their wages, making them more capable of paying the higher interest rate. Both the supply and the demand curves tend to shift upward.

3. In an inflation-protected loan, the repayments are adjusted to compensate for the effect of inflation, so the borrower pays the cost of inflation. The borrower is assuming the risk of inflation.

In a non-inflation-protected loan, the borrower's repayment amounts are fixed and are not affected by inflation. The lender is adversely affected by inflation, since this reduces the value of the repayments that will be received. It is the lender who assumes the risk of inflation. The borrower is actually helped by inflation, if the inflation results in higher wages for the borrower.

4. The difference between interest rates charged on loans and interest rates paid on deposits provides the bank with money to pay for its overhead expenses, to cover the losses on defaulted loans, and to allow for a profit.

5. A business that connects savers/investors with borrowers is called a "financial intermediary." The two major categories of financial intermediaries are banks and savings and loan companies.

(The study note does not mention credit unions, but the total assets of credit unions in the United States are comparable to those of savings and loans. To the extent that some insurance products (such as annuities) can be regarded as savings products, insurance companies also act as financial intermediaries.)

6. When demand for loans increases, banks may increase the interest rates they pay on customer deposits (savings accounts and CDs) in order to attract funds to meet the increased demand for loans.

7. A bank's prime rate is the interest rate that it charges on loans to its most creditworthy customers. Although very few customers qualify for this rate, it is significant to borrowers who are shopping for loans, since a bank with a more attractive (lower) prime rate for prime borrowers is likely to offer more attractive rates to non-prime borrowers as well.

8. The yield curve tends to invert when interest rates are reaching the <u>peak</u> of a cycle.

9. Revenue bonds have payments that are secured by revenue that the issuing entity receives from a particular project (e.g., by tolls collected from a toll road or bridge, or charges for water or sewer service). General obligation bonds are secured by the issuing entity's general authority to levy taxes.

10. The "bid price" is the price that an investment broker who makes a market in the bond is willing to pay for the bond. The "ask price" is the price at which the broker is willing to sell the bond. Naturally, the ask price is higher than the bid price. The difference (or "spread") between these two prices is the margin that the broker can earn by making a purchase *and* a sale of the bond.

Since bond yields move in the opposite direction of prices, the bid yield (which is based on buying the bond at the bid price) is higher than the ask yield (which is based on paying the higher ask price for the bond).

If a bond is actively traded (i.e., if it is more "liquid"), the bid-ask spread tends to be smaller than if it is less actively traded, since the investment broker is taking less risk of being unable to realize the spread. If a bond is thinly traded, then the broker usually sets a larger bid-ask spread to compensate for the risk of losing money when the price of the bond shifts.

11. A bond's seniority is its rank in the corporation's capital structure, in terms of the order in which the corporation's creditors would be paid in the event of insolvency. When a bond issue is offered to the public, it may be explicitly assigned a higher or lower seniority (priority) relative to the corporation's other obligations. Alternatively, the bond may be secured by certain of the corporation's assets, so that the bondholders have first call on the value of those assets if the corporation becomes insolvent.

If a corporation becomes insolvent (bankrupt), its assets are used to pay the creditors in order of seniority, with the owners of common stock having the lowest seniority (and usually receiving nothing, since the corporation's assets have less value than its obligations). A bond with greater seniority is more valuable than a similar bond of the same corporation that has lower seniority. Consequently, the more senior bond sells for a higher price and generates a lower yield.

Seniority affects yield, but the corporation's overall creditworthiness has a greater effect on yield. If the corporation does not default, the bond's seniority doesn't matter, so a potential investor is concerned primarily with whether the corporation is likely to default on the bond (and only secondarily on how much value can be salvaged after default occurs).

12. The two core functions that a country's central bank performs are facilitating the country's payment system (clearing checks and processing credit transactions) and acting as "lender of last resort" to banks authorized to do business in the country. Other functions that a central bank may perform if it is authorized to do so include control of the currency (including issuing currency), supervision of the country's banks, and management of the country's economy.

13. The U.S. Federal Reserve System consists of the Board of Governors, the twelve regional Federal Reserve banks, and the Federal Open Market Committee.

14. The federal funds rate is the rate at which banks lend each other reserves they hold in Federal Reserve accounts. The discount rate is the rate at which banks borrow from their Federal Reserve bank. The discount rate is higher than the federal funds rate, which gives banks an incentive to borrow from each other, rather than from the Federal Reserve.

15. An increase in the target for the federal funds rate (implemented by the Federal Open Market Committee's actions in buying and selling bonds) makes it more expensive for a bank to have a shortfall in its reserves. When that happens, banks keep more money on deposit with the Federal Reserve by issuing fewer loans. This causes the supply curve for loans to shift upward and the equilibrium interest rate for loans to increase.

NOTE: As of the time this manual is being prepared (February 2017), the Society of Actuaries has not yet published sample exam problems for the material in Module 8 (Determinants of Interest Rates), which has recently been added to the syllabus for Exam FM. Consequently, there are no Sample Exam Problems for Module 8.

The student is encouraged to check the Society's website periodically to see whether sample exam problems for this material have been added. The web page for Exam FM is:

https://www.soa.org/education/exam-req/edu-exam-fm-detail.aspx

Section 8.9

Supplemental Exercises

1. How does increased <u>uncertainty</u> about future inflation affect the interest rates that lenders and borrowers find acceptable?
(Answer separately for borrowers and for lenders.)

2. A 2-year inflation-protected loan of 10,000 specifies an annual effective interest rate of 2.5% and level annual repayments (before adjustment for inflation). At times 0, 1, and 2, the price index specified in the loan agreement has the values shown in the following table:

Time (years since loan origination)	0	1	2
Index	191.7	199.3	202.6

What are the amounts of the loan payments at the end of the first and second year of the loan?

3. The compensation lenders require for default risk varies by loan term. Does it tend to increase or decrease as the loan term increases? What is the name for the relation between loan term and the charge for default risk?

4. How do alternative lenders raise the funds to make loans to borrowers?

5. If a bank has a low credit rating from the rating agencies, how might that affect the interest rates that it offers for savings accounts and certificates of deposit?

6. What are the differences among secured, unsecured, and guaranteed loans? Give an example of each type of loan, and describe how interest rates on loans are affected by being secured or having a guarantee.

7. The loan interest rates that a bank advertises for its loans are not necessarily the rates it actually charges. What factors might affect the interest rate charged to a particular loan customer?

8. Name at least 6 factors that affect a bond's yield (and therefore its price).

9. How are zero-coupon bonds (strip bonds) created?

10. Two bonds issued by the same corporation have the same term. One bond has a higher coupon rate than the other. Which bond will typically have the higher yield?

11. Sponsors of pension plans providing benefits that are indexed to inflation often purchase Treasury Inflation-Protected Securities (TIPS) as a hedge against inflation, thus transferring the inflation risk to the federal government. Why is the federal government well-positioned to assume the risk of inflation?

12. Why are the coupon rates and yields for municipal bonds issued by state and local governments in the United States typically lower than the coupon rates and yields on similar corporate bonds?

13. Why does the Canadian government issue some bonds denominated in U.S. currency?

14. The Canadian government has never defaulted on a bond issue, so its bonds that are denominated in U.S. dollars are essentially as risk-free as U.S. Treasury bonds. Nonetheless, Government of Canada bonds denominated in U.S. dollars tend to have slightly higher yields than similar U.S. Treasury bonds. What is the reason for this higher yield?

15. Under what circumstances is a corporation likely to exercise a bond's call provision? Is the price of a callable bond higher or lower than a similar bond without a call provision? If a bond has a "put" provision, will it be priced higher or lower than a bond without a put provision? Under what circumstances would the put provision be exercised?

16. What factors are used to set the amount of a U.S. bank's reserve requirement (the amount of funds it must keep on deposit with its Federal Reserve bank)?

17. The Federal Open Market Committee is responsible for implementing the monetary policy of the federal government. What functions are involved in this role? How do they affect interest rates in the economy?

18. Do bank prefer to meet their reserve shortfalls by borrowing from the Federal Reserve, or by borrowing from each other? Give two reasons for this preference.

19. Under the provisions of the Federal Reserve Act, what are the goals of the Federal Reserve's monetary policy?

Use the following definitions to answer questions 20. and 21.:

- r interest rate that would be charged for a risk-free loan if there were no inflation (i.e., r is compensation for deferred consumption only)
- i_e the expected rate of inflation
- i_u compensation the lender requires for the risk of unexpected inflation
- i_a actual inflation rate during the loan period (determined after the end of the loan period)
- c compensation the borrower requires for providing inflation protection

All of the above are continuously compounded annual rates.

20. Given two loans with identical characteristics except that Loan A has inflation protection and Loan B does not, write an expression (using the above variables) that represents the difference between the higher-rate loan (Loan B) and the lower-rate loan (Loan A).

21. In terms of the above variables, what is the <u>actual real rate of return</u> that is realized on an inflation-protected loan if there is no risk of default? What is the actual real return realized on a similar non-inflation-protected loan?

Section 8.10

Supplemental Exercise Solutions

1. When there is increased uncertainty about future inflation, lenders generally require additional compensation for inflation, so the supply curve for money shifts upward. Borrowers, however, may not be willing to pay higher interest rates, due to their uncertainty regarding future inflation and its effect on their wages. So the demand curve may not move in the same direction as the supply curve.

2. We first calculate the loan payments at the specified interest rate of 2.5%, ignoring the effect of inflation. The amounts we calculate are in "real" dollars. That is, they are amounts that will have the same purchasing power as the same number of dollars would have had at time 0. The calculation is:

$$10{,}000 = Pmt \cdot \left(v + v^2\right) \qquad Pmt = 10{,}000 \div \left(1.025^{-1} + 1.025^{-2}\right) = 5{,}188.2716$$

Then we need to adjust each of the two payments to reflect inflation between the loan date and the date the payment is made. The amounts of the payments are:

Payment at time 1: $5{,}188.2716 \cdot \left(199.3 / 191.7\right) = 5{,}393.96$

Payment at time 2: $5{,}188.2716 \cdot \left(202.6 / 191.7\right) = 5{,}483.28$

3. The charge for default risk (i.e., the difference between the loan interest rate and the risk-free rate) is called the credit spread, or simply the "spread." The relation of credit spread to loan term is called the "spread curve." The spread between the risk-free interest rate and the rate actually charged generally increases with loan term, so the spread curve usually has a positive slope.

4. Alternative lenders do not have depositors. Instead, they raise money from investors to lend to borrowers. *(The study note does not mention it, but these investors may include lenders, as well as equity investors.)*

5. A bank with a low credit rating may need to offer higher interest rates to depositors in order to attract deposits. Potential customers will be concerned that the bank's failure could result in the loss of their deposits (or the inconvenience of waiting for the FDIC to pay their claims). As compensation for this risk, the bank may have to pay higher interest rates on savings accounts and CDs in order to attract deposits.

6. A secured loan involves some form of collateral that the borrower provides in order to "secure" the loan. The most common examples are auto loans and home loans (either a mortgage or a home equity loan or "home equity line of credit"). The lender has a "lien" on the property that is offered as collateral, and has the legal right to "foreclose on" or "repossess" the property if the borrower defaults on the loan payments. This reduces the risk of default losses for the lender, as the value of the collateral is available to repay the outstanding loan balance in the event of default.

An unsecured loan does not have collateral that the lender can use to pay the outstanding balance in the event of default. Consequently, it is a riskier loan for the lender and typically carries a higher interest rate. Examples of unsecured loans include credit card debt and "signature loans" (where the bank requires only the customer's signature in order to lend money).

A guaranteed loan has its repayment guaranteed by a third party. This significantly reduces the lender's risk, so the interest rate is generally lower than that for an unsecured loan (and probably lower than the rate for a loan secured by property). Examples include student loans that are guaranteed by the federal government, and guaranteed mortgages that are guaranteed by a mortgage guaranty insurance company.

7. Loan interest rates are significantly affected by the borrower's creditworthiness. An important factor affecting creditworthiness is the borrower's credit history, including the amount of debt outstanding and repayment history on previous loans. Credit card accounts and the borrower's payment history on those accounts are also significant. In addition, the borrower's assets and income are important factors, including the reliability of the income. (A steady salary is viewed more favorably than commission income or profits from a small business.)

8. Bond yields (and prices) are affected by the issuing entity's creditworthiness*;* the bond's term to maturity*;* the outlook for inflation*;* the tax status of the bond's coupons*;* the bond's seniority in the issuing corporation's capital structure and whether it is secured by specific corporate assets*;* the currency in which the bond is denominated*;* the presence (or absence) of call or put features*;* and the bond's liquidity.

9. Zero-coupon bonds are created by investment banks that purchase the coupon bonds of an issuing corporation or (most frequently) the U.S. Treasury. The investment bank then sells investors zero-coupon bonds that provide a single payment. The single payments for the various zero-coupon bonds consist of the individual coupons and maturity payments that are paid by the underlying bonds of the issuer.

10. The bond with the lower coupons will typically have the higher yield. For the high-coupon bond, the coupons represent a larger share of its total value, and the maturity value represents a smaller share. Since the coupons are paid earlier than the maturity value, the interest rates that would apply to them are generally lower than the rate that would apply to the maturity value (assuming a normal yield curve). If we think of a bond's yield as a weighted average of the yields on each of its cash flows, the high-coupon bond's yield will reflect greater weight on the coupons' yields and less on the higher yield of its maturity value, resulting in a lower weighted average yield.

(In any case, regardless of whether the yield curve is normal or inverted, the bond with the higher coupons will have a higher price, since all of its payments are equal to or greater than the corresponding payments made by the lower-coupon bond.)

11. In the same way that wage-earning borrowers are aided by inflation, since their wages increase with inflation, making it easier for them to make the loan payments, the federal government's income tends to increase with inflation, as it collects taxes based on its citizens' income and profits (and, to some extent, asset values). Thus the federal government can commit to making interest payments that increase with inflation.

12. Subject to certain requirements, the interest earned on municipal bonds in the United States is tax-free or taxed at a preferred rate (lower than the rate that applies to ordinary income). As a result, investors are willing to accept a lower yield on these bonds than on fully taxable corporate bonds. The coupons are usually set at a lower level also, so that the bonds will sell at or near par.

(Note: The interest earned on U.S. Treasury bonds is subject to federal income tax, but is generally not subject to state or local income tax. Since the federal income tax rate is generally higher than the state income tax rate, municipal bonds have a more favorable tax status than Treasury bonds. As a result, the yields on Treasury bonds are frequently higher than those on municipal bonds, even though the risk of default is lower for the Treasury bonds.)

13. When international investors purchase Government of Canada bonds denominated in Canadian dollars, they are subject to currency risk, since they will be affected by changes in the value of the Canadian dollar relative to their local currency. In order to attract U.S. investors who do not want to accept this currency risk (and international investors who choose to accept the risk of U.S. currency but not Canadian), the Canadian government issues some bonds in U.S. dollars.

14. Government of Canada bonds denominated in U.S. dollars trade less frequently than the corresponding U.S. Treasury bonds, so they are somewhat less liquid. This lesser liquidity accounts for their lower price and higher yield than similar U.S. Treasury bonds.

15. A call provision allows the issuer to redeem a bond before its maturity date by paying the "call price" to the bondholder. It is likely to be exercised if the price of the bond is enough higher than the call price to justify the cost of redeeming the existing bonds and issuing new bonds at a lower interest rate. (The fact that the bond's price is higher than the call price indicates that interest rates have fallen and the issuer will be able to borrow money at a lower interest rate.)

A put provision gives the bondholder the power to "put" (sell) the bond back to the issuer at the put price. It is likely to be exercised if the market price for the bond falls below the put price (which is usually lower than the bond's par value). If this happens, it indicates that the yield on the bond has risen, so the investor (the bondholder) can sell the bond at the put price and reinvest the money at the higher current yield rates.

16. A U.S. bank's reserve requirement depends on the amount of money its customers have on deposit with it, the volume of transactions it processes through the payment system on a typical day, and the bank's importance in the country's overall payment system.

17. The Federal Open Market Committee manages the country's monetary policy through its capital market activities. The Committee sets a target for the federal funds rate, which is the rate at which banks make loans to each other. (A bank holding excess reserves with its Federal Reserve bank can lend money (usually overnight) to a bank that has a shortfall.) The interest rate for these loans is determined by the banks transacting the loan, but the Federal Open Market Committee can influence the rate that is charged by purchasing or selling securities (usually Treasury bonds) on the open market.

When the Federal Reserve purchases bonds, it is increasing the amount of cash in the economy, which makes it easier for banks to meet their reserve requirements. This reduces the demand for overnight loans, resulting in lower interest rates on those loans. Selling bonds has the opposite effect, raising the rates banks charge each other. By buying and selling bonds, the Federal Reserve can maintain the federal funds rate close to the target that it sets.

18. Banks prefer to borrow from other banks, rather than from the Federal Reserve. They are able to borrow from each other at the federal funds rate, which is lower than the discount rate charged by the Federal Reserve. Also, borrowing from a peer bank demonstrates that other banks consider the borrowing bank to be creditworthy.

19. The Federal Open Market Committee is directed to conduct monetary policy in a way that promotes maximum employment, stable prices, moderate long-term interest rates, and continuing economic growth.

20. The answer is $i_e + i_u + c$.

The rate for Loan A can be expressed as $r - c$. (The lender would charge *r* if there were no inflation, but given the risk of inflation, the lender accepts an inflation-protected real rate of $r - c$.)

The rate for Loan B can be expressed as $r + i_e + i_u$. The lender charges *r* for deferred consumption, i_e to compensate for expected inflation, and i_u as a risk charge for unexpected inflation.

The difference between the two rates is:

$$(r + i_e + i_u) - (r - c) = i_e + i_u + c$$

(Note: We have ignored the charge for the risk of default (represented in the study note by the variable s, which stands for "spread"), because the default component of the interest rate (the credit spread) would be the same for both loans. Only the inflation components are different. If we included the credit spread, each loan rate would be increased by s, but the difference between the two would still be $i_e + i_u + c$.)

21. The contractual rate for an inflation-protected loan is $r - c$. The borrower assumes the risk of inflation, so $r - c$ is also the real rate that the lender realizes. To express this as an equation, start with the contractual rate of $r - c$. The borrower pays $r - c + i_a$. The real rate the lender earns (after the effect of inflation is $(r - c + i_a) - i_a = r - c$.

The rate realized by the lender for the non-inflation protected loan is $r + i_e + i_u - i_a$. This is equal to the rate charged for the loan, $r + i_e + i_u$, reduced by the actual rate of inflation, i_a.

Module 9

Interest Rate Swaps

The final topic in our study of interest theory for Exam FM is **Interest Rate Swaps**. An interest rate swap is an agreement between two entities (e.g., corporations or individuals) under which each agrees to make a series of interest payments to the other based on a specified principal amount. One of the parties makes interest payments at a fixed interest rate; the other makes payments at a variable interest rate. The issues that we will study include computing the fixed rate for a swap, determining the payment amount each period, and calculating the market value of an existing swap.

Section 9.1

Introduction to Derivative Securities

An interest rate swap is an example of a **derivative security**. You will study a wide range of derivative securities (or "**financial derivatives**") in later actuarial exams. But since interest rate swaps are one type of financial derivative, we will explain briefly what derivatives are.

A **derivative** is defined as "a financial instrument whose value depends on the value of something else." As an example, consider a **call option** that gives its owner the right to buy 100 shares of a particular stock at a price of 40 per share (the "strike price") on or before the option's expiry date. The call option is a derivative, because the value of the option depends on the price of the stock.

In this example, if the stock is trading at 60 per share, then the option to buy 100 shares at 40 has considerable value. The optionholder can buy 100 shares of stock at 40 and sell them at 60, realizing a **payoff** of 2,000 $\left(=100\times(60-40)\right)$. But if the stock is trading at 20, the call option has very little value, since it will not provide a payoff unless the stock price rises above 40 before the option's expiry date.

Note: If the stock is trading at 20, this does not mean that the option has a negative value, since the optionholder is not obligated to exercise the option to buy the stock at 40. A call option's value is never less than zero.

In the case of interest rate swaps, the value of the derivative (the swap) depends on the level of market interest rates. As interest rates change, the value of the swap also changes, as we will see. In studying interest rate swaps for Exam FM, you will be introduced to terminology that will be useful later when you study other types of derivatives.

Section 9.2

Variable-Rate Loans

The motivation for an interest rate swap is that one of the parties would prefer to pay interest at a fixed interest rate, and the other would prefer to pay interest at a variable rate. In order to understand how that situation develops, we need to consider how variable-rate loans work.

In our previous discussions of loans, we have generally assumed that the loan interest rate is determined at the time the loan is made, and that it will not vary during the term of the loan. Such loans are said to have a **fixed interest rate**. However, one of the examples in Module 3 involved a variable-rate mortgage in which the interest rate and the monthly payment could change during the term of the loan. Loans of this type are said to have a **variable interest rate** (sometimes called a "**floating interest rate**"). The interest rate on a variable-rate loan can increase or decrease periodically, based on the level of market interest rates. The loan agreement specifies the *frequency* of interest rate changes (e.g., annually, monthly, or even daily), and it also specifies the *formula* that will be used to calculate the new rate on each change date.

It is common for mortgage loans, home equity loans, and bank loans to corporations to contain a variable interest rate provision. The interest rate for such loans is typically based on an interest index, such as **LIBOR** or the **prime interest rate**. LIBOR (the London Interbank Offered Rate) is a rate used by London banks for interbank loans. The prime interest rate is the rate US banks charge for short-term loans to their most creditworthy customers.

The variable rate formula in a loan agreement specifies that the rate will be equal to the value of the index on the recalculation date, plus a specified **spread**. For example, the formula could be "prime rate plus 225 basis points." (A basis point (bp) is 0.01%, so "prime plus 225 bp" describes an interest rate 2.25% higher than the prime rate. That is, the spread is 225 bp.) The spread for a particular loan is set by the lender, or negotiated between borrower and lender. The size of the spread depends on various considerations, including the borrower's creditworthiness.

In the case of a mortgage, the monthly payment includes both interest and principal, so that the loan is fully paid off with the last payment. When a variable rate mortgage has a change in interest rate, a new payment amount is calculated that will fully pay the remaining principal as of the last payment date. We have examined that situation in Module 3. The type of variable rate loan we will consider now is one in which the periodic payments pay only the interest that is due each period. An interest rate swap allows the borrower (or the lender) to convert variable interest payments to fixed-amount payments.

Example (9.1)

ABC Corporation borrows 200 million dollars for 3 years under an interest-only loan with a variable loan rate provision. Under the loan agreement, ABC will pay interest to the lender at the end of each year for 3 years at an interest rate equal to LIBOR plus 150 basis points. The interest rate will be determined at the *beginning* of each year based on the current LIBOR 1-year rate, and the interest will be paid at the *end* of the year. At the end of the 3rd year, ABC will pay the interest due for the 3rd year, and will also repay the 200 million principal.

Suppose the LIBOR 1-year rate has the following values on the indicated dates:

Time (in years)	0	1	2	3	4
LIBOR 1-yr. rate	3.25%	3.50%	3.05%	2.85%	2.95%

The interest rate for the first year of the loan is determined at time 0 and is equal to:

$$3.25\% + 1.50\% = 4.75\%$$

The amount of interest due at the end of the first year is:

$$4.75\% \times 200,000,000 = 9,500,000$$

By similar calculations, the interest payments due at the end of the 2nd and 3rd years are 10,000,000 and 9,100,000, respectively. The total amount paid by ABC Corporation, including the principal payment at time 3, is:

$$9,500,000 + 10,000,000 + 9,100,000 + 200,000,000 = 228,600,000$$

Exercise (9.2)

BCD Corporation takes out a 4-year interest-only loan for 100 million. The loan agreement provides for a variable loan interest rate equal to "LIBOR plus 125 bp." Based on the LIBOR 1-year rates given in Example **(9.1)**, what is the total amount BCD will pay to the lender?

Answer: 117,650,000

Section 9.3

Example of an Interest Rate Swap

If the borrower under a variable-rate loan would prefer to pay interest at a fixed rate, an **interest rate swap** can be used to replace the variable payments with level payments based on a fixed interest rate. This is demonstrated in the following example. For this example and those that follow, we will show the spot rates that apply to a particular loan or interest rate swap. You may assume that these rates are based on an interest index plus a spread, but for simplicity the index and spread will not be shown.

Example (9.3)

XYZ Corporation borrows 100 million dollars under a 3-year interest-only loan, agreeing to pay interest at the end of each year based on the 1-year spot rate in effect as of the beginning of that year. At the end of 3 years, XYZ will pay the 3rd year's interest and repay the principal of 100 million.

We will assume that on the date of the loan the spot rates (based on the index and the spread specified in the loan agreement) are as follows:

Years to Maturity	Spot Rate
1	6.00%
2	6.50%
3	7.00%

The loan interest rate for the first year of the loan is known to be 6.00%, because that is the 1-year spot rate at the beginning of the first year. However, the interest rates for the 2nd and 3rd years of the loan will not be known until the beginning of those years.

Note: Based on these spot rates, we could calculate forward rates for years 2 and 3. If the interest rates for the 2nd and 3rd years were being determined at time 0, those forward rates would be the loan interest rates for years 2 and 3. However, the loan agreement specifies that the rates for the 2nd and 3rd years will be determined at time 1 and time 2, respectively. At those future times, the 1-year spot rates will not necessarily match today's forward rates for those time periods.

To eliminate the uncertainty regarding the amounts of the future interest payments, XYZ's owners decide to enter into an interest rate swap with a third party (i.e., an individual or corporation other than the lender), which we will call "Contra." Under the terms of the interest rate swap, XYZ will pay interest at a fixed rate to Contra, and will receive payments from Contra equal to the interest due under the variable-rate loan.

Using the spot rates in the above table, we can calculate the fixed interest rate for this swap. We will first calculate the present value of the total amount of interest that would be paid to the lender *based on today's spot rates*. To calculate this value, we will find the amount of interest that would be due at time 3 *if no payments were made at times 1 and 2*, and then calculate its present value:

$$\begin{aligned} \textit{Int due at time } 3 &= 100{,}000{,}000 \cdot \left((1+s_3)^3 - 1 \right) \\ &= 100{,}000{,}000 \cdot \left(1.07^3 - 1 \right) = 22{,}504{,}300 \end{aligned}$$

$$\begin{aligned} \textit{PV (at time 0) of Int due at time } 3 &= \frac{22{,}504{,}300}{1.07^3} \\ &= 18{,}370{,}212.31 \end{aligned}$$

Note: We have used s_t here as the symbol for a spot rate. This is the notation that was introduced in Module 6. The study note that covers interest rate swaps uses r_t for spot rates, and we will adopt that notation later in this module when we develop formulas for swap rates.

Now that we know the present value of the interest for this 3-year loan based on the spot rates at time 0, we can calculate a *level payment amount* to pay an equivalent amount of interest over 3 years, *based on the spot rate yield curve at time 0*.

A level 3-year unit annuity-immediate based on the spot rates at time 0 has a present value of: $1.06^{-1} + 1.065^{-2} + 1.07^{-3} = 2.641353386$. Based on those rates, the interest for this loan can be paid by 3 level payments of: $\frac{18{,}370{,}212.31}{2.641353386} = 6{,}954{,}848.38$.

Under the interest rate swap, Contra agrees to pay XYZ an amount equal to the interest actually due on XYZ's loan each year. In return, XYZ will pay Contra 6,954,848.38 each year. Payments of this level amount have the same present value as the interest on the loan *based on today's spot rates*. By paying 6,954,848.38 to Contra each year, XYZ is paying a level ("fixed") interest rate of 6.95484838% on the 100 million loan. This level interest rate is called the **fixed rate**, or **swap rate** under the 3-year interest rate swap between XYZ and Contra.

At the end of the first year, XYZ will pay Contra 6,954,848.38, and Contra will make a payment to XYZ based on the variable rate for the first year. We know this rate is 6% (the 1-year spot rate as of time 0), so the variable payment made by Contra is 6,000,000. Since the two counterparties are making these payments to each other, there is no reason to write two checks. They will simply note that XYZ owes a net amount of 954,848.38, and XYZ will pay that amount to Contra at the end of the first year. (The amount of this first settlement payment was already known on the date the swap agreement was initiated.) By paying 954,848.38 to Contra and 6,000,000 to the lender (the amount due under the loan agreement), XYZ has made total payments of 6,954,848.38, which equals the interest on 100 million at the swap rate.

At time 1, the 1-year spot rate for the 2nd year will be known, and the amount to be paid at the end of the 2nd year can be calculated. Suppose that the 1-year spot rate at time 1 equals the implied forward rate as of time 0, which is $\dfrac{1.065^2}{1.06}-1=7.00235849\%$. Then the net payment at the end of the second year will be an amount *received* by XYZ (since 7.00235849% is greater than the swap rate). The net payment from Contra to XYZ will be:

$$100{,}000{,}000\left(7.00235849\%-6.95484838\%\right)=47{,}510.11$$

XYZ will pay 7,002,358.49 to the lender, of which 47,510.11 comes from Contra and 6,954,848.38 will be paid from XYZ's assets. So XYZ's final cost in year 2 is again equal to the amount of interest calculated at the swap rate.

Suppose that the 1-year spot rate at time 2 (the spot rate for the 3rd year) is x. Then XYZ must pay $100{,}000{,}000\cdot x$ to the lender, and also pay $100{,}000{,}000\cdot(6.95484838\%-x)$ to Contra. (Note that the payment to Contra could be negative, i.e., it could be an amount received *from* Contra.) So the total amount that XYZ must pay is:

$$100{,}000{,}000\cdot x+100{,}000{,}000\cdot\left(6.95484838\%-x\right)=6{,}954{,}848.38$$

From this analysis, we can see that the combination of the floating rate loan and the interest rate swap gives XYZ a loan for 100,000,000 at a fixed interest rate of 6.95483838%. By entering into an interest rate swap, XYZ has converted a variable-rate loan into a fixed-rate loan. As of the date of the swap, the fixed interest payments that XYZ will make have the same present value as the variable interest payments for the loan. But depending on whether interest rates rise or fall during the 3-year term of the swap, the variable payments made by Contra could have a larger or smaller value than the fixed payments made by XYZ.

It is important to understand that the swap rate we calculated in the above example is fair to both parties because the variable rates and the fixed rate are based on the same source of spot rates (presumably, an index plus a spread). The fixed rate can be determined now (at time 0), but the variable rates for the 2^{nd} and 3^{rd} years will not be known until times 1 and 2, respectively. The swap is fair because the fixed and variable rates are *consistent* (i.e., they are based on the same index and spread).

In the example we just considered, the borrower under a variable-rate loan wanted to pay a fixed rate instead of a variable rate. It can also happen that a *lender* who has made a variable-rate loan decides to use an interest rate swap in order to *receive* interest income based on a fixed rate rather than a variable rate. In that case, the lender would be the *receiver* (of the fixed rate) under the swap. It is also possible that a borrower (or a lender) under a *fixed* interest rate loan would choose to enter into an interest rate swap because of a preference to pay (or receive) interest based on a *variable* rate. In each of these situations, one party to the swap pays a fixed rate (the swap rate), and the other pays a variable rate (the floating rate).

Section 9.4

Interest Rate Swap Terminology

The example in the previous section involved an interest rate swap that allowed a borrower (XYZ) to convert its variable interest payments under a loan to fixed payments by entering into an interest rate swap with a third party (Contra). We used the facts of that situation to determine a fair rate for XYZ to pay to Contra in return for Contra's making payments equal to the variable interest payments on the loan. We will now define some useful terms that relate to that example. These terms will be used in the various situations we will encounter in the following sections.

In Example **(9.3)** XYZ borrowed 100 million at a variable loan interest rate. The lender gave that amount (100 million) to XYZ at time 0 as part of the loan contract. This same amount (100 million) is also used in the interest rate swap agreement to define the payments that each party will make. But the swap agreement does not call for a payment of 100 million by either party. Instead, that number is used to calculate the interest payments that each party will make. It is called the **notional amount** (or **notional principal amount**) for the interest rate swap. "Notional" means "existing only in theory, or as an idea," so it is an appropriate term for this amount that is used in the swap calculations but is not actually paid or received under the terms of the swap.

The party that pays the **swap rate** (the fixed rate) is considered the **payer** under the swap agreement. In our example, XYZ Corporation was the payer. The party that receives the swap rate and pays the variable rate (Contra, in our example) is called the **receiver**. Both of them are referred to as **counterparties**. The term counterparty basically means "the other party" in a financial contract. The payer is a counterparty to the receiver and vice-versa, so we can describe both of them as counterparties.

Note: An interest rate swap could involve <u>non-level</u> fixed rates (i.e., rates that are "fixed" at the time the parties agree to the swap, but are not level from period to period). In that case, the payer would make interest payments at a series of different interest rates that are known in advance and that produce total payments whose present value equals the present value of the interest on the loan (based on the interest rates in effect on the date of the swap). However, for the swaps we study, we will assume that the swap rate (the fixed rate) is level throughout the term of the swap.

In our example, XYZ used an interest rate swap to convert the variable payments it would make under an existing loan to fixed payments that are known and can be budgeted for in advance. This is an example of a **hedge**. When an entity (a corporation or a government or an individual) enters into a contract in order to *reduce the uncertainty* or variability in future cash flows, it is said to be "hedging" its risk. An auto insurance policy is an example of a hedge, because the policyholder is hedging its exposure to the risk of an accident. In each case there is an existing exposure to a risk (the risk of an increase in interest rates or the risk of an automobile accident), and that risk is offset (partially or completely) by implementing a hedge.

In some cases, one or both of the counterparties to an interest rate swap might *not* have an existing exposure to interest rate risk. In our example, Contra may have entered into the swap in the hope of making a profit (based on the expectation that interest rates would decrease). When this occurs, that party is said to be engaging in **speculation** rather than hedging. A speculator takes positions that *increase* his or her exposure to uncertainty or variability, and does so based on the expectation (or hope) that the variability will generate cash flows with a positive net present value. A speculator might take a position as payer (of the fixed rate) in an interest rate swap, in the hope that interest rates will increase, or as receiver in the hope that interest rates will decrease. In some cases, *both* counterparties in an interest rate swap could be speculators. When that occurs, no loan is involved and neither party is hedging. In effect, the two speculators are making a wager; one is betting that interest rates will increase over the term of the swap, while the other is betting that they will decrease.

The date on which a swap agreement begins is called its **inception date**. The fixed interest rate (the swap rate) is determined based on current interest rates on the inception date. The period of time from the inception date until the end of the last interest period (3 years in our example) is called the **swap term** or the **swap tenor**. The dates on which interest payments are exchanged by the counterparties are called the **settlement dates**. (In our example, the settlement dates were times 1, 2, and 3.) The length of time between settlement dates (1 year in our example) is called the **settlement period**. Usually, all of the settlement periods in an interest rate swap are of equal length. The floating interest rate for a settlement period is determined as of the beginning of that settlement period. The rate is said to **adjust** or **reset** on that date.

Note: Since the interest rate for each settlement period is known at the beginning of that period, the parties could make the settlement payment at the beginning of the period instead of at the end. In that case, the net settlement amount would be discounted (at the current 1-period spot rate) to reflect that it is being paid one period before the interest is earned. In the study note (and in this manual), all of the swaps will assume that the net settlement payment is made at the end of the settlement period.

On each settlement date, each of the counterparties is required to make an interest payment to the other counterparty. One pays the fixed rate, and the other pays the variable rate. As was explained in the example, there is no reason for both parties to write checks, so they calculate a **net swap payment** that one party owes to the other, and their obligations are "**net settled**" with a single payment. The net payment made by the payer (or received by the payer, if it is negative) is equal to: $(\text{notional amount}) \times (\text{fixed rate} - \text{variable rate})$.

In our example, XYZ was the borrower under a loan agreement with variable interest payments that matched the payments XYZ would receive from Contra under the interest rate swap. When one of the counterparties is a borrower under a loan agreement, the borrower's **net interest payment** is defined as the amount of interest paid to the lender, plus any net swap payment made to the counterparty (or *minus* any net swap payment *received* from the counterparty). XYZ entered into the swap contract in order to have fixed, rather than variable, interest payments. Therefore, XYZ's net interest payment each year will be level. The amount that XYZ pays to the lender increases or decreases as the 1-year spot rate moves up or down, but the total of the amount paid to the lender plus (or minus) the net settlement amount with Contra is level.

Note: In some cases, a borrower might enter into an interest rate swap where the payments from the counterparty do not exactly match the interest payments on the loan. For example, suppose that the variable loan interest rate is LIBOR plus 150 bp, and the variable rate in the swap is LIBOR (plus 0 bp). In that case, the borrower receives payments from the counterparty that are less than the amount of interest due, but the shortfall is a constant amount: 1.5% of the notional amount. Thus the borrower must pay the fixed swap rate (to the counterparty) and also pay 1.5% of the notional amount to the lender.

Under that arrangement, the borrower's total payment amount is fixed, even though the variable rates specified in the swap and the loan do not match. The important thing (in order to assure that the borrower's net interest payments are level) is that the difference between the two variable rates be constant (150 bp in the example of the preceding paragraph). Incidentally, we can assume that the fixed rate the borrower pays under this interest rate swap (with variable payments equal to LIBOR) would be about 150 bp lower than the fixed rate under a similar swap based on "LIBOR plus 150 bp." So the net interest payments by the borrower should be approximately the same in either case.

Most interest rate swaps have a *level* notional amount (as was true in our example), but the counterparties could agree to have an **amortizing swap**, in which the notional amount decreases over time, following the decreasing outstanding balance on a loan. Or a swap could have an increasing notional amount, in which case it is referred to as an **accreting swap**. This would be appropriate if the amount of the loan is expected to increase over time.

The first settlement period for a swap usually begins on the inception date (time 0). However, in a **deferred swap**, the first settlement period begins after a deferral period. In our example, the first settlement period ran from time 0 to time 1, so it was *not* a deferred swap. Since the variable rate for that first period was known at time 0, XYZ's first loan payment was not at risk for an unanticipated adverse movement (upward) in interest rates. So XYZ might have preferred a *1-year-deferred* swap, covering only the 2^{nd} and 3^{rd} years of the loan (the years for which the interest rates were uncertain). In that case, XYZ's 3 interest payments (to the lender at time 1, and to Contra at times 2 and 3) would not be level. The first payment would be lower than in the example, and the other two would be higher. However, this deferred swap would nonetheless eliminate the *variability* in all 3 payments, so XYZ would know in advance the amount of the net interest payment in each year and could budget for it.

Generally, neither counterparty makes a payment to the other counterparty when an interest rate swap is initiated. In our example, the swap rate was calculated so that the present value of the fixed payments was equal to the present value of the expected variable payments as of the swap contract's inception date. Since the present value of each party's expected net payments is zero, neither party needs to make a payment to the other in order to initiate the swap contract.

In some customized contracts, however, the fixed and variable payments do *not* have equal present values. In that case, the value of the swap contract is *not* zero on the inception date, and one of the counterparties must pay a **premium** to the other for agreeing to the terms of the swap. We will not examine such contracts in our studies for Exam FM. The swap rate formulas we develop will balance the fixed and variable payments, so that neither party has to pay the other a premium as an incentive to enter into the contract.

As just explained, for the swaps we will study, the present values of the fixed and variable payments will be equal at time 0. However, at any time *after* the inception date, interest rates are likely to have changed, and that will affect the present values of the future payments that each counterparty will make. As a result, the swap will have a **market value** for each counterparty, equal to the present value of the future payments that will be received, less the present value of the amounts that will be paid out. If that value is positive, it represents the amount that another investor would have to pay to assume that counterparty's position in the swap. If it is negative, the counterparty would have to pay that amount to another investor in order for that investor to take over the position.

Example (9.4)

In Example **(9.3)**, XYZ was the payer in an interest rate swap with a notional amount of 100 million and a (fixed) swap rate of 6.95484838%. If the variable rate in the 2^{nd} year of the swap is 6.5%, what payments will be made or received by XYZ, Contra, and the lender?

Solution.
Contra has agreed to pay the variable rate to XYZ, so it owes XYZ 6.5% of 100 million, which is 6,500,000. At the same time, XYZ must pay the fixed rate of 6.95484838% to Contra, so it owes 6,954,848.38 to Contra. In practice, these two payments will be netted, and XYZ will pay 454,848.38 to Contra. The amount XYZ must pay the lender, based on the variable rate of 6.5%, is 6,500,000.

This means that:

XYZ will pay 6,500,000 to the lender and 454,848.38 to Contra.
Contra will receive 454,848.38 from XYZ.
The lender will receive 6,500,000 from XYZ.

Exercise (9.5)

In Example **(9.3)**, if the 1-year spot rate at time 2 is 7.4%, what payments will be made or received by XYZ, Contra, and the lender?

Answer: XYZ to lender: 7,400,000 Contra to XYZ: 445,151.62

Section 9.5

Calculating the Swap Rate

In Example **(9.3)**, we analyzed a particular interest rate swap and calculated the fixed rate of interest (the swap rate) that the payer would pay to the receiver. In return, the receiver paid the variable rate (the floating rate) to the payer. In this section we will develop a general formula that can be used to calculate the fixed rate for any interest rate swap. In the following section, we will develop some simpler formulas that can be applied in certain circumstances.

The underlying concept in calculating a swap rate is that the present value of the payments each counterparty will make should equal the present value of the payments that counterparty will receive, *based on the spot rates in effect on the inception date of the swap agreement*.

We will examine a swap agreement whose notional amount can vary from one settlement period to another. It may or may not be a deferred swap agreement. We will assume that the settlement periods are of equal length.

We start by defining the following variables:

Let t_{k-1} and t_k be the beginning and end dates, respectively, of the k^{th} settlement period (measured in years). Note that t_0 will not necessarily be time 0.

Let n be the number of settlement periods in the interest rate swap, so that the last settlement period extends from time t_{n-1} to time t_n.

Let Q_{t_k} be the notional amount during the k^{th} settlement period (i.e., the period ending at time t_k).

Let r_t be the t-year spot rate (as of time 0), expressed as an annual effective rate. (Note: t is not necessarily an integer.) In Module 6, we used s_t as the symbol for this rate, but for our discussion of interest rate swaps, we will use the same notation as the study note on interest rate swaps.

Let P_t be the t-year present value factor. That is, $P_t = (1 + r_t)^{-t}$. (Note that P_t is also the price of a zero-coupon bond that matures for 1 in t years.)

Let R be the swap rate (the fixed interest rate in the swap), expressed as an effective rate *per settlement period*. (For example, if the swap agreement has annual settlement periods, R is an annual effective rate. If it has quarterly settlement periods, R is a quarterly effective rate, etc.)

Let $f^*_{[t_{k-1},t_k]}$ be the forward interest rate (determined as of time 0) for the period from time t_{k-1} to time t_k, expressed as an effective rate *per settlement period*. (For example, if the settlement period is one month, $f^*_{[t_{k-1},t_k]}$ is a monthly effective rate.) Then $f^*_{[t_{k-1},t_k]} = \dfrac{\left(1+r_{t_k}\right)^{t_k}}{\left(1+r_{t_{k-1}}\right)^{t_{k-1}}} - 1 = \dfrac{P_{t_{k-1}}}{P_{t_k}} - 1$.

*Note: The study note also defines $f_{[t_{k-1},t_k]}$ (the same symbol, but with no *) as the forward rate for this same time period, but expressed as an annual effective rate. Thus $\left(1+f_{[t_{k-1},t_k]}\right)^{(t_k-t_{k-1})} = 1 + f^*_{[t_{k-1},t_k]}$. The forward rates used in Module 6 of this manual correspond to $f_{[t_{k-1},t_k]}$. That is, $i_{t_{k-1},t_k} = f_{[t_{k-1},t_k]}$. The symbol $f_{[t_{k-1},t_k]}$ (without the *) will not be needed for our formulas.*

The amount of interest to be paid by the payer during the k^{th} settlement period is $Q_{t_k} \cdot R$. The present value of *all* amounts to be paid by the payer is $\sum_{k=1}^{n}\left(Q_{t_k} \cdot R \cdot P_{t_k}\right)$, and since R is constant, this is equal to $R \cdot \sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)$.

The variable payments under the interest rate swap will, of course, depend on the level of interest rates in the future. However, we can calculate the present value of these payments based on the *current yield curve*. That is, we can assume that the future interest rate for each period will equal the *implied forward rate* for that period, calculated *using today's spot rates*. The resulting present value of the variable payments is then $\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)$.

Since the fixed payments and the variable payments must have the same present value as of the inception date, we can write:

$$R \cdot \sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right) = \sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)$$

Solving this equation for R, we have a formula for calculating the swap rate:

(9.6)

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

Example (9.7)

A 4-quarter accreting interest rate swap has a notional amount of 20 million in the first quarter, increasing by 5 million in each subsequent quarter. The swap is based on the following spot rates. (Values of P_{t_k} are also shown, as they will be used in calculating the swap rate.)

Years to Maturity	Ann. Eff. Spot Rate	PV Factor P_{t_k}
0.25	5.00%	0.98788
0.50	5.45%	0.97382
0.75	5.75%	0.95894
1.00	6.00%	0.94340

The values of Q_{t_1} to Q_{t_4} (in millions) are 20, 25, 30, and 35.

We need to calculate values for $f^*_{[t_0,t_1]}$ to $f^*_{[t_3,t_4]}$, which are forward rates expressed as effective rates *per settlement period* (i.e., per quarter):

$$f^*_{[t_0,t_1]} = \left(1+s_{0.25}\right)^{0.25} - 1 = 1.05^{0.25} - 1 = 0.01227$$

$$f^*_{[t_1,t_2]} = \frac{\left(1+s_{0.50}\right)^{0.50}}{\left(1+s_{0.25}\right)^{0.25}} - 1 = \frac{1.0545^{0.50}}{1.05^{0.25}} - 1 = 0.01444$$

$$f^*_{[t_2,t_3]} = \frac{\left(1+s_{0.75}\right)^{0.75}}{\left(1+s_{0.50}\right)^{0.50}} - 1 = \frac{1.0575^{0.75}}{1.0545^{0.50}} - 1 = 0.01552$$

$$f^*_{[t_3,t_4]} = \frac{\left(1+s_{1}\right)}{\left(1+s_{0.75}\right)^{0.75}} - 1 = \frac{1.06}{1.0575^{0.75}} - 1 = 0.01647$$

Applying Equation **(9.6)**, we have:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

$$= \frac{20 \cdot .012 \cdot .988 + 25 \cdot .014 \cdot .974 + 30 \cdot .016 \cdot .959 + 35 \cdot .016 \cdot .943}{20 \cdot 0.98788 + 25 \cdot 0.97382 + 30 \cdot 0.95894 + 35 \cdot 0.94340}$$

$$= \frac{1.58427}{105.88987} = 0.01496$$

The quarterly effective swap rate is R = 1.496%.

Note: Figures shown in the numerator of the "large" fraction above have been rounded in order to fit the available space on this page, but the calculations have been done with full precision.

Exercise (9.8)

In Exercise **(9.7)**, if the interest rate swap has a term of 3 quarterly periods instead of 4, what is the swap rate? (The notional amounts for the three periods are 20 million, 25 million, and 30 million.)

Answer: 1.428%

Example (9.9)

ZYX Corporation enters into a 4-year loan agreement under which it will receive 100 million at the beginning of each year for 4 years. ZYX will pay interest at the end of each year based on the 1-year spot rate as of the beginning of that year. At the end of 4 years, ZYX will pay the 4th year's interest and repay the principal of 400 million.

We will assume that on the date of the loan the yield curve is as follows:

Years to Maturity	Spot Rate
1	6.00%
2	6.50%
3	7.00%
4	7.25%

ZYX arranges an interest rate swap in order to eliminate the uncertainty about the amounts of future interest payments. Since the loan interest rate for the first year is known to be 6.00%, ZYX decides there is no risk that needs to be hedged during the first year. Therefore, ZYX enters into a 1-year-deferred interest rate swap covering the interest payments at the end of years 2, 3, and 4.

To calculate the fixed rate for this swap using Equation **(9.6)**, we will need to calculate the forward rates for years 2, 3, and 4. Since the settlement periods are 1 year in length, these will be annual effective rates. Based on the above spot rates, the forward rates are as follows:

$$f^*_{[1,2]} = \frac{(1+r_2)^2}{1+r_1} - 1 = \frac{1.065^2}{1.06} - 1 = 7.00236\%$$

$$f^*_{[2,3]} = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = \frac{1.07^3}{1.065^2} - 1 = 8.00705\%$$

$$f^*_{[3,4]} = \frac{(1+r_4)^4}{(1+r_3)^3} - 1 = \frac{1.0725^4}{1.07^3} - 1 = 8.00351\%$$

We will also need the present value factors (zero-coupon bond prices) for times 2, 3, and 4:

$$P_2 = (1+r_2)^{-2} = 1.0650^{-2} = 0.881659$$

$$P_3 = (1+r_3)^{-3} = 1.0700^{-3} = 0.816298$$

$$P_4 = (1+r_4)^{-4} = 1.0725^{-4} = 0.755807$$

The notional amount for the swap is 200 million in the 2nd year, 300 million in the 3rd, and 400 million in the 4th; it is an "accreting swap." To save space, we will use 2, 3, and 4 as the loan amounts in our calculations. (Note: The swap is deferred 1 year, so the 1st year is not part of the calculation. Also, there are 3 years in the swap, so the k values are 1, 2, and 3; we therefore have $t_0 = 1$, $t_1 = 2$, $t_2 = 3$, $t_3 = 4$.)

Applying Equation **(9.6)**, we have:

$$R = \frac{\sum_{k=1}^{3}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{3}\left(Q_{t_k} \cdot P_{t_k}\right)} = \frac{Q_2 \cdot f^*_{[1,2]} \cdot P_2 + Q_3 \cdot f^*_{[2,3]} \cdot P_3 + Q_4 \cdot f^*_{[3,4]} \cdot P_4}{Q_2 \cdot P_2 + Q_3 \cdot P_3 + Q_4 \cdot P_4}$$

$$= \frac{2 \cdot 0.0700236 \cdot 0.881659 + 3 \cdot .0800705 \cdot 0.816298 + 4 \cdot 0.0800351 \cdot 0.75}{2 \cdot 0.881659 + 3 \cdot 0.816298 + 4 \cdot 0.755807}$$

$$= 0.07760723$$

The swap rate (or "fixed" rate) for this swap is 7.760723%.

Exercise (9.10)

Suppose that ZYX's loan in Example **(9.9)** is 400 million at time 0, and that ZYX has agreed to pay interest at the 1-year spot rate and also repay 100 million of principal at the end of each of the 4 years. ZYX enters into an interest rate swap (as the "payer") that covers years 2, 3, and 4 of this loan. Because the principal decreases over the term of the swap, this is an "amortizing" swap. Based on the spot rates in Example **(9.9)**, what is the swap rate for this deferred amortizing swap?

Answer: 7.478567%

Example (9.11)

Based on the swap rate calculated in Example **(9.9)**, find the amount of the net settlement payment at the end of the 2nd year if the 1-year spot rate at time 1 is 7.6%.

Solution.
At the end of the first year of the loan, ZYX will pay 6 million to the lender (based on a 100 million loan and the 6% 1-year spot rate that was in effect at time 0).At the end of the 2nd year, ZYX will pay 15,521,446 $(=200\,\text{million}\times 7.760723\%)$ to the counterparty under the swap, and the counterparty will pay ZYX the amount due under the loan, 200 million times the 1-year spot rate as of time 1 (the beginning of the 2nd year).

Based on a 7.6% 1-year spot rate at time 1 (covering the period from time 1 to time 2), the counterparty will pay ZYX 15,200,000 $(=0.076\cdot 200{,}000{,}000)$ at time 2. In practice, these payments at time 2 will be "net settled" by a single payment from one counterparty to the other. Since ZYX's interest payment (15,521,446) is larger than the counterparty's payment (15,200,000), ZYX will pay the counterparty a net amount of:

$$15{,}521{,}446-15{,}200{,}000=321{,}446$$

At the end of year 2, ZYX pays 321,446 to the counterparty and 15,200,000 $(=200\,\text{million}\times 7.6\%)$ to the lender.

Note: Even though the interest rate for year 2 is higher than the 2nd year forward rate that was calculated at time 0, it is not as high as the swap rate, so ZYX must make a net payment under the swap agreement. The reason this occurred is that the swap agreement not only fixed the interest rates that ZYX would pay in years 2, 3, and 4 (eliminating the uncertainty), it also levelled them. Even though the forward rates as of time 0 indicated an increasing interest rate each year, ZYX agreed to pay a level interest rate each year. That level rate is higher than the year 2 forward rate (and, as it turns out, it is also higher than the year 2 actual rate). The level swap rate is lower than the year 3 and year 4 forward rates, but it may or may not be lower than the actual rates in those future years.

Exercise (9.12)

Based on the swap rate calculated in Exercise **(9.10)**, calculate the net settlement payment at the end of the 2nd year if the 1-year spot rate at time 1 is 7.6%. Will this amount be paid by or received by XYZ?

Answers: 364,300 Received

Section 9.6

Simplified Formulas for the Swap Rate

Since the great majority of interest rate swaps have a level notional amount, it is useful to develop a less complicated formula for the swap rate that can be used in this more common situation.

From Formula **(9.6)**, we see that if the notional amount is a constant (Q), we can write:

$$R = \frac{\sum_{k=1}^{n}\left(Q \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q \cdot P_{t_k}\right)} = \frac{\sum_{k=1}^{n}\left(f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n} P_{t_k}}$$

The numerator of the above expression can be simplified. Note that:

$$f^*_{[t_{k-1},t_k]} = \frac{P_{t_{k-1}}}{P_{t_k}} - 1$$

Substituting this value into the first equation, we have:

$$R = \frac{\sum_{k=1}^{n}\left(f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n} P_{t_k}} = \frac{\sum_{k=1}^{n}\left(\left[\frac{P_{t_{k-1}}}{P_{t_k}} - 1\right] \cdot P_{t_k}\right)}{\sum_{k=1}^{n} P_{t_k}} = \frac{\sum_{k=1}^{n}\left(P_{t_{k-1}} - P_{t_k}\right)}{\sum_{k=1}^{n} P_{t_k}} = \frac{P_{t_0} - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$

Thus, if the interest rate swap has a level notional amount, the following formula applies:

(9.13)

For an interest rate swap with a level notional amount:

$$R = \frac{P_{t_0} - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$

Using this formula saves considerable time, since it doesn't require calculation of the forward rates.

Example (9.14)

WXY Corporation borrows 10 million dollars under a 5-year interest-only loan, agreeing to pay interest at the end of each year based on the 1-year spot rate as of the beginning of that year. At the end of 5 years, WXY will pay the 5th year's interest and repay the principal of 10 million.

On the date of the loan, the spot rate yield curve is as follows:

Years to Maturity	Spot Rate
1	6.00%
2	6.50%
3	7.00%
4	7.25%
5	7.40%

WXY Corporation will pay 600,000 of interest on the loan at the end of year 1 (6% interest on 10 million), but its interest payments for years 2 through 5 are unknown. To remove this uncertainty, WXY enters into a 1-year-deferred interest rate swap covering years 2 through 5. What is the fixed interest rate that WXY will pay to the counterparty in return for receiving payments at times 2 through 5 based on the spot rates for the 2nd through 5th years?

Solution.
Because the notional amount is level, we can use Formula **(9.13)**:

$$R = \frac{P_{t_0} - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}} = \frac{P_1 - P_5}{P_2 + P_3 + P_4 + P_5}$$

$$= \frac{1.06^{-1} - 1.074^{-5}}{1.065^{-2} + 1.07^{-3} + 1.0725^{-4} + 1.074^{-5}} = \frac{0.243589}{3.153571} = 0.0772422$$

The fixed rate for this 1-year-deferred interest rate swap is 7.72422%.

Exercise (9.15)

In Example **(9.14)**, if WXY enters into an interest rate swap that covers only years 3, 4, and 5, what will the swap rate be?

Answer: 8.00435%

Formula **(9.13)** can be used for a deferred swap, where the first settlement period begins after the inception date (i.e., $t_0 > 0$). Example **(9.14)** and Exercise **(9.15)** involved deferred swaps. For interest rate swaps that are *not* deferred (i.e., where $t_0 = 0$), if the notional amount is level, we can further simplify the formula for *R*. If the swap is not deferred, $P_{t_0} = P_0 = 1$, so we have:

(9.16)

For a non-deferred swap with level notional amount:

$$R = \frac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$

The swap rate calculated by this formula is also the par coupon bond rate. That is, an *n*-year bond with a coupon rate equal to *R* would have a price equal to its face amount. This can be seen by rearranging Formula **(9.16)**:

$$R \cdot \sum_{k=1}^{n} P_{t_k} = 1 - P_{t_n} \qquad R \cdot \sum_{k=1}^{n} P_{t_k} + P_{t_n} = 1$$

Notice that $R \cdot \sum_{k=1}^{n} P_{t_k}$ is the present value of a payment of *R* at the end of each period for *n* periods, and P_{t_n} is the present value of 1 paid at the end of *n* periods. If we substitute our usual symbols for those two values, we have the following equation:

$$R \cdot a_{\overline{n|}} + v_{s_n}^{n} = 1$$

This equation shows that an *n*-period bond paying coupons of *R* and maturing for 1 has a present value of 1. In other words, the swap rate (*R*) for a non-deferred swap with level notional amount is equal to the coupon rate for an *n*-period par bond.

Example (9.17)

We can use Formula **(9.16)** to calculate the swap rate for Example **(9.3)**. The facts of that example are repeated here for convenience:

XYZ Corporation borrows 100 million dollars under a 3-year interest-only loan, agreeing to pay interest at the end of each year based on the 1-year spot rate as of the beginning of that year. At the end of 3 years, XYZ will pay the 3rd year's interest and repay the principal of 100 million.

On the date of the loan, the spot rate yield curve is as follows:

Years to Maturity	Spot Rate
1	6.00%
2	6.50%
3	7.00%

We will calculate the swap rate for a 3-year interest rate swap that XYZ can use to convert the floating interest rate to a fixed rate.

Because the notional amount is level and this is not a deferred swap, we can use Formula **(9.16)**:

$$R = \frac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}} = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 1.07^{-3}}{1.06^{-1} + 1.065^{-2} + 1.07^{-3}}$$

$$= \frac{0.183702}{2.641353} = 0.0695484838$$

The swap rate is 6.95484838%. This is the same rate we found in Example **(9.3)** using general reasoning. It is also the coupon rate for a 3-year annual-coupon bond selling at par (i.e., the par coupon bond rate), which we can confirm by using the spot rates to calculate the price of a bond that pays annual coupons at a 6.955% coupon rate:

$$\text{Price} = 69.55 \cdot (P_1 + P_2 + P_3) + 1{,}000 \cdot P_3$$

$$= 69.55 \cdot \left(1.06^{-1} + 1.065^{-2} + 1.07^{-3}\right) + 1{,}000 \cdot 1.07^{-3}$$

$$= 69.55 \cdot 2.641353 + 1{,}000 \cdot (0.816298) = 1{,}000.00$$

Exercise (9.18)

Using the facts of Example **(9.17),** suppose that XYZ Corporation took out a 4-year interest-only loan for 100 million. If the 4-year spot rate is 7.25%, what is the swap rate for a 4-year interest rate swap that XYZ could use in order to pay a fixed interest rate?

Answer: 7.18816%

Example (9.19)

QRS Corporation borrows 1 million dollars for 1 year, agreeing to pay interest at the end of each quarter based on the 3-month spot rate as of the beginning of that quarter. At the end of the year, QRS will pay the interest due for the 4th quarter and repay the principal of 1 million.

On the date of the loan, zero-coupon bonds are priced as follows:

Years to Maturity	Zero-Coupon Bond Price (per 100 face)
0.25	98.62
0.50	96.93
0.75	95.21
1.00	93.54

We could convert these bond prices to interest rates and then calculate the present value factors (P_t) that we need to apply Formula **(9.16)**. However, the zero-coupon bond prices are already the present value factors that we need (multiplied by 100), so we can apply Formula **(9.16)** directly:

$$R = \frac{1 - P_{t_4}}{\sum_{k=1}^{4} P_{t_k}} = \frac{1 - 0.9354}{0.9862 + 0.9693 + 0.9521 + 0.9354} = \frac{0.0646}{3.8430} = 0.016810$$

The swap rate *R*, which is expressed as an effective rate per quarter, is 1.6810%. So QRS Corporation will pay its counterparty 16,810 at the end of each quarter, and will receive a payment from the counterparty based on the 1-quarter spot rate at the beginning of that quarter and a notional principal of 1 million.

The payment from the counterparty at the end of the first quarter (based on a 1-quarter zero-coupon bond price of 98.62 at the beginning of that quarter) will be:

$$\left(\frac{100}{98.62} - 1\right) \cdot 1{,}000{,}000 = 13{,}993$$

QRS will make a net settlement payment to the counterparty in the amount of $16{,}810 - 13{,}993 = 2{,}817$.

Exercise (9.20)

In Example **(9.19)**, if QRS and the counterparty create a 3-quarter (instead of 4-quarter) swap what is the swap rate (expressed as an effective rate per quarter)? What is the amount of the net settlement payment at the end of the first quarter? Would QRS pay or receive that amount?

Answers: 1.6474% 2,481 Pay

Section 9.7

Market Value of a Swap

With rare exceptions, swap rates are set so that the present value of the fixed payments equals the present value of the variable payments based on the yield curve *as of the inception date of the swap*. This is the concept that has been used in all of the swap rate calculations we have considered.

After the inception date, however, and particularly after one or more settlement periods have passed, it is unlikely that the present values of the remaining fixed payments and variable payments will continue to be equal. The difference between these two values (present value of variable payments, less present value of fixed payments) is called the **market value** of the swap. More specifically, this is the market value *for the payer*, since the payer will receive the variable payments and pay the fixed payments. The market value for the receiver will be the negative of this value (i.e., present value of fixed, less present value of variable). Notice that the market value can be either positive or negative, and it would correspondingly appear in the company's accounting as either an asset or a liability.

Either counterparty can sell or close its position in a swap agreement at any time. This is done by agreeing with the other counterparty to terminate the agreement, or by finding another investor who will assume the terminating party's responsibility for future payments under the swap. In either case, the terminating party receives the market value (or pays it, if it is negative) and then has no further obligations under the swap.

The concept of the market value of a swap can be summarized by the following conceptual formula:

$$MV_{\text{payer}} = PV(\text{var. pmts to be received}) - PV(\text{fixed pmts to be paid}) = -MV_{\text{receiver}}$$

It would be possible to develop mathematical formulas for these present values as of t_k years after the swap's inception date. However, it is generally more practical to compute the two present values from first principles to find the market value. This is the approach we will take in the following examples.

Example (9.21)

Using the facts from Example **(9.14)**, where WXY enters into a 1-year-deferred swap for a 5-year loan of 10 million dollars, we will calculate the market value of the swap at the end of the 3rd year (immediately after the net settlement payment for the 3rd year has been made).

Let us assume that the yield curve at time 3 is:

Years to Maturity	Spot Rate
1	6.10%
2	6.60%
3	7.05%
4	7.30%
5	7.50%

The fixed interest rate we calculated in Example **(9.14)** was 7.72422%, so WXY's remaining annual payments (at times 4 and 5) are each 772,422. The present value of these two payments, using the current interest rates as of time 3 is:

$$772,422 \cdot \left(1.061^{-1} + 1.066^{-2}\right) = 1,407,749$$

This is the present value of the fixed payments at times 4 and 5. We also need to find the present value of the variable payments on those dates, based on the spot rates at time 3. We could calculate this value by first calculating the amounts of the two payments and then finding their total present value. But a simpler approach is just to calculate the present value of the interest on 10,000,000 for the next 2 years, which is equal to the difference in value between 10,000,000 payable *now* (at time 3) and 10,000,000 payable in 2 years (at time 5). This calculation is:

$$10,000,000 \cdot \left(1 - P_2\right) = 10,000,000 \cdot \left(1 - 1.066^{-2}\right) = 1,199,941$$

The market value of WXY's position in the swap is the present value of the variable payments that will be received (1,199,941), minus the present value of the fixed payments to be paid out (1,407,749). So the market value of WXY's position in the swap is:

$$1,199,941 - 1,407,749 = -207,808 .$$

Since this market value is negative, WXY would have to pay the counterparty 207,808 to terminate the swap, or else find another investor who will agree to assume WXY's position in return for receiving 207,808 from WXY.

Exercise (9.22)

Based on the facts in Example **(9.21)**, calculate the market value of WXY's position in the swap at the end of the 2nd year, given that the spot rates at time 2 are as shown below.
Will WXY pay this amount or receive it?

Years to Maturity	Spot Rate
1	7.10%
2	7.60%
3	8.05%
4	8.30%
5	8.50%

Answers: 71,995 Receive

If an interest rate swap's notional value is not level, the calculation of market value is based on the same principles, but the calculations are somewhat more complex. The following example illustrates this.

Example (9.23)

In Example **(9.9)**, ZYX Corporation entered into a 1-year-deferred swap that covered years 2, 3, and 4 of a 4-year loan with principal amounts of 200 million, 300 million, and 400 million in those 3 years. The swap rate that ZYX pays is 7.760723%. We will calculate the market value of this swap as of the end of the first year, assuming the spot rates at that time are as follows:

Years to Maturity	Spot Rate
1	6.80%
2	7.30%
3	7.70%
4	8.00%

To facilitate later calculations, we will first calculate the present value factors for the 3 remaining years (P_1, P_2, and P_3):

$$P_1 = 1.068^{-1} = 0.93632959$$
$$P_2 = 1.073^{-2} = 0.86856146$$
$$P_3 = 1.077^{-3} = 0.80048443$$

Next, we calculate the present value of ZYX's fixed payments as of the end of the first year. (To avoid showing a large number of 0's, the following calculations are performed in millions.)

$$\left(200 \cdot P_1 + 300 \cdot P_2 + 400 \cdot P_3\right) \cdot 0.07760723 = 59.604535$$

The present value of ZYX's 3 fixed payments as of time 1 is 59,604,535.

The projected variable payments (in millions) as of time 1 are based on the forward rates at that time:

$$f^*_{[0,1]} = r_1 = 0.068 \qquad 200 \cdot 0.068 = 13.600000$$

$$f^*_{[1,2]} = \frac{P_1}{P_2} - 1 = 0.07802341 \qquad 300 \cdot 0.07802341 = 23.407022$$

$$f^*_{[2,3]} = \frac{P_2}{P_3} - 1 = 0.08504479 \qquad 400 \cdot 0.08504479 = 34.017916$$

The present value of the variable payments is thus:

$$13{,}600{,}000 \cdot P_1 + 23{,}407{,}022 \cdot P_2 + 34{,}017{,}916 \cdot P_3 = 60{,}295{,}332$$

The market value of ZYX's swap as of time 1 is the present value of the variable payments ZYX will receive, less the present value of the fixed payments it must pay to the counterparty:

$$MV = 60{,}295{,}332 - 59{,}604{,}535 = 690{,}798$$

This is a positive market value, indicating that ZYX's position in the swap is worth 690,798, and that is the amount the counterparty should pay to ZYX in order to terminate the swap. Alternatively, ZYX could sell its position to another investor for that amount, and the investor would then have the privilege of receiving the variable payments and the obligation of making the fixed payments under the swap.

Note that, even though no payments have been made under the swap as of time 1, its value has changed from 0 to 690,798 (or to -690,798 for the counterparty). This change is due to the fact that the spot rates at time 1 are higher than would have been forecast based on the yield curve at time 0 when the swap rate was determined. This increases the projected amounts for the variable payments, which has a favorable effect for ZYX and an adverse effect for the counterparty.

Exercise (9.24)

Calculate the market value of ZYX's position in the swap from Example **(9.9)** at time 2 if the spot rates at time 2 are:

Years to Maturity	Spot Rate
1	6.30%
2	6.70%
3	7.00%
4	7.15%

Is this a positive or negative market value for ZYX?

Answer: -6,438,568 Negative

Section 9.8

Formula Sheet

Definitions:

Spread - the amount added to a specified interest rate index to determine the variable rate under a loan or a swap

Inception date - the date as of which an interest rate swap takes effect

Swap term (or swap "tenor") - the length of time from the swap's inception date to the end of the last settlement period

Swap rate - the fixed interest rate under a swap, expressed as an effective rate per settlement period

Variable rate - the variable (or "floating") rate under a swap, expressed as an effective rate per settlement period; this is the spot rate per period as of the beginning of a settlement period, and is typically determined as the value of an interest index plus a spread

Notional amount - the assumed "principal amount" on which interest payments are calculated.

Payer - the party that pays the fixed interest rate

Receiver - the party that pays the variable interest rate

Net swap payment - the net amount paid by the counterparty owing the larger interest payment for a settlement period

Net interest payment - the amount paid "out of pocket" by the borrower, consisting of interest paid to the lender, plus or minus the net settlement amount from the interest rate swap

Amortizing swap - a swap with a notional amount that decreases over time

Accreting swap - a swap with a notional amount that increases over time

Deferred swap - a swap whose first settlement period begins at a time after the inception date

Market value - the amount that a counterparty could receive for selling its position in an existing swap; the market value can be positive or negative

Variables:

t_k end date of k^{th} settlement period

n number of settlement periods

Q_{t_k} notional amount for k^{th} settlement period

R swap rate (fixed interest rate),
expressed as *an effective rate per settlement period*

r_t t-year spot rate, expressed as an annual effective rate

$f^*_{[t_{k-1},t_k]}$ forward rate for the period from t_{k-1} to t_k,
expressed as *an effective rate per settlement period*

P_t t-year present value factor (and the price of a zero-coupon bond maturing for 1 in t years)

Formulas:

$$P_t = \left(1+r_t\right)^{-t}$$

$$f^*_{[t_{k-1},t_k]} = \frac{\left(1+r_{t_k}\right)^{t_k}}{\left(1+r_{t_{k-1}}\right)^{t_{k-1}}} - 1 = \frac{P_{t_{k-1}}}{P_{t_k}} - 1$$

General formula for swap rate:

$$R = \frac{\sum_{k=1}^{n} Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

For swaps with a level notional amount:

$$R = \frac{P_{t_0} - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$

For non-deferred swaps with a level notional amount:

$$R = \frac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$ (This is also the par coupon bond rate for an n-period bond.)

Net Settlement Payment made by "payer" to "receiver":

$$(\text{notional amount}) \times (\text{fixed rate} - \text{variable rate})$$

Market Value of the Swap for the "payer":

PV(variable payments to be received) – PV(fixed payments to be paid)

Section 9.9

Basic Review Problems

For problems 1-7, use the following table of spot rates:

Yrs. to Maturity	Ann. Eff. Spot Rate
1	3.00%
2	3.60%
3	3.85%
4	4.05%
5	4.20%

1. What is the swap rate for a 3-year interest rate swap with a level notional amount and annual settlements?

2. What is the swap rate for a 2-year-deferred, 3-year interest rate swap with a level notional amount and annual settlements?

3. For the swap in problem 2., if the level notional amount is 1 million and the 1-year spot rate at time 3 is 3.00%, what is the amount of the net settlement payment at time 4?
   Is this amount paid by the payer, or by the receiver?

4. What is the swap rate for a 3-year amortizing swap where the notional amounts are 500,000; 400,000; and 300,000 in years 1, 2, and 3, respectively?

5. For the swap in problem 4., if the yield curve at time 1 is unchanged from the yield curve at time 0 (which is given above), what is the market value at time 1 of the receiver's position in this swap?

6. What is the swap rate for a 1-year-deferred, 3-year accreting swap with notional amounts of 2 million, 3 million, and 4 million for years 2, 3, and 4, respectively (i.e., for the three years after the deferral period)?

7. For the swap in problem 6. if the 1-year spot rate at time 3 is 5%, what is the amount of the net settlement payment at time 4?
   Is this amount paid by the payer, or by the receiver?

Section 9.10

Basic Review Problem Solutions

1. The swap rate (the fixed interest rate) is the 5-year par coupon bond rate, which can be calculated as follows:

$$R = \frac{1-P_5}{\sum_{t=1}^{5} P_t} = \frac{1-1.0420^{-5}}{1.03^{-1}+1.036^{-2}+1.0385^{-3}+1.0405^{-4}+1.042^{-5}} = 0.04166$$

   The fixed rate for a 5-year swap is 4.166%.

2. The swap rate for this deferred swap is:

$$R = \frac{P_2-P_5}{\sum_{t=3}^{5} P_t} = \frac{1.036^{-2}-1.0420^{-5}}{1.0385^{-3}+1.0405^{-4}+1.042^{-5}} = 0.04595$$

   The fixed rate for a 2-year-deferred, 3-year swap is 4.595%.

3. The payer pays the fixed rate of 4.595%.
   The receiver pays the variable rate (the 1-year spot rate) of 3%.
   The net settlement payment is:

$$1{,}000{,}000 \cdot (4.595\% - 3\%) = 15{,}950\ .$$

   Because the fixed rate is higher than the variable rate, the payer will pay this amount to the receiver.

   Note: If you maintained full precision for the fixed rate that was calculated in problem 2., your answer should be 15,951.61.

4. To calculate the swap rate, we will use 5, 4, and 3 as the notional amounts in years 1, 2, and 3 (i.e., the notional amount will be expressed in 100,000's). In order to calculate the swap rate, we will need the forward rates for the 3 years of the swap:

$$f^*_{[0,1]} = r_1 = 0.0300$$

$$f^*_{[1,2]} = \frac{(1+r_2)^2}{(1+r_1)} - 1 = \frac{1.0360^2}{1.0300} - 1 = 0.0420$$

$$f^*_{[2,3]} = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = \frac{1.0385^3}{1.0360^2} - 1 = 0.0435$$

We also need the present value factor (zero-coupon bond price) for each of the 3 years:

$$P_1 = (1+r_1)^{-1} = 1.03^{-1} = 0.97087$$

$$P_2 = (1+r_2)^{-2} = 1.036^{-2} = 0.93171$$

$$P_3 = (1+r_3)^{-3} = 1.0385^{-3} = 0.89285$$

The swap rate is then calculated as follows:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

$$= \frac{5 \cdot 0.03 \cdot 0.97087 + 4 \cdot 0.042 \cdot 0.93171 + 3 \cdot 0.0435 \cdot 0.89285}{5 \cdot 0.97087 + 4 \cdot 0.93171 + 3 \cdot 0.89285} = 0.03720$$

The swap rate is 3.72%.

Note: If we had calculated the present value factors first, we could have used them to compute the forward rates for the 2nd and 3rd years:

$$f^*_{[1,2]} = \frac{P_1}{P_2} - 1 = \frac{0.97087}{0.93171} - 1 = 0.0420$$

$$f^*_{[2,3]} = \frac{P_2}{P_3} - 1 = \frac{0.93171}{0.89285} - 1 = 0.0435$$

5. The market price of the swap at time 1 is equal to the present value of what the counterparty will receive at times 2 and 3, minus the present value of what he or she will pay on those dates. In this case, we are determining the *receiver's* market value, so we will calculate the present value of the fixed-rate payments (which will be received), minus the present value of the variable-rate payments (which will be paid out).

   The fixed rate (from problem 4.) is 3.72% in both year 2 and year 3. The variable rate (calculated as of time 1) consists of the forward rates for each of the two remaining years. Since the problem states that the yield curve is the same as at time 0, we can use the forward rates that were calculated for problem 4. (Note that the forward rates calculated for the 1st and 2nd years now apply to the 2nd and 3rd years, since we are at time 1 and the spot rates haven't changed.) So the variable rates for years 2 and 3 are 3% and 4.2%.

   The market value for the receiver at time 1 can be calculated as follows:

$$\begin{aligned} MV &= Q_2 \cdot (\text{fixed} - \text{variable}_2) \cdot 1.03^{-1} + Q_3 \cdot (\text{fixed} - \text{variable}_3) \cdot 1.036^{-2} \\ &= 400{,}000 \cdot (0.0372 - 0.03) \cdot 0.97087 + 300{,}000 \cdot (0.0372 - 0.042) \cdot 0.93171 = 1{,}444.16 \end{aligned}$$

   The market value for the receiver is 1,444.16.

6. Because the notional value is not level, the swap rate calculation requires us to use the forward rates and present value factors for each year of the swap (i.e., years 2, 3, and 4). Except for $f^*_{[3,4]}$ and P_4, these were already calculated for problem 4. The two missing values are:

$$f^*_{[3,4]} = \frac{(1+r_4)^4}{(1+r_3)^3} - 1 = \frac{1.0405^4}{1.0385^3} - 1 = 0.0465$$

$$P_4 = (1+r_4)^{-4} = 1.0405^{-4} = 0.85316$$

   Then the swap rate is:

$$\begin{aligned} R &= \frac{\sum_{t=2}^{4} Q_t \cdot f^*_{[t-1,t]} \cdot P_t}{\sum_{t=2}^{4} Q_t \cdot P_t} \\ &= \frac{2 \cdot 0.0420 \cdot 93171 + 3 \cdot 0.0435 \cdot 0.89285 + 4 \cdot 0.0465 \cdot 0.85316}{2 \cdot 93171 + 3 \cdot 0.89285 + 4 \cdot 0.85316} = 0.04446 \end{aligned}$$

   The swap rate is 4.446%.

7. The notional amount in the 4^{th} year is 4,000,000. The fixed rate is 4.446%, and the problem gives us the variable rate: 5%.

   The variable rate is higher than the fixed rate, so the receiver (who pays the variable rate and receives the fixed rate) will have to pay a net settlement payment. The amount of this payment is:

   $$4{,}000{,}000 \cdot (0.05 - 0.04446) = 22{,}160$$

   The receiver pays 22,160 to the payer at time 4.

   Note: If you performed the calculations without rounding, you should have calculated a payment amount of 22,160.51.

Section 9.11

Sample Exam Problems

1. (Sample question 23)
   You are given the following spot rates from the latest upward-sloping yield curve:

Years to Maturity	1	2	3	4	5
Spot Rate	4.00%	4.50%	5.25%	6.25%	7.50%

   You enter into a 5-year interest rate swap (with a notional amount of 100,000) to pay a fixed rate and to receive a floating rate based on future 1-year LIBOR rates. If the swap has annual payments, what is the fixed rate you should pay?

   A) 5.20%   B) 5.70%   C) 6.20%   D) 6.70%   E) 7.20%

2. (Sample question 63)
   Company ABC has an existing debt of 2,000,000 on which it makes annual payments at an annual effective rate of LIBOR plus 0.5%.

   ABC decides to enter into a swap with a notional amount of 2,000,000, on which it makes annual payments at a fixed annual effective rate of 3% in exchange for receiving annual payments at the annual effective LIBOR rate.

   The annual effective LIBOR rates over the first and second years of the swap contract are 2.5% and 4.0%, respectively. ABC does not make or receive any other payments.

   Calculate the net interest payment that ABC makes in the second year.

   (A) 50,000   (B) 60,000   (C) 70,000   (D) 80,000   (E) 90,000

Section 9.12

Sample Exam Problem Solutions

1. The correct rate is the par coupon bond rate:

$$R = \frac{1 - \dfrac{1}{1.075^5}}{\dfrac{1}{1.04} + \dfrac{1}{1.045^2} + \dfrac{1}{1.0525^3} + \dfrac{1}{1.0625^4} + \dfrac{1}{1.075^5}} = 0.07197$$

   Answer: E

2. ABC will pay the fixed rate of 3% to the counterparty. In addition, because the loan rate is LIBOR plus 0.5% and the variable rate received from the counterparty is LIBOR, it will be necessary for ABC to pay the other 0.5% (50 bp) of loan interest.

   Thus the net amount ABC will pay is 3.5% of 2,000,000, which is 70,000. This is the amount of net interest ABC will pay *every* year during the swap, regardless of the level of LIBOR each year.

   Note: We could have used the LIBOR rate of 4% in the 2nd year to determine that ABC pays 4.5% on the loan, receives 4% from the counterparty, and pays the fixed rate of 3% to the counterparty. The net interest rate paid is 4.5% 4% + 3% = 3.5%. So the net payment is 3.5% of 2,000,000, which is 70,000, as calculated above.

   Answer: C

Section 9.13
Supplemental Exercises

For problems 1-3, use the following table of zero-coupon bond prices:

Quarter	1	2	3	4
Zero-coupon bond price	0.985	0.972	0.954	0.933

1. What is the fixed quarterly rate on a four-quarter interest rate swap?

   A) 0.0117 B) 0.0136 C) 0.0155 D) 0.0174 E) 0.0193

2. A 3-quarter interest rate swap (beginning at time 0) has a notional amount of 100,000. What is amount of the net settlement payment that the payer will receive at the end of the second quarter if the 1-quarter spot interest rate at the beginning of the second quarter is 0.017 (a quarterly effective rate)?

   A) 100 B) 120 C) 150 D) 170 E) 190

3. What is the market value of the payer's position in the three-quarter interest rate swap of problem 2. at the end of the first settlement period if the spot rates at that time are as shown in the following table?

Years to Maturity	0.25	0.50	0.75	1.00
Spot Rate (expressed as an ann. eff. rate)	6.23%	6.29%	6.32%	6.33%

   A) 85 B) 87 C) 89 D) 92 E) None of A, B, C, D

4. Company A borrows 1,000,000 for 2 years and agrees to make semi-annual interest payments at LIBOR plus 50 bp. Suppose that the LIBOR 6-month interest rates during the term of the loan are as follows:

Time (in years)	0	0.5	1.0	1.5	2.0
LIBOR 6-mo. rate	0.0275	0.0290	0.0265	0.0325	0.0340

Note: The rates in the table are 6-month effective rates.

   What is the amount of Company A's interest payment at the end of the first year of the loan?

   A) 26,500 B) 29,000 C) 31,500 D) 32,500 E) 34,000

5. Bank B lends 10 million for 5 years with annual interest payments at a fixed rate of 6%. In order to match the bank's liability cash flows (which rise and fall with interest rates), the bank's managers would prefer to receive variable interest payments. They enter into a 5-year interest rate swap as the payer. The swap has a notional amount equal to the amount of the loan. The swap rate is calculated based on 1-year LIBOR (with no spread), and the LIBOR yield curve on the swap's inception date is as follows:

Term (in years)	LIBOR
1	0.040
2	0.046
3	0.049
4	0.051
5	0.052

If at time 1 (the beginning of the 2nd year) the LIBOR yield curve is unchanged from the above values, what net interest payment will the bank receive at time 2 from the combination of the loan and the swap?

A) 484,000 B) 544,000 C) 604,000 D) 656,000 E) 716,000

Section 9.14

Supplemental Exercise Solutions

1. The fixed interest rate is the four-quarter par coupon bond rate, which can be calculated as follows:

$$R = \frac{1 - P_4}{P_1 + P_2 + P_3 + P_4} = \frac{1 - 0.933}{0.985 + 0.972 + 0.954 + 0.933} = 0.01743$$

   Answer: D

2. The fixed interest rate is the 3-quarter par coupon bond rate:

$$R = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 0.954}{0.985 + 0.971 + 0.954} = 0.01580$$

   The payer will pay 0.01580 and receive the spot rate of 0.017, so the payer will *receive* a net settlement payment of:

$$100{,}000 \cdot (0.0170 - 0.01580) = 120$$

   Answer: B

3. The market value at the end of the first settlement period (time 0.25) equals the present value of the two remaining settlement payments, based on the interest rates at that time. These present values are calculated based on the forward rates for periods 2 and 3 (the 2nd and 3rd quarters):

$$f^*_{[0.00,0.25]} = (1+r_{0.25})^{0.25} - 1 = 1.0623^{0.25} - 1 = 0.01522$$

$$f^*_{[0.25,0.50]} = \frac{(1+r_{0.50})^{0.50}}{(1+r_{0.25})^{0.25}} - 1 = \frac{1.0629^{0.50}}{(1.0623)^{0.25}} - 1 = 0.01551$$

Using these forward rates as the variable rates for the 2nd and 3rd quarters of the swap, and using a fixed rate of 0.01580 (which was calculated in problem 2.), the net settlement amounts at times 2 and 3 are $100{,}000 \cdot (0.01580 - 0.01522) = 58$, and $100{,}000 \cdot (0.01580 - 0.01551) = 29$.

Since we subtracted the variable rates from the fixed rate, these are net amounts that the payer (the fixed rate payer) must pay to the receiver.

The total present value of these two payments as of time 1 is:

$$58 \cdot (1+r_{0.25})^{-0.25} + 29 \cdot (1+r_{0.50})^{-0.50} = 58 \cdot 1.0623^{-0.25} + 29 \cdot 1.0629^{-0.50} = 85.26$$

This is the present value of the remaining net settlement payments the payer must make. Therefore, the payer's position in this swap has a market value of -85.26. Since Answer choices A through D are all positive, the answer is "none of the above."

Answer: E

4. At the end of the first year, Company A will pay interest for the period from time 0.5 to time 1.0. The interest rate for this period is determined at the beginning of the period, which is time 0.5. At time 0.5, 6-month LIBOR is 0.0290. The loan interest rate for the 6-month period is LIBOR plus 50 bp, which is 2.90% + 0.50% = 3.40%.

The interest payment at time 1 is 3.40% of 1,000,000, which is 34,000.

Answer: E

5. To determine the amount received by the bank at time 2, we first need to calculate the interest rate for the swap. Since this is a non-deferred swap with a level notional amount, the swap rate can be calculated as follows:

$$R = \frac{1 - P_5}{\sum_{t=1}^{5} P_t} = \frac{1 - 1.052^{-5}}{1.04^{-1} + 1.046^{-1} + 1.049^{-3} + 1.051^{-4} + 1.052^{-5}} = 0.051618$$

The problem states that the LIBOR yield curve at time 1 is unchanged from time 0, so the variable interest rate for the 2nd year of the swap (determined at time 1) is 1-year LIBOR from the table, which is 4%. The net settlement payment between the bank and the counterparty is: $10{,}000{,}000 \cdot (0.051618 - 0.04) = 116{,}180$

The bank pays the variable rate (5.1618%) and receives the fixed rate (4%), so it pays 116,180 to the counterparty.
The bank also receives 6% interest from the borrower, which provides a positive cash flow of 600,000.
The net interest payment received by the bank is:

$$600{,}000 - 116{,}180 = 483{,}820$$

Answer: A

Note: The interest rate for this loan is a fixed rate (6%) that is unrelated to LIBOR. Nonetheless, the combination of the loan and the swap give the bank a series of interest payments that will rise and fall with LIBOR. Each year the bank will receive a payment based on a 6% fixed interest rate and will pay the counterparty an amount based on a fixed swap rate of 5.1618%. In addition, the bank will receive an interest payment from the counterparty based on 1-year LIBOR. Thus the bank receives LIBOR plus 6%, minus 5.1618%, which amounts to LIBOR plus 0.8382%. The combination is equivalent to receiving interest on the loan at LIBOR plus 83.82 bp.

"Midterm" #3

Questions

1. Company A's stock pays semi-annual dividends. The next dividend payment, which is due 4 months from now, is expected to be 5.00 per share. Each subsequent dividend is expected to be 3% larger than the dividend paid 6 months earlier.

Based on an annual effective interest rate of 15%, what is the value of a share of Company A's stock?

A) 112 B) 115 C) 118 D) 121 E) 124

2. A bank makes a secured loan of 1,000 that is to be repaid with interest at time 4. The bank's required return before adjusting for inflation is a 5% annual effective rate (in other words, a 5% annual gain in purchasing power). It is anticipated that 15 loans per 1,000 of this type will default, and that, in the event of default, the bank will (on average) recover 50% of the amount owed by the borrower.

Assuming that inflation over the next 4 years is certain to be 2.5% per year (an annual effective rate), what is the minimum annual effective interest rate that the bank could charge for the loan, consistent with these facts?

A) 7.54% B) 7.64% C) 7.74% D) 7.83% E) 7.92%

3. A company has an obligation to pay 10,000 at the end of each of the next 3 years. To match these 3 liability cash flows exactly, the company will purchase appropriate amounts of the following 3 bonds, each of which is available in any desired face amount.

- a 1-year zero-coupon bond that is trading for 96.50 per 100 face
- a 2-year bond paying 4% annual coupons with a 5% yield to maturity
- a 3-year bond paying 6% annual coupons, trading at 99.10 per 100 face

What is the total cost of the bonds the company will purchase?

A) 27,005 B) 27,259 C) 27,334 D) 27,412 E) 27,490

4. An investor enters into a 4-year interest rate swap with a level notional amount. She will pay a fixed rate and receive a floating rate. Zero-coupon bond prices on the swap's inception date are as shown in the following table:

Term (in years)	1	2	3	4
Zero-coupon bond price (per 100)	98.91	96.68	93.41	90.60

Which of the following is closest to the fixed rate she will pay?

A) 2.36% B) 2.41% C) 2.44% D) 2.48% E) 2.50%

5. A bank makes a 2-year inflation-protected loan of 10,000. Principal and interest is to be repaid at the end of 2 years. The real rate of return for the loan (before adjustment for inflation) is 6.6%, compounded monthly.

On the date of the loan, the inflation index specified in the loan contract had a value of 190.7. Its values 1 year and 2 years later were 198.3 and 202.4, respectively.

What is the amount the borrower must pay at the end of 2 years?

A) 11,643 B) 11,862 C) 12,061 D) 12,107 E) 12,168

6. A 1-year-deferred 3-year interest rate swap with a notional amount of 1,000,000 is based on the following spot rates:

Term (in years)	1	2	3	4
Spot rate	0.045	0.053	0.058	0.061

The 1-year spot rates at time 2 and time 3 are 0.060 and 0.059, respectively. To the nearest 100, what is the net settlement amount that the payer will pay at time 3? (Time 3 is three years after the inception date, and is the end of the swap's second settlement period.)

A) 5,900 B) 6,200 C) 6,500 D) 6,900 E) 7,200

7. An investor owns a bond that will mature in 16 years. The bond has a face amount and maturity value of 1,000 and pays semi-annual coupons at an 8% annual rate. It does not have a call or put provision. The bond's current price is 932.

The investor calculates the bond's yield to maturity convertible semi-annually, and wonders how sensitive the price is to changes in that yield. He manages to calculate the derivative of the bond's price with respect to its yield (at its current price of 932). What is the value of that derivative?

A) -7,200 B) -7,450 C) -7,600 D) -7,750 E) -8,100

8. A 3-quarter accreting swap has notional amounts of 6 million, 8 million, and 10 million during the three quarters of its tenor. The fixed rate for the swap is based on the following forward rates (which are expressed as quarterly effective rates):

t	$f^*_{[t-0.25,t]}$
0.25	0.022
0.50	0.031
0.75	0.035
1.00	0.037

At the end of the second quarter, the 3-month zero-coupon bond is priced at 97.50 per 100 face. To the nearest 100, what is the market value of the swap for the receiver at the end of the second quarter?

A) 34,600 B) 35,500 C) 39,800 D) 45,300 E) 46,500

9. A company has an obligation to pay 100,000 five years from now. It intends to create an immunized portfolio by purchasing suitable amounts of two zero-coupon bonds, which are available for any face amount. The two bonds will mature in 4 years and in 7 years.

If the liability and both of the bonds are valued at an annual effective interest rate of 6%, what is the face amount of the 4-year zero-coupon bond that the company will purchase?

A) 49,817 B) 62,893 C) 65,118 D) 66,667 E) 74,726

10. The following variables are defined as shown. Assume there is no risk of default.

$R_1^{(a)}$ is the rate actually realized on an inflation-protected loan, based on an actual inflation rate of i_a.
r is the compensation for deferred consumption (i.e., the rate the lender would charge if there were no inflation and no risk of default).
c is the cost of inflation protection.
i_a is the *actual* rate of inflation (determined after the fact).

Based on these definitions, which of the following are true?

I) r is always greater than or equal to c.
II) $R_1^{(a)}$ is equal to $r + i_a$
III) The values of r and c cannot be observed individually.

A) I only B) II only C) III only
D) II and III E) Another combination

11. A 1-year-deferred, 2-year interest rate swap is based on the spot rates in the following table:

Term (in years)	1	2	3	4
Spot rate	0.03	0.04	x	0.05

It is an amortizing swap, with notional amounts of 2 million and 1 million during its two settlement periods (years 2 and 3). The swap rate is $R = 0.05267$. Find the 3-year spot rate x.

A) 0.043 B) 0.044 C) 0.045 D) 0.046 E) 0.047

Solutions

1. This problem can be solved using the dividend growth model, but with some adjustments. The dividend period is a half-year, so we need i, the effective interest rate per half-year, and g, the rate of dividend growth per half-year. We are given that the dividend increases by 3% each half-year, so $g = 0.03$.

The annual effective interest rate is 0.15, so the half-year effective rate is:

$$i = 1.15^{0.5} - 1 = 0.07238$$

Then we can calculate the value of the stock as of one period before the next dividend:

$$P_0 = \frac{Div}{i-g} = \frac{5}{0.07238 - 0.03} = 117.9787$$

This is the value as of 6 months before the next dividend. Since the next dividend will be paid in 4 months, we need to accumulate this value by 2 months, or 1/3 of a dividend period:

$$P_{1/3} = P_0 \cdot 1.07238^{1/3} = 117.9787 \cdot 1.07238^{1/3} = 120.76$$

Answer: D

2. The bank's target expected real return is an annual effective rate of at least 5%. On a 4-year loan of 1,000, the bank must receive a repayment amount of $1{,}000 \cdot 1.05^4 = 1{,}215.51$. Since this is the amount of purchasing power (the number of "year 0" dollars) the bank wants to receive, this number must be adjusted for the known 2.5% annual effective rate of inflation: $1{,}215.51 \cdot 1.025^4 = 1{,}341.69$. The bank will need to receive 1,341.69 at time 4 in order for the repayment to have the same purchasing power that 1,215.51 would have had at time 0.

Because 15 loans per 1,000 are expected to default, the expected amount of the repayment to be received at time 4 is only 0.985 of the amount specified in the loan agreement. However, the bank estimates it can recover 50% of the remaining 0.015, so the expected value of the total amount collected (as a proportion of the loan amount due) is $0.985 + 0.5 \cdot 0.015 = 0.9925$. In order to receive an amount of 1,341.69 (as calculated above), the bank must specify a repayment amount of $1{,}341.69 \div 0.9925 = 1{,}351.83$.

The annual effective interest rate that will result in a contractual repayment amount of 1,351.83 is:

$$\left(\frac{1{,}351.83}{1{,}000}\right)^{1/4} - 1 = 7.82775\%$$

At this interest rate, the bank's <u>expected</u> real return on the loan equals its target of 5%. However, the default/recovery risk introduces an element of uncertainty in regard to the amount that will actually be received. This rate is the <u>minimum</u> the bank could charge based on the facts of the problem. In practice, the bank would typically charge a higher rate in order to provide compensation for the bank's agreeing to take on the risk that the borrower will default on the loan.

This answer can also be arrived at using continuously compounded rates. The continuously compounded rates of interest and inflation are $\ln 1.05 = 0.04879$ and $\ln 1.025 = 0.024693$, respectively. The loss to default can be converted to a continuously compounded annual rate as follows:

$$(\ln 0.9925)/4 = -0.001882$$

The minimum rate the bank could charge is a continuously compounded rate of: $0.048790 + 0.024693 - (-0.001882) = 0.075365$. This translates to an annual effective rate of $e^{0.075365} - 1 = 0.0782775$

Answer: D

3. As usual with problems of this type, we will start with the longest-term bond (the only one that can contribute to the final payment at time 3) and will work backward to the shortest-term bond.

To meet the payment of 10,000 at time 3, the company will need to purchase $10,000 / 1.06 = 9,433.96$ of the 3-year bond. This will provide 10,000 at time 3 because of its 9,433.96 face amount and its 566.04 $(= 10,000 - 9,433.96)$ coupon. It will also pay coupons of 566.04 at time 1 and time 2 (which means we still need another 9,433.96 $(= 10,000 - 566.04)$ at time 1 and time 2.

The total face amount of the 2-year bonds will be $9,433.96 / 1.04 = 9,071.11$, and they will pay coupons at time 1 and time 2 of 362.85 $(= 9,433.96 - 9,071.11)$. At time 1, we will still need to provide another 9,071.11 $(= 10,000 - 566.04 - 362.85)$ using the 1-year bond. Since the 1-year bond is a zero-coupon bond, 9,071.11 is the face amount of the 1-year bond that is needed.

(Notice the pattern of the face amounts when the liability payments are level (10,000, in this case). For the longest-term bond, we divide the payment amount by (1 + coupon rate) to determine the face amount to buy. Then we divide that result (the face amount of the longest bond) by (1 + coupon rate) for the second-longest bond to find its face amount. We then continue with each successively shorter-term bond until we have found all the face amounts).

We now know the amounts of each of the three bonds. The problem has given the prices per 100 for the 1- and 3-year bonds, so we need to calculate a price per 100 for only the 2-year bond. Using the BA II Plus, we set N=2, I/Y=5, PMT=4, and FV=100. Then CPT PV = -98.14.

The cost for each of the 3 bonds is:

3-year bond	$9,433.96 \cdot 99.10 / 100 = 9,349.06$
2-year bond	$9,071.11 \cdot 98.14 / 100 = 8,902.45$
1-year bond	$9,071.11 \cdot 96.50 / 100 = 8,753.63$

The total cost is $9,349.06 + 8,902.45 + 8,753.63 = 27,005.13$.

Answer: A

4. The swap rate is the 4-year par coupon bond rate:

$$R = \frac{1 - P_4}{P_1 + P_2 + P_3 + P_4} = 0.024776$$

Answer: D

5. The repayment amount for an inflation-indexed loan is adjusted by the percentage change in the index during the period of the loan. For this loan, the index increased from 190.7 on the inception date to 202.4 on the date the loan matured, so the repayment amount will be increased by a factor of $202.4/190.7=1.06135$.

At 6.6% convertible monthly, the principal of 10,000 increases by a factor of $(1+0.066/12)^{24}=1.14070$ over the loan's 2-year term. This accumulation factor is increased by the inflation factor of 1.06135, so the repayment amount due on this inflation-indexed loan is:

$$10{,}000\times 1.14070\times 1.06135=12{,}107$$

Answer: D

6. Because this swap has a level notional amount, we can use the simplified formula for the swap rate: $R=\dfrac{P_{t_0}-P_{t_n}}{\sum_{k=1}^{n}P_{t_k}}$

For the facts of this problem, we have:

$$R=\frac{P_1-P_4}{P_2+P_3+P_4}=\frac{1.045^{-1}+1.061^{-4}}{1.053^{-2}+1.058^{-3}+1.061^{-4}}=0.066194$$

The swap rate is 6.6194%, so the payer (the payer of the fixed rate) will pay $6.6194\%\times 1{,}000{,}000=66{,}194$ every year.

The receiver (the payer of the variable rate will pay the 1-year spot rate each year. At time 3, the receiver will pay the 1-year spot rate that applies to year 3. This rate is determined at the beginning of year 3, which is the end of year 2 (i.e., time 2). The 1-year spot rate at time 2 is 0.060. So the receiver's payment equals $6.0\%\times 1{,}000{,}000=60{,}000$.

The net payment by the payer is $66{,}194-60{,}000=6{,}194$

Answer: B

7. The definition of modified duration is $D_{\text{mod}} = -\dfrac{dP/di}{P}$, so we can use the bond's modified duration to find dP/di. This is the derivative of price with respect to *i*, the annual effective yield. However, the investor in this problem found the value of $dP/di^{(2)}$, the derivative of price with respect to the bond's yield convertible semi-annually. In order to calculate this value, we need to work with the bond's modified duration based on its yield convertible semi-annually (see Formula **7.36** on page M7-18):

$$D_{\text{mod}}\left(i^{(2)}\right) = -\frac{dP/di^{(2)}}{P} = \frac{D_{\text{mac}}}{\left(1+\frac{i^{(2)}}{2}\right)} \qquad dP/di^{(2)} = -P\cdot\frac{D_{\text{mac}}}{\left(1+\frac{i^{(2)}}{2}\right)}$$

To calculate this derivative, we need to find values for $i^{(2)}$ and D_{mac}.
Start by solving for the yield on the BA II Plus:
Set N=32, PV=-932, PMT=40, and FV=1,000. CPT I/Y = 4.40006.
The yield convertible semi-annually is 8.8%.

More importantly, $i^{(2)}/2 = 0.044$. This is the semi-annual effective rate. Calling this value *j*, we can use it to calculate D_{mac}:

$$D_{\text{mac}} = \frac{40\cdot(Ia)_{\overline{32}|j} + 1{,}000\cdot 32\cdot v_j^{32}}{932} = \frac{40\cdot\left[\ddot{a}_{\overline{32}|j} - 32v_j^{32}\right]/j + 32{,}000\cdot v_j^{32}}{932}$$

$$= \frac{40\cdot\left[\dfrac{1-1.044^{-32}}{0.044/1.044} - 32\cdot 1.044^{-32}\right]/0.044 + 32{,}000\cdot 1.044^{-32}}{932}$$

$$= 9.04796$$

Now that we have values for $i^{(2)}/2$ and D_{mac}, we can calculate the required derivative:

$$dP/di^{(2)} = -P\cdot\frac{D_{\text{mac}}}{\left(1+\frac{i^{(2)}}{2}\right)} = -932\cdot\frac{9.04796}{1.044} = -8{,}077$$

(Note: This result, -8,077, sounds like an enormous value for the rate of change of the bond's price, since its price is only 932. However, this is the amount by which the bond's price would change (assuming the price function were linear) for a change of 1 in the yield. In this case, 1 means 100%!! For a more modest change in the yield, such as a change of 10 bp (0.1%) from 8.8% to 8.9%, the derivative indicates that the bond's price would change by $-8{,}077\cdot 0.1\% = -8.08$*, which is a more reasonable value. The actual price change if the yield increased by 0.1% is -8.03, so the estimated decrease of 8.08 predicts a price lower than the actual price. This is the usual result with a first-order modified approximation (an underestimate of the price). It results from the "convexity" of the price function.)*

Answer: E

8. Because the swap's notional amount is not level, we must use the general formula for the swap rate: $R = \dfrac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$

We are given the forward rates and the notional amounts, but we need to calculate the present value factors for 1, 2, and 3 quarters:

$$P_{0.25} = 1.022^{-1} = 0.978474$$
$$P_{0.50} = P_{0.25} \cdot 1.031^{-1} = 0.949053$$
$$P_{0.75} = P_{0.50} \cdot 1.035^{-1} = 0.916959$$

Using these values in the swap rate formula, we have:

$$R = \frac{6(0.022)\cdot 0.97847 + 8(0.031)\cdot 0.94905 + 10(0.035)\cdot 0.91696}{6\cdot 0.978474 + 8\cdot 0.949053 + 10\cdot 0.916959} = 0.030286$$

The fixed rate for the swap is 3.0286%.

At the end of the second quarter, the variable interest rate for the third quarter can be determined based on the 1-quarter zero-coupon bond price, which is 97.50 per 100. The variable rate is $100/97.50 - 1 = 2.5641\%$.

The net settlement payment at the end of the third (and final) quarter of the swap, based on the 10 million notional amount for that quarter is:

$$10{,}000{,}000 \cdot (0.030286 - 0.025641) = 46{,}450$$

This is the net amount that the payer will pay to the receiver at the end of the third quarter. The problem asks for the market value of the swap as of the end of the second quarter. This is the present value of 46,450, discounted one quarter at the one-quarter spot rate, which is 2.5641% (or we can discount it using the present value factor of 0.975). This value is:

$$46{,}450 \cdot 0.975 = 45{,}289$$

Answer: D

9. To create an immunized portfolio, the bond portfolio must match the present value and modified duration of the liability. This is an example of a <u>fully</u> immunized portfolio, since the liability's cash flow (at time 5) occurs between the times of the bonds' two cash flows.

Because 6% is the valuation rate for each of the securities, we can match modified durations by matching Macaulay durations. The requirement to match present values and Macaulay durations gives us the following two equations in two unknowns (where the unknowns, B_4 and B_7, are the face amounts of the two zero-coupon bonds):

$$\text{Present Value:} \quad 100{,}000 \cdot 1.06^{-5} = B_4 \cdot 1.06^{-4} + B_7 \cdot 1.06^{-7}$$

$$\text{Macaulay Duration:} \quad 5 \cdot 100{,}000 \cdot 1.06^{-5} = 4 \cdot B_4 \cdot 1.06^{-4} + 7 \cdot B_7 \cdot 1.06^{-7}$$

This leads to:

$$100{,}000 \cdot 1.06^{-5} = 3 \cdot B_7 \cdot 1.06^{-7}$$

$$B_7 = 37{,}453.33 \qquad B_4 = 62{,}893.08$$

Answer: B

10. Only III. is true.

I. is false because the cost of inflation protection is sometimes greater than the compensation for deferred consumption. This can be observed when real interest rates on inflation-protected loans are negative. This is most likely to occur on short-term loans when interest rates are low (especially if inflation is volatile, so that the cost of inflation protection is high).

II. is false because it ignores c, the cost of inflation protection. The correct formula is $R_1^{(a)} = r - c + i_a$.

III. is true. We don't know how much a lender would charge in the absence of inflation (r), or how much the lender has to pay for inflation protection (c), but the difference, $r - c$, can be observed in the rate for an inflation-protected risk-free loan, such as a real-return U.S. Treasury bond.

Answer: C

11. The swap rate (which is given as 0.05267) was calculated based on the general formula for a swap rate:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)} = \frac{Q_2 \cdot f^*_{[1,2]} \cdot P_2 + Q_3 \cdot f^*_{[2,3]} \cdot P_3}{Q_2 \cdot P_2 + Q_3 \cdot P_3}.$$

We can fill in all of the values except $f^*_{[2,3]}$ and P_3. And we can express $f^*_{[2,3]}$ as a function of P_3: $f^*_{[2,3]} = \frac{P_2}{P_3} - 1 = \frac{1.04^{-2}}{P_3} - 1$.

Using this expression for $f^*_{[2,3]}$, P_3 is the only unknown in the formula. So we can solve for the value of P_3 and use that value to calculate x, the missing 3-year spot rate.

$$0.05267 = \frac{2 \cdot \left(\frac{1.04^2}{1.03} - 1\right) \cdot 1.04^{-2} + 1 \cdot \left(\frac{1.04^{-2}}{P_3} - 1\right) \cdot P_3}{2 \cdot 1.04^{-2} + 1 \cdot P_3} = \frac{0.092635 + 0.924556 - P_3}{1.849112 + P_3}$$

$$0.097393 + 0.05267 \cdot P_3 = 1.0171914 - P_3$$

$$1.05267 \cdot P_3 = 1.0171914 - 0.097393 = 0.919799$$

$$P_3 = \frac{0.964521}{1.05267} = 0.873777$$

$$P_3 = (1+x)^{-3} \quad \rightarrow \quad x = (P_3)^{-1/3} - 1 = 0.873777^{-1/3} - 1 = 0.04600$$

Answer: D

About the Practice Exams

After learning the material in each module and reviewing past modules and the three midterm exams, you will be ready to tackle these practice exams.

a) The first 6 practice exams are relatively straightforward to enable you to review the basics. You may want to attempt them in a *non-timed* environment to evaluate your skills and understanding.

b) The final 5 practice exams introduce more difficult questions in order to replicate the exam experience. You should take each of these in a *timed* environment to give yourself experience with exam conditions.

Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like the ones in these practice exams.

Practice Exam 1

Exam FM

Questions

1. You are given the following yield curve:

Year	Spot Rate
1	5.5%
2	5.0%
3	5.0%
4	4.5%
5	4.0%

What is the 4-year forward rate?

A) 1.8% B) 1.9% C) 2.0% D) 2.1% E) 2.2%

2. Find the Macaulay duration of a 10-year 1,000 par value bond with 8% annual coupons at an annual effective interest rate of 6.5%.

A) 7.2 B) 7.4 C) 7.6 D) 7.8 E) 8.0

3. At an annual effective rate of interest i, a person can pay off a loan of K in two ways:

1) 475 now and 475 in 1 year, or
2) 570 in 2 years and 570 in 3 years.

Calculate K.

A) 893 B) 901 C) 909 D) 917 E) 925

4. A 10-year annuity-immediate pays 100 quarterly for the first year. In each subsequent year, the quarterly payment amount is increased by 5% over the payment amount during the previous year. At a nominal annual interest rate of 8% convertible quarterly, what is the present value of this annuity?

A) 2,997 B) 3,075 C) 3,108 D) 3,225 E) 3,333

5. The present value at interest rate i of a 10-year annuity-immediate with level annual payments is X. At the same interest rate, the present value of a 20-year annuity-immediate with the same annual payment amount is $1.5X$. Find i.

A) 7.2% B) 7.4% C) 7.6% D) 7.8% E) 8.0%

For Problems 6 and 7 use the following account summary:

Date	Balance Before Activity	Deposits	Withdrawals
January 1	100,000		
March 1	105,000	10,000	
September 1	112,000		30,000
December 31	95,000		

6. Find the time-weighted yield for this account.

A) 17.2% B) 17.5% C) 17.9% D) 18.1% E) 18.5%

7. Find the dollar-weighted yield for this account.

A) 14.9% B) 15.3% C) 15.6% D) 16.1% E) 16.4%

8. An investor has 3,000 worth of 5-year bonds with a modified duration of 4.615; 7,000 worth of 10-year bonds with a modified duration of 9.323; and 10,000 worth of 20-year bonds with a modified duration of 19.085. What is the modified duration of this entire portfolio?

A) 13.5 B) 13.7 C) 13.9 D) 14.1 E) 14.3

9. A company has liabilities requiring payments of 1,000; 3,000; and 5,000 at the end of years 1, 2 and 3, respectively. The investments available to the company are the following zero-coupon bonds:

Maturity (years)	Annual Effective Yield	Par
1	7%	1,000
2	8%	1,000
3	9%	1,000

Determine the cost to match these liability cash flows exactly.

A) 6,918 B) 7,024 C) 7,165 D) 7,368 E) 7,522

10. A man creates a retirement fund by making deposits at the end of each month for 20 years. For the first 10 years the deposits are 100 per month, and for the last 10 years the deposits are 200 per month. The fund earns interest at a nominal rate of 6% per year convertible monthly. At the end of 20 years, he uses the proceeds to purchase a 30-year annuity-immediate with monthly payments. The annuity is priced based on a nominal rate of 8% convertible monthly. What are the monthly payments from this annuity?

A) 408 B) 425 C) 437 D) 441 E) 459

11. An annuity makes annual payments at the beginning of each year for 20 years. For the first 10 years the payments are 100. Starting with the 11^{th} payment, each payment is increased by 6% over the previous payment. At an annual effective rate of 8%, what is the present value of this annuity?

A) 1,177 B) 1,190 C) 1,202 D) 1,213 E) 1,225

12. An annual-coupon corporate bond has a yield of 7.2% at its current price of 972.48. At 7.2%, the bond's Macaulay duration is 7.1245. Using the modified approximation method, estimate the change in price that would cause the bond's yield to increase by 0.10%.

A) -6.463 B) -6.685 C) -6.814 D) -7.012 E) -7.163

13. A 40-year loan is repaid by level annual payments at the end of each year. The principal paid in the 20^{th} payment is 166.59 and the principal paid in the 25^{th} payment is 244.78. Find the interest rate for this loan.

A) 7.7% B) 8.0% C) 8.2% D) 8.5% E) 8.8%

14. Linus deposits 100 into an account at the end of each year for 20 years. This account earns interest at an annual effective rate of 5%. Lucy deposits money into an account at the end of each year for 20 years. Her account also earns interest at an annual effective rate of 5%. The amount of her deposit increases each year in the following pattern: P, $2P$, $3P$, …. , $20P$. At the end of 20 years the balances in the two accounts are the same. Find P.

A) 10.93 B) 11.05 C) 11.12 D) 11.23 E) 11.35

15. Schroeder borrows money to buy a new piano. He agrees to pay back the loan with level annual payments at the end of each year for 30 years. The annual effective interest rate is 7%. The interest in his 10th payment is 366.74. What is the interest in his 20th payment?

A) 221.86 B) 229.64 C) 244.18 D) 250.72 E) 253.80

16. A woman makes a deposit into an account. For the first 5 years the account accumulates at a force of interest of 0.05. For the next 10 years the fund accumulates at a nominal annual rate of discount of 6% convertible quarterly. For the 15-year period, what is the equivalent nominal annual interest rate convertible monthly?

A) 5.59% B) 5.71% C) 5.83% D) 5.96% E) 6.04%

17. Violet purchases a 10-year 1,000 par bond with 8% semi-annual coupons. The bond is priced to yield 7.5% convertible semi-annually. She reinvests the coupon payments in a fund that pays a nominal rate of 7% convertible semi-annually. Over the 10-year period, what is Violet's nominal annual yield convertible semi-annually?

A) 7.36% B) 7.41% C) 7.48% D) 7.56% E) 7.63%

18. You are given the following yield curve:

Year	Spot Rate
1	4.0%
2	4.2%
3	4.6%
4	--
5	5.1%

If $i_{4,5} = 6.1\%$, find s_4.

A) 4.81% B) 4.83% C) 4.85% D) 4.87% E) 4.89%

19. A 20-year annuity-immediate has annual payments. The first payment is 100 and subsequent payments increase by 100 each year until they reach 1,000. The remaining payments remain at 1,000. At an annual effective interest rate of 7.5%, what is the present value of this annuity?

A) 6,201 B) 6,372 C) 6,413 D) 6,584 E) 6,700

20. A woman buys a 1,000-par 5-year zero-coupon bond priced to yield 6%. At the same time she buys a 1,000-par 5-year bond with 8% semi-annual coupons that is priced to yield 7% convertible semi-annually. The coupon payments are reinvested at 6.5% convertible semi-annually. What is her annual effective yield for the combination of these investments?

A) 6.0% B) 6.2% C) 6.4% D) 6.6% E) 6.8%

21. Account A earns compound interest at an annual effective rate i (where i is greater than 0). Account B earns compound interest at an annual effective rate equal to $1.1 \cdot i$.

An amount of 1,000 is deposited into each of these accounts at t=0. No other deposits or withdrawals occur. At the end of 20 years (at t=20), the balance in Account B will be 10% larger than the balance in the Account A.

What will be the difference between the balances in the two accounts at the end of 10 years (at t=10)?

A) 75 B) 80 C) 85 D) 90 E) 95

22. A homebuyer borrows 200,000 under a 20-year variable-rate mortgage with monthly payments. The initial interest rate for this mortgage loan is 3.6% (compounded monthly).

At the end of 2 years, the mortgage interest rate increases from 3.6% to 4.5% and a new monthly payment amount (for the 3rd and later years) is calculated, based on the new interest rate and the outstanding loan balance on that date. There are no other interest rate changes during the first 5 years of the loan.

To the nearest 100, what is the outstanding balance of the loan at the end of 5 years (immediately after the 60th monthly payment)?

A) 162,600 B) 162,900 C) 164,000 D) 164,300 E) 165,400

23. A 20-year bond has a face amount (and maturity value) of 1,000. It pays semi-annual coupons at a 6% (annual) coupon rate. The bond is callable on any coupon date on or after its 10th anniversary, with a 5% call premium.

If an investor purchases this bond on its issue date at a price of 1,060, and holds the bond until it matures or is called, what is the minimum yield that the investor could earn (expressed as a nominal rate, convertible semi-annually)?

A) 5.50% B) 5.59% C) 5.76% D) 5.96% E) 6.37%

24. A corporation has liabilities that require it to pay 1,000 in one year (at t=1), 2,000 one year later (at t=2), and 3,000 one year after that (at t=3).

The corporation will purchase high-quality bonds to fund these liabilities. The following 3 bonds are available:

1-year zero-coupon bond with a 4% yield to maturity
2-year zero-coupon bond with a 6% yield to maturity
3-year bond with 7% annual coupons and an 8% yield to maturity
(Assume that each bond is available for purchase in any face amount. All rates are annual effective rates.)

The corporation purchases a combination of these bonds that will exactly fund its 3 liability payments.
What is the total cost of the bonds that are purchased?

A) 5,101 B) 5,110 C) 5,120 D) 5,123 E) 5,128

25. A corporation's stock pays a semi-annual dividend of 20 per share, payable on June 15 and December 15 each year. Assume that this dividend will remain level in the future.

Using the dividend discount method and an annual effective interest rate of 12%, what is the value of one share of this stock as of April 15?

A) 325 B) 333 C) 340 D) 350 E) 356

26. A 2-quarter interest rate swap is entered into when spot interest rates (expressed as effective rates **per quarter**) are as follows:

Term:	1 quarter	2 quarters
Spot rate (eff. rate per qtr.):	1.4%	1.7%

The notional amount of the swap is 1,000,000.
To the nearest 1,000, what is the amount of the fixed payment for this swap?

A) 14,000 B) 15,000 C) 16,000 D) 17,000 E) 18,000

27. A U.S. Treasury bill with a maturity value of 1,000 will mature 260 days from today. If its current price is 948, what is the quoted rate for this T-bill?

A) 5.5% B) 7.2% C) 7.3% D) 7.6% E) 7.7%

28. Barb Dwyer invests 500 at t=0 at a nominal annual interest rate of 6% compounded quarterly. What additional amount will Barb need to invest at t=2 in order to have a total of 1,000 at t=5?

A) 242 B) 273 C) 278 D) 290 E) 327

29. Lisa Carr purchased a newly-issued 25-year bond that pays semi-annual coupons at an 8% (annual) rate. The bond has a par value (and a redemption value) of 1,000. The bond is callable on or after its 10th anniversary with a 10% call premium (i.e., 1,100 is payable if it is called).

Lisa purchased this bond at issue at a price that will assure her a rate of return of at least 7.5% (a nominal rate, convertible semi-annually). If the bond is called on its 12^{th} anniversary, what is Lisa's actual rate of return?

A) 7.5% B) 7.6% C) 7.7% D) 7.8% E) 7.9%

30. Suppose you enter into a 3-quarter interest rate swap with a level notional amount as the receiver. The swap rate is based on the following zero-coupon bond prices:

Bond term (in quarters)	1	2	3	4
Zero-coupon bond price per 100 face	98.40	96.90	95.30	93.50

What net interest rate will be paid to you in the second quarter if the spot interest rate for the second quarter is 0.018?

A) 0.0010 B) 0.0012 C) 0.0016 D) 0.0018 E) 0.0020

31. The following table gives the term structure of spot interest rates.

Term (in years)	**Spot interest rate**
1	4.25%
2	4.80%
3	X
4	5.40%

If the two-year forward rate is 6.5%, calculate X.

A) 5.36% B) 5.20% C) 5.04% D) 4.95% E) 11.61%

32. Sara takes out a loan for 50,000 with 30 quarterly payments. For the first 10 payments, Sara will pay only the interest due at the end of each quarter. For the last 20 payments, Sara will pay X at the end of each quarter. If the annual effective interest rate is 5.8%, what is the total of all of Sara's payments for this loan?

A) 50,709 B) 57,785 C) 64,882 D) 65,330 E) 67,050

33. For one year Robert deposits 300 at the beginning of odd months (January, March, etc.) and withdraws 250 at the beginning of even months (February, April, etc.). If Robert earns an interest rate of 6.3% convertible monthly, what is the value of his fund at the end of December?

A) 310 B) 319 C) 308 D) 337 E) 299

34. A corporation has an obligation to repay a loan 5 years from now at a cost of 100,000. The present value of this obligation (based on the 5-year spot rate) is 76,513.44.

The company will purchase bonds to support this liability. The available bonds are 3-year and 8-year zero-coupon bonds that are currently priced at 88.39 and 60.88 (per 100 of maturity value), respectively.

What is the total *face amount* of the bonds that the company needs to purchase so that the present value and modified duration of its bond assets will match the present value and modified duration of its liability?

A) 97,250 B) 100,000 C) 100,250 D) 102,250 E) 105,250

35. Assuming no inflation and a 2% risk of default (with no recovery), what is the minimum rate a lender would charge for a 1-year loan if the lender's required continuously compounded rate of return is 6%? Express your answer as a continuously compounded annual rate.

A) 7.98% B) 8.00% C) 8.02% D) 8.04% E) 8.06%

Solutions

1. The four-year forward rate $i_{4,5}$ is given by

$1 + i_{4,5} = (1 + s_5)^5/(1 + s_4)^4 = 1.04^5/1.045^4 = 1.020$
$i_{4,5} = 0.02$
(Note the unusually low value, due to the inverted yield curve.)

Answer: C

2.

$$D_{\text{mac}} = \frac{\left[80\left(Ia\right)_{\overline{10}|} + 10(1{,}000)\,v^{10}\right]}{\text{Bond Price}}$$

$$\left(Ia\right)_{\overline{10}|} = \frac{\left(\ddot{a}_{10} - 10v^{10}\right)}{i}$$

$v^{10} = 1.065^{-10} = 0.532726$

$\ddot{a}_{\overline{10}|} = 7.6561$ (be sure calculator is in BGN mode)

$\left(Ia\right)_{\overline{10}|} = 35.8284$

Bond price $= 1{,}107.83$

(Reset calculator to END mode. N = 10, PMT = 80, I/Y =6.5, FV = 1,000. CPT PV = -1,107.83)

$D_{\text{mac}} = [80(35.8284) + 5{,}327.26]/1{,}107.83 = 7.396$

Answer: B

3. $K = 475 + 475v = 570v^2 + 570^3$
$v^2 = [475(1 + v)]/[570(1 + v)] = 0.8333 \Rightarrow v = 0.91287$

Note that each payment of 570 occurs 2 years after each payment of 475, so we could have written: $570v^2 = 475$ $\quad v = \left(475/570\right)^{0.5} = 0.91287$

$K = 475(1 + v) = 475(1.91287) = 908.61$

Answer: C

4. The accumulated amount at the end of year one is 412.16.
(N = 4, I/Y = 2, PMT = 100, PV =0. CPT FV = - 412.16)

We can view the annuity as a 10-year annuity-immediate with annual payments, the first being 412.16 and subsequent payments increasing by 5% each year. The annual effective rate is $i = (1.02)^4 - 1 = 0.08243$.

The present value of this annuity is:

$$(412.16/1.08243)[1 + (1.05/1.08243) + \ldots + (1.05/1.08243)^9]$$
$$= (412.16/1.08243)[1 - (1.05/1.08243)^{10}]/(1 - 1.05/1.08243)$$
$$= 3{,}333.28$$

Alternatively, calculate the value of a geometric annuity-immediate with <u>annual</u> payments that begin at 400 and increase 5% per year, and multiply that value by $i / i^{(4)}$ to reflect quarterly payments:

$$i = 1.02^4 - 1 = 0.082432$$
$$400a_{\overline{10|}}^{5\%} = 400 \cdot \frac{1 - (1.05 / 1.082432)^{10}}{0.082432 - 0.05} = 3{,}234.93$$
$$3{,}234.93 \cdot \left(i / i^{(4)}\right) = 3{,}234.93 \cdot (0.082432 / 0.08) = 3{,}333.28$$

Answer: E

5. $X = (1 - v^{10})/i$, $1.5X = (1 - v^{20})/i$ $\left[\left(1 - v^{20}\right) / i\right] \div \left[\left(1 - v^{10}\right) / i\right] = 1.5$
Hence $1 + v^{10} = 1.5$, $v^{10} = 0.5$, $i = 0.5^{-0.1} - 1 = 0.072$

Answer: A

6. For the time-weighted yield:
$1 + j = (105{,}000/100{,}000)(112{,}000/115{,}000)(95{,}000/82{,}000) = 1.185$
$j = 0.185$

Answer: E

7. For the dollar-weighted yield:
$I = 95{,}000 - 100{,}000 - (10{,}000 - 30{,}000) = 15{,}000$
$i = 15{,}000/[100{,}000 + (1 - 2/12)(10{,}000) - (1 - 8/12)(30{,}000)]$
$= 0.153$

Answer: B

8. The weights are 3/20, 7/20 and 1/2 respectively for the 5-year, 10-year, and 20-year bonds. The modified duration is:

$D_{\text{mod}} = (3/20)(4.615) + (7/20)(9.323) + (1/2)(19.085) = 13.498$

Answer: A

9. The company must invest the present values of:
1,000 in one year at 7%, 3,000 in 2 years at 8%, and 5,000 in 3 years at 9%
The cost is $1{,}000/1.07 + 3{,}000/1.08^2 + 5{,}000/1.09^3 = 7{,}367.51$.

Answer: D

10. The deposits can be viewed as payments of 100 into a 20-year annuity-immediate and 100 into a 10-year-deferred 10-year annuity-immediate.

The accumulated amount in the first annuity is 46,204.09.
(N = 240, I/Y = 0.5, PMT = -100, PV = 0. CPT FV = 46,204.09)

The accumulated amount in the second annuity is 16,387.94.
(Reset N = 120. CPT FV = 16,387.94)

Total accumulation is 62,592.02.

The monthly payments from the 30-year annuity are 459.30.
(N = 360, I/Y = 0.6667, PV = - 62,592.02, FV = 0. CPT PMT = 459.30)

Answer: E

11. The present value of this annuity is
$$100\ddot{a}_{\overline{10}|} + (106/1.08^{10})\cdot \ddot{a}_{\overline{10}|}^{6\%}$$

To get the value of the first term set the BA II Plus to BGN mode.
Set N = 10, I/Y = 8, PMT = -100, and FV = 0. CPT PV = 724.69.

The value of the second term (using the artificial interest rate method) is:
$$\frac{106}{1.08^{10}}\cdot \ddot{a}_{\overline{10}|j}, \text{ where } j = \frac{1.08}{1.06} - 1 = 0.018868$$
$$\frac{106}{1.08^{10}}\cdot \frac{1-\left[1.018868\right]^{-10}}{0.018868/\left[1.018868\right]} = 452.03$$

Present value is 724.69 + 452.03 = 1,176.72

Answer: A

12. Based on the modified approximation method, the change is:
$\Delta P = -(D_{\text{mod}})P(i)\Delta i = -[(D_{\text{mac}})/(1+i)]P(i)\Delta i$
$= -[(7.1245)/(1.072)](972.48)(0.001)$
$= -6.463$

Note that using the Macaulay approximation method, the new price is:

$$972.48 \cdot \left(\frac{1.072}{1.073}\right)^{7.1245} = 966.041$$

This gives a price change of -6.439, leading to the same answer choice.

Answer: A

13. PRin_k is the amount of principal repaid in the k^{th} period.
$\text{PRin}_{k+n} = (1+i)^n\,\text{PRin}_k$.

In this problem, $k = 20$ and $n = 5$.
$244.78 = (1+i)^5\,(166.59)$.
$i = (244.78/166.59)^{1/5} - 1 \Rightarrow i = 0.08$

Answer: B

14. The accumulation in Linus's account is $100s_{\overline{20|}} = 3{,}306.60$.
(N = 20, I/Y = 5, PV = 0, PMT = -100. CPT FV = 3,306.60)

The accumulation in Lucy's account is $P(Is)_{\overline{20|}}$.

$$(Is)_{\overline{20|}} = \frac{(\ddot{s}_{\overline{20|}} - 20)}{0.05} = 294.385$$

$$P = \frac{3{,}306.60}{294.385} = 11.23$$

Answer: D

15. Let P be the annual payment. The interest paid in the 10th payment is
$P(1 - v^{30-10+1}) = P(1 - v^{21}) = P(1 - 0.24151) = 366.74$
$P = 483.51$

For the 20th payment the interest is $483.51(1 - v^{11}) = 483.51(0.52491) = 253.80$

If you don't remember the above formula for the amount of interest in the kth payment, you can reason as follows:
The interest in the 10th payment equals 7% of the loan balance at time 9, so that balance is 366.74/0.07=5,239.14. At time 9, 21 payments remain, so we can calculate the payment amount by setting N=21, I/Y=7, PV=5,239.14, and FV=0. Then CPT PMT=-483.52. Then change N to 11 (the number of payments remaining) to find the balance after 19 payments.
Then CPT PV=3,625.72. So the interest paid at time 20 is:
$0.07(3{,}625.72) = 253.80$

Answer: E

16. If D is the amount deposited, then the accumulation is:
$$A = De^{0.05(5)}(1 - 0.015)^{-40} = 2.3503D$$

There are 180 months in the 15-year period. If j is the monthly interest then
$j = 2.3503^{1/180} - 1 = 0.00476$
$i^{(12)} = 12(0.00476) = 0.0571$

Answer: B

17. The price of the bond is 1,034.74.
(N = 20, I/Y = 3.75, PMT = 40 and FV = 1,000. CPT PV = - 1,034.74)

The accumulated amount of reinvested coupon payments is $40s_{\overline{20|}} = 1{,}131.19$.
The total accumulation is 2,131.19.

The semi-annual yield on the investment is:
$j = (2{,}131.19/1{,}034.74)^{1/20} - 1 = 0.0368$.

The annual yield is 2(.0368) = 0.0736.

Answer: A

18. $1 + i_{4,5} = (1 + s_5)^5/(1 + s_4)^4$
$(1 + s_4)^4 = (1 + s_5)^5/(1 + i_{4,5}) = (1.051)^5/(1.061) = 1.20864$
$1 + s_4 = 1.20864^{1/4} = 1.0485$, $s_4 = 0.0485$

Answer: C

19. This can be viewed as a 10-year increasing annuity and a 10-year-deferred 10-year annuity.

The present value of the deferred annuity is:
$1{,}000v^{10}\,a_{\overline{10}|} = 1{,}000(0.48519)(6.8641) = 3{,}330.39$.

The present value of the increasing annuity is $100(Ia)_{\overline{10}|}$.

$$(Ia)_{\overline{10}|} = \frac{\left(\ddot{a}_{\overline{10}|} - 10v^{10}\right)}{i} = \frac{7.3789 - 4.8519}{0.075} = 33.6933$$

Total cost is 3,330.39 + 3,369.33 = 6,699.72.

Answer: E

20. The price of the zero-coupon bond is $1{,}000/1.06^5 = 747.26$.

To find the price of the second bond using the BA II Plus, set
N = 10, I/Y = 3.5, PMT = - 40, and FV = -1,000. CPT PV = 1,041.58.

The accumulation of the reinvested coupon payments is 463.87.
(N = 10, I/Y =3.25, PMT = -40, and PV =0. CPT FV = 463.87)

Total investment is 747.26 + 1,041.58 = 1,788.84.

Total accumulation is 1,000 + 1,000 + 463.87 = 2,463.87.

Annual effective yield is $\left(\frac{2{,}463.87}{1{,}788.84}\right)^{\frac{1}{5}} - 1 = 0.066$.

Answer: D

21. The balances at the end of 20 years are $1{,}000\cdot(1+i)^{20}$ and $1{,}000\cdot(1+1.1i)^{20}$. The balance at a rate of 1.1i is 10% larger than the balance at rate i, so we have: $\left(\frac{1+1.1i}{1+i}\right)^{20}=1.1$. Then:

$$1+1.1\cdot i=(1+i)\cdot 1.1^{1/20}=1.004777\cdot(1+i)$$
$$(1.1-1.004777)\cdot i=1.004777-1$$
$$i=\frac{0.004777}{1.1-1.004777}=0.050165$$

The difference in the balances in the two accounts after 10 years will be:

$$1{,}000\cdot\left(\left[1+1.1(0.050165)\right]^{10}-1.050165^{10}\right)=79.63$$

Answer: B

22. First calculate the initial monthly payment.

The equation of value is $200{,}000=P_1\cdot a_{\overline{240}|0.3\%}=P_1\cdot\frac{1-1.003^{-240}}{0.003}$

We can solve this equation for P_1 (the monthly payment in the first year), or we can use the BA II Plus's TVM functions. In either case, P_1 = 1,170.22.

At the end of 2 years (24 payments), the outstanding balance of the loan is the present value of the remaining 216 (=240-24) payments:

$$Bal_{24}=P_1\cdot a_{\overline{216}|}=1{,}170.22\cdot\frac{1-1.003^{-216}}{0.003}=185{,}831.95$$

(This value can be calculated by formula or by using the TVM functions.)

The new monthly payment in the 3rd year is the amount that will repay the outstanding balance in the remaining period (18 years) at the new interest rate (4.5%): $185{,}831.95=P_3\cdot a_{\overline{216}|0.375\%}$.

$$P_3=\frac{185{,}831.95}{\left(1-1.00375^{-216}\right)/0.00375}=1{,}256.83$$

Finally, the outstanding balance at the end of 5 years is the present value of the remaining 15 years (180 months) of payments:

$$Bal_{60}=1{,}256.83\cdot a_{\overline{180}|0.375\%}=164{,}292.58$$

Answer: D

23. If the bond is held to maturity, its yield is determined as follows:
N = 40, PV = -1,060, PMT = 30, FV = 1,000
CPT I/Y = 2.7508% (semi-annual effective yield)
The yield to maturity is: $i^{(2)} = 0.027508 \times 2 = 0.055016$ or 5.50%

If the bond is called at the 10th anniversary, the entries are:
N = 20, PV = -1,060, PMT = 30, FV = 1,050 (including the call premium)
CPT I/Y → 2.79434% (semi-annual effective yield)
The yield to maturity is: $i^{(2)} = 0.0279434 \times 2 = 0.0558868$ or 5.59%

The minimum yield that the bondholder could earn is the smaller of these 2 values, 5.50%.

Answer: A

24. First determine the *amounts* of the bonds that must be purchased.

For the 3-year bond with 7% coupons, the amount needed to cover the 3,000 payment required at time 3 is: $\frac{3{,}000}{1.07} = 2{,}803.74$ (or 28.0374 bonds with face amounts of 100 each). This will generate coupon payments at times 1 and 2 of $2{,}803.74 \cdot (0.07) = 196.26$.

At time 2, in addition to receiving 196.26 from the 3-year bond, the corporation will need an additional 1,803.74 from the 2-year bond (which pays no coupons), so the 2-year bond's face amount is 1,803.74.
Similarly, the face amount of the 1-year bond at time 1 is 803.74.

The prices for the 3 bonds are calculated as follows:

1-year: $\frac{803.74}{1.04} = 772.83$

2-year: $\frac{1{,}803.74}{1.06^2} = 1{,}605.32$

3-year: N=3, I/Y=8, PMT=7, FV=100; CPT PV=-97.4229

$28.0374 \cdot (97.4229) = 2{,}731.48$

Total Cost $= 772.83 + 1{,}605.32 + 2{,}731.48 = 5{,}109.63$

Answer: B

25. Since dividends are paid semi-annually, we need a semi-annual effective interest rate to calculate the present value of dividends. Let i be this rate.

$$i = 1.12^{1/2} - 1 = 0.0583$$

The dividend discount model (with level dividends) tells us that the value of a share of stock is:

$$P = \frac{Div}{i} = \frac{20}{0.0583} = 343.05$$

This is the value as of a date one full period before the next dividend date. We are asked for the value as of April 15 (2 months before a dividend payment). We must accumulate this value forward 4 months:

$$343.05 \cdot 1.0583^{4/6} = 356.26$$

Answer: E

26. This is a non-deferred swap with level notional amount, so we can use the simplified formula: $R = \dfrac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}} = \dfrac{1 - P_2}{P_1 + P_2} = \dfrac{1 - 1.017^{-2}}{1.014^{-1} + 1.017^{-2}} = 0.016975$

The swap rate (the *fixed* rate) is 1.6975%, so the amount of the fixed payment is 16,975.

Answer: D

27. The formula for the quoted rate of a U.S. Treasury bill is:

$$\text{Quoted rate} = \frac{360}{\text{Days to maturity}} \times \frac{\text{Amount of interest}}{\text{Maturity value}}$$

$$= \frac{360}{260} \times \frac{1{,}000 - 948}{1{,}000} = 0.072$$

Answer: B

28. Using a valuation date of t=2, the equation of value is:

$$500 \cdot \left(1 + \frac{i^{(4)}}{4}\right)^{8} + x = 1{,}000 \cdot \left(1 + \frac{i^{(4)}}{4}\right)^{-12}$$

The amount invested at time 2 is:

$$x = 1{,}000 \cdot \left(1 + \frac{0.06}{4}\right)^{-12} - 500 \cdot \left(1 + \frac{0.06}{4}\right)^{8} = 273.14$$

Answer: B

29. To find the purchase price, we must calculate the price that would produce a 7.5% return based on a call at 10 years, and also the price based on the bond's <u>not</u> being called. The lower of these two numbers is the price paid.

Call at 10 years: N=20, I/Y=3.75, PMT=40, FV=1,100, CPT PV=-1,082.63

No call: N=50, I/Y=3.75, PMT=40, FV=1,000, CPT PV=-1,056.09

The actual price paid is the lower number: 1,056.09.

If the bond is called after 12 years, Lisa's yield (convertible semi-annually) is calculated as follows:

N=24, PV=-1,056.09, PMT=40, FV=1,100, CPT I/Y=3.895

The nominal annual yield is $3.895\% \times 2 = 7.79\%$.

Answer: D

30. The fixed interest rate for the swap is the par coupon bond rate for a 3-quarter coupon bond.

$$R = c = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 0.953}{0.984 + 0.969 + 0.953} = 0.0162$$

The net rate paid to you will be 0.018 - 0.0162 = 0.0018

Answer: D

31. We are given that $i_{2,3} = 0.065$, so we have:

$$1 + i_{2,3} = \frac{(1 + s_3)^3}{(1 + s_2)^2}$$

$$1 + 0.065 = \frac{(1 + X)^3}{(1 + 0.048)^2}$$

$$1.169694 = (1 + X)^3$$

$$X = 0.0536363$$

X = 5.364%

Answer: A

32. Since Sara is paying only the interest due for the first 10 payments, the balance will remain at 50,000. We need to find what the payment would be for a 50,000 loan with 20 payments where the effective interest rate per payment period is:

$$i = (1.058)^{1/4} - 1 = 0.014194887$$

Set N=20, I/Y=1.4194887, PV=50,000, AND FV=0. CPT PMT=2,990.2252.

Sara is making 20 payments of 2,889.23 and 10 payments of 709.74, for a total of 20(2889.23)+10(709.74)=64,882.

Answer: C

33. Find the future value of the deposits minus the future value of the withdrawals. We will assume that each month is approximately the same length. First find the monthly effective rate, $i = \dfrac{0.063}{12} = 0.00525$.

The future value of the deposits at the beginning of odd months would be:

$$300(1+i)^{12} + 300(1+i)^{10} + 300(1+i)^{8} + 300(1+i)^{6} + 300(1+i)^{4} + 300(1+i)^{2}$$

$$= 300(1+i)^2 \left[\frac{1-\left((1+i)^2\right)^6}{1-(1+i)^2} \right] = 1,867.4997$$

The future value of the withdrawals at the beginning of even months would be:

$$250(1+i)^{11} + 250(1+i)^{9} + 250(1+i)^{7} + 250(1+i)^{5} + 250(1+i)^{3} + 250(1+i)$$

$$= 250(1+i) \left[\frac{1-\left((1+i)^2\right)^6}{1-(1+i)^2} \right] = 1,548.1221$$

Alternatively, we can view these cash flows as 6-period annuities with a period of 2 months. The 2-month effective rate is $1.00525^2 - 1 = 0.0105276$, so the accumulated value of this 6-period annuity as of its last payment date is:

$$s_{\overline{6}|0.0105276} = \frac{1.0105276^6 - 1}{0.0105276} = 6.160148$$

The deposits' accumulated value on Nov. 1 (the last deposit date) is:

$$300 \cdot 6.160148 = 1,848.04$$

Their value at year-end is $1,848.04 \cdot 1.0105276 = 1,867.50$.

The withdrawals' accumulated value on Dec. 1 (the last deposit date) is:

$$250 \cdot 6.160148 = 1,540.04$$

Their value at year-end is $1,540.04 \cdot 1.00525 = 1,548.12$

The balance at the end of the year would be the accumulated value of the deposits minus the accumulated value of the withdrawals:

1,867.50-1,548.12 = 319.38

Answer: B

34. Let x and y be the face amounts of the 3-year and 8-year bonds, respectively. To match the present value of the assets and the liability, we have:

$$0.8839x + 0.6088y = 76{,}513.44$$

The 3-year, 5-year, and 8-year spot rates are as follows:

3-year: $s_3 = 0.8839^{-\frac{1}{3}} - 1 = 4.2\%$

5-year: $s_5 = \left(\dfrac{100{,}000}{76{,}513.44}\right)^{\frac{1}{5}} - 1 = 5.5\%$

8-year: $s_8 = 0.6088^{-\frac{1}{8}} - 1 = 6.4\%$

The modified durations for the bonds and the liability are:

3-year: $D_{\text{mod}}^{3\text{-yr.}} = \dfrac{3}{1.042} = 2.879$

5-year: $D_{\text{mod}}^{5\text{-yr.}} = \dfrac{5}{1.055} = 4.739$

8-year: $D_{\text{mod}}^{8\text{-yr.}} = \dfrac{8}{1.064} = 7.519$

To match the modified duration of the combined assets to that of the liability, we have:

$$2.879 \times 0.8839x + 7.519 \times 0.6088y = 4.739 \times 76{,}513.44$$

Solving the two equations for x and y produces:

$$x = 51{,}857 \qquad y = 50{,}390 \qquad x + y = 102{,}247$$

Answer: D

35. At a continuously compounded rate of 6%, the lender would receive an amount of $e^{0.06} = 1.0618$ at the end of 1 year for each unit lent.
Assuming a 2% default rate and no recovery on defaulted loans, the lender must request a payment of $1.0618 \cdot \dfrac{1}{1-0.02} = 1.0835$. This corresponds to a continuously compounded rate of $\ln 1.0835 = 0.0802$.

Answer: C

Practice Exam 2

Exam FM

Questions

1. A man borrows 1,000 for 2 years at an annual effective rate *i*. He has two payment options:

1. Pay 560 at the end of each year, or
2. Pay *K* at the end of year 1 and 800 at the end of year 2.

Find *K*.

A) 329.42 B) 331.66 C) 334.82 D) 337.57 E) 341.65

2. A company has liabilities that require payments of 2,000 in 1 year and 5,000 in 3 years. The investments available to the company are the following zero-coupon bonds:

Maturity (years)	Annual Effective Rate	Par Value
1	6.5%	1,000
3	7.5%	1,000

Determine the cost to match the liability cash flows exactly.

A) 5,903 B) 5,935 C) 5,952 D) 5,970 E) 5,988

3. A woman has a 30-year fixed-rate mortgage on her home. Her monthly payments are level and made at the end of the month. The total principal repaid during the 20^{th} year of the loan is 3 times the total principal repaid during the 5^{th} year. Find the annual effective interest rate for this mortgage.

A) 6.8% B) 7.0% C) 7.2% D) 7.4% E) 7.6%

4. You are given the following n-year forward rates:

Year (n)	Forward Rate ($i_{n,n+1}$)
0	2.9%
1	3.7%
2	4.4%
3	5.2%

Find s_4.

A) 3.92% B) 4.05% C) 4.17% D) 4.31% E) 4.46%

5. A man buys a 20-year annuity-immediate for 10,000. He receives annual payments of 910. He invests these payments in a fund that earns 7.5% annually. What is his annual effective yield on this 20-year investment?

A) 6.5% B) 6.7% C) 6.9% D) 7.1% E) 7.3%

6. An investment pays 2,000 at the end of year one and 4,000 at the end of year three. It is purchased to yield a 7.2% annual effective rate. What is the Macaulay duration for this investment?

A) 2.270 B) 2.301 C) 2.334 D) 2.358 E) 2.515

7. A woman buys two 5-year 1,000 par bonds. The first has 7.5% semi-annual coupons and is priced to yield 8% convertible semi-annually. The second has 6% semi-annual coupons and is priced to yield 7% convertible semi-annually. The coupon payments from the two bonds are deposited in a fund that pays 6.8% convertible semi-annually.

What is her annual effective yield for this combined investment?

A) 7.3% B) 7.5% C) 7.7% D) 7.9% E) 8.1%

8. The spot rate for year k is given by the equation:

$s_k = 0.08 + 0.003k - 0.0015k^2$.

Find the three-year forward rate implied by this yield curve.

A) 4.36% B) 4.41% C) 4.58% D) 4.65% E) 4.74%

9. A 10-year annuity-due pays 50 each quarter for the first 5 years and 100 each quarter for the last 5 years. At a nominal rate of 6% convertible quarterly, what is the present value of this annuity?

A) 1,978 B) 2,034 C) 2,077 D) 2,119 E) 2,165

10. A 20-year annuity-immediate pays 100 a year for the first 10 years. Starting with the 11^{th} payment, each payment is increased by 6% over the previous one.

Find the present value of this annuity at an annual effective rate of 7%.

A) 1,150 B) 1,185 C) 1,235 D) 1,262 E) 1,288

11. A 3-year 1,000 par bond has 8% annual coupons and an annual effective yield of 7%. Find the Macaulay duration of this bond.

A) 2.5 B) 2.6 C) 2.7 D) 2.8 E) 2.9

12. A new company expects the dividend on its common stock to be 1 the first year and to increase by 1 each year until it reaches 10. Thereafter it expects the dividend to grow by 3% each year.

Using the dividend growth model and assuming an annual effective interest rate of 5%, calculate the value of this stock.

A) 344 B) 351 C) 356 D) 365 E) 372

13. A company has a loan of 100,000 to be repaid with 30 level end-of-year payments. The principal and the interest in the 21^{st} payment are the same. Find the principal repaid in the 10^{th} payment.

A) 1,862 B) 1,871 C) 1,884 D) 1,901 E) 1,913

14. A man deposits money into a fund. For the first four years the fund accumulates at a nominal interest rate of 6% convertible quarterly. For the next six years the fund accumulates at a nominal rate of discount of 8% convertible semi-annually.

For the 10-year period what is the equivalent force of interest?

A) 0.0719 B) 0.0728 C) 0.0731 D) 0.0737 E) 0.0742

15. A 20-year annuity-immediate has annual payments. The first payment is 1,000. Subsequent payments decrease by 100 each year until they reach 100. The remaining payments stay at 100. At an annual effective interest rate of 6.5%, what is the present value of this annuity?

A) 4,708 B) 4,765 C) 4,815 D) 4,853 E) 4,894

16. A man borrows 100,000 to buy a house. His 30-year mortgage has level end-of-month payments and a fixed interest rate of 7.5% convertible monthly. After 10 years he refinances the outstanding principal balance for 15 years at 6% convertible monthly.

Calculate his new monthly payments after refinancing the mortgage.

A) 702.45 B) 717.68 C) 732.43 D) 750.65 E) 762.38

17. A woman is asked to invest 20,000 in a project. She is promised returns of 5,000 in one year, 6,000 in two years, 7,000 in three years and 10,000 in four years. Find the IRR for this investment.

A) 12.71% B) 12.84% C) 12.96% D) 13.11% E) 13.23%

18. Consider the following yield curve:

Year	Spot Rate
1	2.0%
2	2.5%
3	3.0%
4	4.0%

A 4-year 1,000 par bond has an annual coupon rate of 3.5%. Use the spot rates of the yield curve to find the price of this bond.

A) 980 B) 984 C) 989 D) 994 E) 999

19. A man buys a 10-year 1,000 par bond with 7% semi-annual coupons. The bond is priced to yield 6.5% convertible semi-annually. The coupon payments are invested in a fund that earns 6% convertible semi-annually.

His wife makes annual end-of-year payments of K into a fund that earns a 6.5% annual effective rate. At the end of 10 years their accumulated funds are the same. Find K.

A) 126.28 B) 131.45 C) 139.25 D) 143.80 E) 151.38

20. For an unknown interest rate i, the following payments have the same present value:

1. 675 at the end of two years
2. 200 at the end of one year and 500 at the end of three years

Find the value of i. (Assume $i < 100\%$.)

A) 9.0% B) 9.2% C) 9.4% D) 9.6% E) 9.8%

21. Bart Ender invests 500 at $t=0$ at a 6% annual rate of simple interest. At $t=2$ he invests an additional amount so that the balance in his account at $t=5$ will be 1,000. What annual effective interest rate will Bart's account earn during the 5th year (from $t=4$ to $t=5$)?

A) 4.84% B) 5.02% C) 5.20% D) 5.44% E) 6.00%

22. Bert Nurney borrows 10,000 and agrees to repay the loan over 10 years with interest at an annual effective rate of 8%. A loan payment is to be made at the end of each year for 10 years. The payments are to be level for the first 5 years; then, in the 6^{th} through 10^{th} years, each payment will be twice as large as the payment amount during the first 5 years. (Thus the repayment schedule consists of 5 payments of P, followed by 5 payments of $2P$.)

What is the outstanding balance of Bert's loan at the end of 8 years? In other words, how much does Bert still owe immediately after he makes the 8^{th} loan payment?

A) 1,892 B) 2,121 C) 3,131 D) 3,783 E) 4,243

23. A 25-year bond has a face amount (and maturity value) of 1,000. It pays semi-annual coupons at an 8% (annual) coupon rate. The bond is callable on any coupon date on or after its 10^{th} anniversary, with a 5% call premium.

An investor purchases this bond at a price such that it will yield 7.2%, compounded semi-annually if it is held to maturity and is not called.

What is the earliest coupon date on which the bond could be called and the investor would earn a rate of return of at least 7.2% (compounded semi-annually)?

A) 30th B) 31st C) 32nd D) 33rd E) 34th

24. The following table provides data for a portion of a company's liabilities and for the assets that are assigned to support these liabilities.

	Value	Modified Duration	Modified Convexity
Asset 1	5,000	2	14
Asset 2	2,000	4	19
Asset 3	3,000	7	34
Liability 1	4,000	2.25	16
Liability 2	6,000	5	22

Which of the following statements are true?

I. The modified durations of the assets and liabilities are equal.

II. The modified convexities of the assets and liabilities are equal.

III. This portion of the company's balance sheet is immunized against a small "parallel shift" in interest rates.

A) All B) I and II only C) I and III only D) I only E) None

25. A Government of Canada Treasury bill with a maturity value of 1,000 will mature 180 days from today. If its current price is 982, what is the quoted rate for this T-bill?

A) 4.9% B) 5.0% C) 5.1% D) 5.2% E) 5.3%

26. Based on the following forward rates, what is the fixed interest rate for a 3-year interest rate swap with a level notional amount? (All rates are expressed as annual effective rates.)

$$f^*_{[0,1]} = 0.03200$$
$$f^*_{[1,2]} = 0.04202$$
$$f^*_{[2,3]} = 0.04603$$

A) 0.0363 B) 0.0398 C) 0.0418 D) 0.0436 E) 0.0456

27. Rhoda Kammel takes out a 20-year mortgage to purchase a home. It is a conventional mortgage with level monthly payments that fully repay the amount borrowed. The 60th monthly payment consists of 360.17 of interest and 197.58 of principal.

To the nearest 100, what is the amount that Rhoda originally borrowed?

A) 71,000 B) 72,100 C) 72,500 D) 72,900 E) 73,500

28. A 20-year bond with a 1,000 par value pays semi-annual coupons at a 6% (annual) coupon rate. The bond is callable on any coupon date on or after its 10th anniversary with a 4% call premium. If the bond is not called, it will be redeemed at par on its 20th anniversary.

The bond is purchased on its 5th anniversary at a price that assures a yield of at least 6.5% convertible semi-annually. If the bond is called on its 12th anniversary, what is the purchaser's actual yield (convertible semi-annually)?

A) 6.01% B) 6.33% C) 6.67% D) 6.99% E) 7.33%

29. A retiree intends to purchase bonds that will provide payments of exactly 50,000 at the end of each of the next 3 years. The following 3 bonds are available:

1-year zero-coupon bond with a 6% yield to maturity
2-year bond with 7% annual coupons and an 8% yield to maturity
3-year bond with 7% annual coupons and a 9% yield to maturity
(All rates are annual effective rates.)

To the nearest 100, what is the <u>cost</u> of the <u>1-year zero coupon bond</u> that will be purchased as part of the portfolio to provide the desired payments?

A) 40,100 B) 41,200 C) 42,400 D) 43,700 E) A different amount

30. Given the following table of zero-coupon bond prices (per dollar of maturity value), what is the guaranteed quarterly interest rate on a 4-quarter interest rate swap with a level notional amount?

Term (in quarters)	1	2	3	4
Zero-coupon bond price	0.984	0.969	0.953	0.935

A) 0.0118 B) 0.0137 C) 0.0158 D) 0.0169 E) 0.0195

31. Andy deposits 200 at the end of each month into an account earning a nominal interest of 6% convertible monthly.

Scott deposits 500 at the end of each quarter into an account earning an annual effective interest rate of i. At the end of 4 years, both Andy and Scott have the same accumulated value. How much interest did Scott earn during the last year of his investment?

A) 387 B) 955 C) 1,113 D) 1,352 E) 3,350

32. Brett purchases a perpetuity that provides payments at the end of each 6-month period. Each of the first 18 payments is for an amount X.

Starting with the 19^{th} payment, the payments increase by one each period, so the 19^{th} payment is $X+1$, the 20^{th} is $X+2$, etc. At a nominal rate of 6% convertible semi-annually, the present value of the perpetuity is 3,000. What is the amount of the 30^{th} payment?

A) 189 B) 173 C) 82 D) 81 E) 70

33. A company has liability payments of X due one year from now, 10,000 two years from now, and 15,000 three years from now. The company can purchase any amount of each of the following three annual-coupon bonds in order to exactly match the required payments.

Bond A: a 1-year bond with a face amount of 1,000, a 7% annual coupon rate, and a 5% annual yield rate.

Bond B: a 2-year bond with a face amount of 1,000, a 4% annual coupon rate, and a 3% annual yield rate.

Bond C: a 3-year bond with a face amount of 1,000, a 6% annual coupon rate, and a 5% annual yield rate.

If the total cost of the bonds purchased is 25,122, calculate X.

A) 122 B) 2,000 C) 2,900 D) 3,500 E) 5,000

34. An investor makes a 2-year loan of 1,000 at a continuously compounded interest rate of 6.5%. Assuming there will be no inflation during the 2-year term of the loan, what probability of default (with no recovery) would give the investor a 6% expected continuously compounded rate of return?

A) 0.5% B) 0.7% C) 0.9% D) 1.0% E) 1.1%

35. A speculator enters into a 3-year interest rate swap as the payer. The swap has a level notional amount of 1,000,000 and is based on the spot rates in the following table:

Term (in years)	1	2	3
Spot Rate	0.030	0.040	0.047

If the 1-year spot rate at the end of the 2nd year of the swap is 5.2%, what net settlement amount (to the nearest 100) will the speculator pay to the counterparty at the end of the 3rd year?

A) 5,600 B) 5,480 C) -5,480 D) -5,600 E) A different amount

Solutions

1. We first need to find i. We can use the BA II Plus and set
N = 2, PMT = 560, PV = -1,000 and FV =0. CPT I/Y = 7.9

To find K, we have:
$1{,}000 = K/1.079 + 800/1.079^2$. $K = 337.57$

Answer: D

2. The company must invest the present value of 2,000 payable in 1 year at 6.5% plus the present value of 5,000 payable in 3 years at 7.5%

The cost is
$2{,}000/1.065 + 5{,}000/1.075^3 = 1{,}877.93 + 4{,}024.80 = 5{,}902.73$

Answer: A

3. If we know the amount of principal repaid in payment k, we can find the principal repaid in payment n+k: $PRin_{n+k} = PRin_k \cdot (1+i)^n$, where i is the effective interest rate per period. In this case, we are interested in the total amount of principal repaid in the 12 monthly payments during the 5th year, and in the 12 payments during the 20th year. Since each of the 12 payments in year 20 occurs 180 periods after the corresponding payment during year 5, the principal in each payment during the 20th year equals $(1+i)^{180}$ times the principal in the corresponding 5th-year payment, so the total amount of principal repaid during the 20th year is also larger than the total amount during the 5th year by a factor of $(1+i)^{180}$.

Since we are given that 3 times as much principal was repaid during the 20th year, we have: $(1+i)^{180} = 3$, so $i = 3^{1/180} - 1 = 0.006122$, which is the monthly effective rate for the loan. However, the problem asks for the *annual* effective rate, which is $1.006122^{12} - 1 = 1.07599$.

Answer: E

4. $(1 + s_4)^4 = (1 + i_{0,1})(1 + i_{1,2})(1 + i_{2,3})(1 + i_{3,4})$
$= (1.029)(1.037)(1.044)(1.052) = 1.17195$
$s_4 = 1.17195^{1/4} - 1 = 0.0405$

Answer: B

5. To find the accumulated value at t=20 using the BA II Plus, set N = 20, I/Y = 7.5, PV = 0, PMT = -910. CPT FV = 39,407.26

The annual yield rate is $i = (39{,}407.26/10{,}000)^{1/20} - 1 = 0.071$.

Answer: D

6. The present values of these investments are:
$2{,}000/1.072 = 1{,}865.67$ and $4{,}000/1.072^3 = 3{,}246.95$

The total is 5,112.62. The weights for the Macaulay duration are
$w_1 = 1{,}865.67/5{,}112.62 = 0.3649$ and $w_2 = 3{,}246.95/5{,}112.62 = 0.6351$.

$D_{\text{mac}} = (1)(0.3649) + (3)(0.6351) = 2.270$

Answer: A

7. Using the BA II Plus to get the price of the first bond, set
N = 10, I/Y = 4, PMT = 37.5, FV = 1,000. CPT PV = -979.72.

To get the price of the second bond, set
N = 10, I/Y = 3.5, PMT = 30, FV = 1,000. CPT PV = -958.42.

The total price of the bonds is 1,938.14.

To get the accumulated value of the coupon payments, set
N = 10, I/Y = 3.4, PMT = -67.5, PV = 0. CPT FV = 788.22.

Accumulated value of the coupons plus the bonds' redemption value is:
2,000 + 788.22 = 2,788.22.

$(1 + i)^5 = 2{,}788.22/1{,}938.14 = 1.4386$
$i = 1.4386^{1/5} - 1 = 0.075$

Answer: B

8. We need to find $i_{3,4}$.

$1 + i_{3,4} = (1 + s_4)^4/(1 + s_3)^3$
$s_3 = 0.08 + 0.003(3) - 0.0015(9) = 0.0755$
$s_4 = 0.08 + 0.003(4) - 0.0015(16) = 0.068$
$i_{3,4} = (1.068)^4/(1.0755)^3 - 1 = 0.0458$

Answer: C

9. This annuity can be viewed as a 10-year annuity-due with payments of 100, minus a 5-year annuity-due with payments of 50.

To get the present values of these annuities using the BA II Plus, first set the mode to BGN. For the 10-year annuity:
Set N = 40, I/Y = 1.5, PMT = -100, and FV = 0. CPT PV = 3,036.46.

For the 5-year annuity:
N = 20, I/Y =1.5, PMT = -50, and FV = 0. CPT PV = 871.31.

The present value of the annuity is 3,036.46 – 871.31 = 2,165.15.

Note: This problem can also be solved with the calculator in END *mode. Enter the same values as shown above and calculate a present value of 2,133.15 based on end-of-quarter payments. Then multiply by 1.015 to find the value with the payments occurring one quarter of a year earlier: 2,165.15.*

Answer: E

10. The present value of the annuity is $100a_{\overline{10|}} + \dfrac{106}{1.07^{11}}\left[1 + \dfrac{1.06}{1.07} + \ldots + \left(\dfrac{1.06}{1.07}\right)^9\right]$.

$$100a_{\overline{10|}} = 702.36$$

$$\left(\frac{106}{1.07^{11}}\right)\left[1 + \frac{1.06}{1.07} + \ldots + \left(\frac{1.06}{1.07}\right)^9\right] = \left(\frac{106}{1.07^{11}}\right)\frac{\left[1 - \left(\frac{1.06}{1.07}\right)^{10}\right]}{\left[1 - \left(\frac{1.06}{1.07}\right)\right]} = 482.94$$

The present value is 702.36 + 482.94 = 1,185.30

Or we can use the geometric annuity formula for the payments after the first 10 years:

$$\frac{106}{1.07^{10}} \cdot a_{\overline{10|}}^{6\%} = \frac{106}{1.07^{10}} \cdot \frac{1 - \left(\frac{1.06}{1.07}\right)^{10}}{0.07 - 0.06} = 482.94$$

Answer: B

11. The Macaulay duration is:
$D_{\text{mac}} = [80v + 2(80)v^2 + 3(1{,}080)v^3]/(80v + 80v^2 + 1{,}080v^3)$
$v = 1/1.07 = 0.93458$
$D_{\text{mac}} = 2{,}859.32/1{,}026.24 = 2.786$

Answer: D

12. The dividends for the first 10 years form an increasing arithmetic sequence. The present value of these dividends is

$$(Ia)_{\overline{10}|} = \frac{\left(\ddot{a}_{\overline{10}|} - 10v^{10}\right)}{i} = \frac{\left[8.1078 - 10(0.6139)\right]}{0.05} = 39.376$$

The dividends thereafter form a geometric perpetuity. The present value of these dividends at time $t = 10$ years is

$P = Div/(i - r) = 10(1.03)/(0.05\text{-}0.03) = 515$

This part of the dividend stream is deferred 10 years, so the stock price is:
$39.376 + 515/1.05^{10} = 355.54$

Answer: C

13. If P is the annual payment, the amount of principal repaid in the 21^{st} payment is $Pv^{30-21+1} = Pv^{10} = P/2$. Hence $v^{10} = ½$.

So $(1 + i)^{10} = 2$. Then $i = 2^{1/10} - 1 = 0.0718$.

Using the BA II Plus to get the payment, set
N = 30, I/Y = 7.18, PV = -100,000, FV = 0. CPT PMT = 8,204.84

The principal repaid in the 10^{th} payment is
$8{,}204.84v^{30\text{-}10+1} = 8{,}204.84(1.0718)^{-21} = 1{,}912.85$

Answer: E

14. If D is the amount deposited into the fund, the accumulation at the end of 10 years is $D(1.015)^{16}/(0.96)^{12} = D(2.0711)$.

To find the equivalent force of interest, δ:
$e^{10\delta} = 2.0711$ $\delta = (1/10)\ln(2.0711) = 0.0728$

Answer: B

15. The present value of this annuity is $100(Da)_{\overline{10}|} + 100v^{10}\, a_{\overline{10}|}$.
$a_{\overline{10}|} = 7.189$ $v^{10} = 0.5327$.

$$(Da)_{\overline{10}|} = \frac{\left(10 - a_{\overline{10}|}\right)}{0.065} = 43.246$$

Present value of the annuity is: 4,324.60 + 100(7.189)(0.5327) = 4,707.56.

Answer: A

16. To compute the initial payment with the BA II Plus, set
N = 360, I/Y = 0.625, PV = 100,000, and FV = 0. CPT PMT = -699.215.

To get the outstanding principal after 10 years, reset N = 240.
Then CPT PV = 86,794.987.
To get the new payment, reset N = 180 and I/Y = 0.5.
Then CPT PMT = -732.425.

Answer: C

17. To find the IRR put the BA II Plus in CF mode. Then enter the following cash flows: C_0 = -20,000, C_1 = 5,000, C_2 = 6,000, C_3 = 7,000 and C_4 = 10,000. Then IRR CPT = 13.23

Answer: E

18. The price of the bond is
$P = 35/1.02 + 35/1.025^2 + 35/1.03^3 + 1{,}035/1.04^4 = 984.38$.

Answer: B

19. The man's accumulation is $35s_{\overline{20}|3\%} + 1{,}000 = 1{,}940.46$.
(Note that the bond's yield was not needed for this calculation.)

The wife's accumulation is $Ks_{\overline{10}|6.5\%} = K(13.4944)$.
Therefore, $K = 1{,}940.46/13.4944 = 143.797$.

Answer: D

20. Equating the present values of the two payments, we have:
$675v^2 = 200v + 500v^3$. Dividing by v we get the following quadratic equation:
$500v^2 - 675v + 200 = 0$.

Using the quadratic formula we get 2 positive values for v, 0.911 and 0.439. Only $v = 0.911$ gives a value for i that is less than 100%.

$i = (1/0.911) - 1 = 0.098$.

Answer: E

21. First find the amount (call it x) that was deposited at t=2:

$$500\cdot(1+5\cdot 6\%)+x\cdot(1+3\cdot 6\%)=1{,}000$$

$$x=\frac{1{,}000-500\cdot(1.30)}{1.18}=296.61$$

To find the effective interest rate for the 5th year, we need to know the balance at the beginning and end of the year. The ending balance is 1,000, and the beginning balance is:

$$500\cdot(1+4\cdot 6\%)+296.61\cdot(1+2\cdot 6\%)=620+332.20=952.20$$

The annual effective rate earned during the 5th year is:

$$\frac{1{,}000}{952.20}-1=5.0196\%$$

Answer: B

22. Find the value of P by starting with the equation of value for the loan:

$$10{,}000=2P\cdot a_{\overline{10}|}-P\cdot a_{\overline{5}|}=P\cdot\left(2\cdot\frac{1-1.08^{-10}}{0.08}-\frac{1-1.08^{-5}}{0.08}\right)$$

$$P=10{,}000\div\left(2\cdot\frac{1-1.08^{-10}}{0.08}-\frac{1-1.08^{-5}}{0.08}\right)=1{,}060.73$$

We can calculate the loan balance as of the end of the 8th year by the prospective method, as follows:

$$Bal_8=2P\cdot a_{\overline{2}|}=2\cdot(1{,}060.73)\cdot\left(1.08^{-1}+1.08^{-2}\right)=3{,}783.13$$

Answer: D

23. We are given that the bond's yield to maturity is 7.2%, so we can calculate the purchase price by setting N=50, I/Y=3.6, PMT=40, and FV=1,000. Then CPT PV=-1,092.15.

Because the bond's purchase price of 1,092.15 is greater than the call price of 1,050, an early call would be disadvantageous to the bondholder, resulting in a yield less than the 7.2% yield to maturity. However, after the bond has been held for a period of time (and coupons have been received at an 8% annual rate), its "book value" (based on 7.2% and assuming it will not be called) will be <u>less</u> than 1,050. At that point, a call will result in a yield-to-call <u>greater</u> than 7.2%. The problem asks for the <u>first</u> coupon date when this will be the case.

To find that date, change FV to 1,050 (the call price) and CPT N=33.096. This result tells us that the bond will yield <u>less</u> than 7.2% if called on the 33rd coupon date (or earlier), but it will yield <u>more</u> than 7.2% if called at the 34th coupon date (or later).

Answer: E

24. To determine whether the statements are true, we need to calculate the modified durations and convexities for the assets and for the liabilities.

Assets:

$$D_{\text{mod}}^{A} = \frac{5,000 \cdot 2 + 2,000 \cdot 4 + 3,000 \cdot 7}{10,000} = 3.9$$

$$C_{\text{mod}}^{A} = \frac{5,000 \cdot 14 + 2,000 \cdot 19 + 3,000 \cdot 34}{10,000} = 21$$

Liabilities:

$$D_{\text{mod}}^{L} = \frac{4,000 \cdot 2.25 + 6,000 \cdot 5}{10,000} = 3.9$$

$$C_{\text{mod}}^{L} = \frac{4,000 \cdot 16 + 6,000 \cdot 22}{10,000} = 19.6$$

Statement I is true, since both portfolios have a modified duration of 3.9.

Statement II is false, because the modified convexities are 21 and 19.6.

Statement III is true, since the total value of each portfolio is 10,000, the modified duration of each is 3.9, and the modified convexity of the Assets is <u>greater than</u> the that of the Liabilities.

Answer: C

25. For a Treasury bill issued by the Government of Canada, the formula for the quoted rate is:

$$\text{Quoted Rate} = \frac{365}{\text{Days to Maturity}} \times \frac{\text{Amount of Interest}}{\text{Current Price}}$$

Since the amount of interest will be the difference between the current price of 975 and the maturity value of 1,000, we can write:

$$\text{Quoted Rate} = \frac{365}{180} \times \frac{1{,}000 - 975}{975} = 5.199\%$$

Answer: D

26. Since this is a non-deferred interest rate swap with a level notional amount, we can use the simplified formula for the swap rate:

$$R = \frac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}.$$

Using the given forward rates, we can calculate the present value factors that we need:

$$P_1 = \left(1 + f^*_{[0,1]}\right)^{-1} = 1.032^{-1} = 0.96899$$

$$P_2 = \left[\left(1 + f^*_{[0,1]}\right)\left(1 + f^*_{[1,2]}\right)\right]^{-1} = \left[1.032 \cdot 1.04202\right]^{-1} = 0.92992$$

$$P_3 = \left[\left(1 + f^*_{[0,1]}\right)\left(1 + f^*_{[1,2]}\right)\left(1 + f^*_{[2,3]}\right)\right]^{-1} = \left[1.032 \cdot 1.04202 \cdot 1.04603\right]^{-1} = 0.88900$$

Then the swap rate is:

$$R = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 0.88900}{0.96899 + 0.92992 + 0.88900} = 0.03982$$

Answer: B

27. The amount of principal repaid in the k^{th} monthly payment is $Pmt \cdot v^{n-k+1}$.
In this case, we know the amount of principal repaid (197.58), the total amount of the payment (360.17+197.58=557.75), k (60), and n (240), so we can solve for v:

$$PRin_{60} = Pmt \cdot v^{240-60+1}$$
$$197.58 = 557.75 \cdot v^{181}$$
$$v = \left(\frac{197.58}{557.75}\right)^{1/181} = 0.99428$$

Then: $i = \dfrac{1}{v} - 1 = \dfrac{1}{0.99428} - 1 = 0.00575$.

Now we know that the 20-year mortgage has monthly payments of 557.75 and a monthly effective interest rate of 0.575%. We can find the original amount of the mortgage on the BA II Plus by setting N=240, I/Y=0.575, PMT=557.75, and FV=0. Then CPT PV=72,500.

Answer: C

28. First calculate the purchase price. Because the bond was purchased at a discount, we know that the price should be calculated assuming no call:
Set N=30, I/Y=3.25, PMT=30, and FV=1,000. Then CPT PV=-952.55.
(Note: N=30 because the bond was purchased on its 5th anniversary, with 15 years (30 coupon periods) remaining.)

If the bond is called on its 12th anniversary, then it was held for 7 years (14 coupon periods) and was redeemed for 1,040 (including the 4% call premium). To calculate the yield, change N to 14 and FV to 1,040.
Then CPT I/Y=3.663.

The yield, convertible semi-annually is $2 \times 3.663\% = 7.326\%$.

Answer: E

29. To generate 50,000 at t=3, the 3-year bond must have a face amount of:
$50,000/1.07 = 46,728.97$.

This 3-year bond will also provide coupons of $46,728.97 \cdot 0.07 = 3,271.03$ at t=1 and t=2. This means that the other two bonds will need to generate an additional 50,000 – 3,271.03 = 46.728.97 on each of those two dates.

At t=2, the 2-year bond must provide the additional 46,728.97, so its face amount must be $46,728.97/1.07 = 43,671.94$.

By similar reasoning, at t=1 the 1-year bond must generate:
50,000 – 3,271.03 – 43,671.94(0.07) = 43,671.94
Since it is a zero-coupon bond, 43,671.94 is also its face amount. The problem asks for the cost (or price) of this bond, which is $43,671.94/1.06 = 41,199.94$.

Answer: B

30. The guaranteed interest rate is the four year par coupon bond rate:

$$R = \frac{1-P_4}{P_1+P_2+P_3+P_4} = \frac{1-0.935}{0.984+0.969+0.953+0.935} = 0.0169$$

Answer: D

31. First calculate the future value of Andy's payments.
$200s_{\overline{48}|0.06/12} = 10,819.5664$

Now, set this equal to the future value of Scott's payments to solve for the interest rate:

N=16, PMT=-500, FV=10,819.5664. CPT I/Y=3.8934%

To solve for the interest Scott earned during the last year, we can use the previous values on the calculator and just change N=12, and CPT FV.

FV = 7,467.1173

Scott's investment increased from 7,467.1173. to 10,819.5664 during the last year, and he deposited 500*4 = 2,000 during that year. The difference is the interest earned: 10,819.5664 – 7,467.1173 – 2,000 = 1,352.4491.

Answer: D

32. We can think of this stream of payments as a level perpetuity of *X*, plus an 18-year-deferred unit increasing perpetuity. The total value of the payments is 3,000, so we have:

$$X \cdot a_{\overline{\infty}|} + v^{18} \cdot (Ia)_{\overline{\infty}|} = \frac{X}{i} + \frac{v^{18}}{d \cdot i} = \frac{X}{0.03} + \frac{1.03^{-18}}{(0.03/1.03) \cdot 0.03} = 3{,}000$$

$$X = 0.03 \cdot \left(3{,}000 - \frac{1.03^{-18}}{(0.03/1.03) \cdot 0.03}\right) = 69.83$$

Each of the 1st through 18th payments is 69.83. The 19th is 1 larger, so it is about 71, and the amount of the 30th payment is 69.83 + 12, or about 82.

Answer: C

33. We start with the 3-year bond. At time 3, the company will receive 1,000 + 60 for each of bond C purchased. They need 15,000 at time 3, so we have:

$$\frac{15{,}000}{1{,}060} = 14.1509 = \text{the number of bond C that should be purchased}$$

At time 2, the company will receive 1,040 for each of bond B purchased. They will also receive 14.1509(60) = 849.06 from the coupons of bond C. They need a total of 10,000, so an additional 9,150.946 is needed at time 2.

$$\frac{9{,}150.946}{1{,}040} = 8.7990 = \text{the number of bond B that should be purchased}$$

At time 1, the company will receive 1,070 for each of bond A purchased. They will also receive 14.1509(60) from the coupons of bond C and 8.7990(40) from the coupons of bond B. They originally needed *X*, but subtracting the coupons from bonds C and B, they need another *X* – 1,201.0160 at time 1.

$$\frac{X - 1{,}201.0160}{1{,}070} = Y\text{, where Y stands for the number of bond A purchased}$$

The price of Bond A is 1,070/1.05 = 1,019.0476.
The prices for bonds B and C are 1,019.1347 and 1,027.2325.
(N=2, I/Y=3, PMT=40, and FV=1,000. CPT PV=-1,019.1347.
N=3, I/Y=5, PMT=60, and FV=1,000. CPT PV=-1,027,2325.)

The total cost of the bonds is given as 25,122, so we have:

$$25{,}122 = 14.1509(1{,}027.2325) + 8.7990(1{,}019.1347) + Y(1{,}019.0476)$$
$$Y = 1.5881$$

This means that 1.5881 of bond A were purchased. Now we can solve for X:

$$\frac{X - 1{,}201.0160}{1{,}070} = 1.5881 \qquad X = 2{,}900.2745$$

Answer: C

34. A continuously compounded rate of return of 6% would provide the investor a repayment amount of $1{,}000 \cdot e^{2 \cdot (0.06)} = 1{,}127.50$.

The 6.5% loan specifies a repayment amount of $1{,}000 \cdot e^{2 \cdot (0.065)} = 1{,}138.83$.

In order for the investor to have an expected repayment amount of 1,127.50, the default rate, D, would be:

$$D = 1 - 1{,}127.50 / 1{,}138.83 = 0.00995$$

Answer: D

35. The fixed rate for this swap is:

$$R = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 1.047^{-3}}{1.030^{-1} + 1.040^{-2} + 1.047^{-3}} = 0.04652$$

The net settlement payment at the end of the second year equals the notional amount times the difference between the spot rate at the beginning of that year (5.2%) and the swap rate (4.652%). The amount is:

$$1{,}000{,}000 \cdot (0.052 - 0.04652) = 1{,}000{,}000 \cdot 0.00548 = 5{,}480$$

The speculator, as payer, pays 4.652% and receives 5.2%, so the amount to be paid is negative: -5,480.

Answer: C

Practice Exam 3

Exam FM

Questions

1. Two annuities-immediate have the same interest rate and the same level payment amount. The first is a 30-year annuity and the second is a 15-year-deferred 15-year annuity. The present value of the first annuity is 4 times the present value of the second. Find the interest rate.

A) 7.3% B) 7.4% C) 7.5% D) 7.6% E) 7.7%

2. A 10-year 1,000 par bond has 6% semi-annual coupons. The bond is sold at a premium of 35. What is the bond's nominal annual yield convertible semi-annually?

A) 5.48% B) 5.54% C) 5.62% D) 5.71% E) 5.79%

3. A company has liability payments of 2,000 and 5,000 due at the end of years one and three respectively. The investments available to the company are two zero-coupon bonds. The first is a one-year 1,000 par value bond with an annual effective yield of 5.6%. The second is a three-year 1,000 par bond. If the cost of exactly matching the liability cash flows is 6,068.36, what is the annual effective yield of the second bond?

A) 5.2% B) 5.5% C) 5.8% D) 6.0% E) 6.2%

4. An investment pays 2,000 at the end of year one, 4,000 at the end of year 3, and 6,000 at the end of year 5. It was purchased to yield an annual effective rate of 6.2%. Find the Macaulay duration of this investment.

A) 3.28 B) 3.44 C) 3.49 D) 3.53 E) 3.56

5. A 10-year 1,000 par bond with 6.5% semi-annual coupons is priced to yield an annual rate of j convertible semi-annually. The amount of premium amortized in period 7 is 2.346, and the amount amortized in period 12 is 2.706. Find j.

A) 5.8% B) 6.0% C) 6.2% D) 6.4% E) 6.6%

6. Money is deposited in a bank. For the first 4 years interest accumulates at a nominal annual rate of 6% convertible monthly. For the next 6 years it accumulates at a force of interest of 5%. For the 10-year period what is the equivalent nominal rate of discount convertible quarterly?

A) 4.9% B) 5.2% C) 5.4% D) 5.7% E) 5.9%

7. A man planning to work for the next 30 years sets up a retirement account by making monthly end-of-month payments into a fund. The first payment is 100 and each subsequent payment is 1 more than the previous one. The fund earns a nominal rate of 7.2% convertible monthly. At the end of the 30 years he plans to make end-of-month withdrawals of 2,000 per month. If interest rates stay the same, how many withdrawals will he be able to make (including a final withdrawal of less than 2,000)?

A) 292 B) 302 C) 312 D) 322 E) 332

8. For a given yield curve the implied forward rates are $i_{0,1} = 0.030$ and $i_{1,2} = 0.032$. The spot rate $s_3 = 0.04$. Find $i_{2,3}$.

A) 0.0542 B) 0.0547 C) 0.0553 D) 0.0561 E) 0.0582

9. Consider the following account summary:

Date	Balance Before Activity	Deposits	Withdrawals
January 1	10,000		
April 1	10,500	2,000	
September 1	12,800		2,600
December 31	X		

If the time-weighted yield is 6.466%, what is the dollar-weighted yield?

A) 6.58% B) 6.62% C) 6.65% D) 6.71% E) 6.74%

10. A man buys a home by borrowing 200,000 under a 30-year mortgage with monthly payments and an interest rate of 5.4% convertible monthly. At the end of 15 years he decides to add 500 a month to each subsequent payment. Assuming there are no penalties, how many more payments, including the final partial payment, must he make?

A) 102 B) 108 C) 111 D) 115 E) 120

11. A woman buys a 10-year 1,000 par bond with 7.0% semi-annual coupons priced at 1,000. The coupon payments are deposited into a bank account that pays 6.6% convertible semi-annually. After the 10^{th} deposit the bank drops its rate to 5.8% convertible semi-annually. At the end of the 10-year period, what is her annual effective yield for this investment?

A) 6.5% B) 6.7% C) 6.9% D) 7.1% E) 7.3%

12. A man has a 30-year loan with level annual end-of-year payments. The principal repaid in the 10^{th} payment is 408.12, and the principal in the 20^{th} payment is 766.10. What amount of principal is repaid in the 15^{th} payment?

A) 540.33 B) 544.02 C) 548.65 D) 552.25 E) 559.16

13. Tom has a 10-year increasing annuity-immediate that pays 100 the first year, and the payments increase by 100 each year thereafter. Jerry has a 10-year decreasing annuity-immediate that pays X the first year and payments decrease by $X/10$ each year thereafter. At an annual effective interest rate of 6.5%, the two annuities have the same present value. Find X.

A) 821 B) 828 C) 835 D) 842 E) 849

14. Sally and Linus each make annual end-of-year deposits into savings accounts that have the same annual interest rate. Sally's annual deposits are 100. Linus deposits 100 per year for the first 10 years and 200 per year thereafter. At the end of 20 years Linus has accumulated 4/3 the amount that Sally has. What is their common, non-zero, interest rate?

A) 6.0% B) 6.5% C) 6.9% D) 7.2% E) 7.5%

15. You are given the following yield curve:

Year	Spot Rate
1	4.5%
2	4.0%
3	3.8%
4	3.6%

A 3-year 1,000 par bond has a 5% annual coupon rate. Use the spot rates in the yield curve to find the price of the bond.

A) 1,033 B) 1,038 C) 1,042 D) 1,046 E) 1,051

16. A 3-year 1,000 par bond with 5.8% annual coupons is priced to yield 6.4%. What is the bond's Macaulay duration?

A) 2.795 B) 2.801 C) 2.837 D) 2.862 E) 2.890

17. A man invests 1,000 at the beginning of each year into a fund that pays an annual interest rate of 5.6%. The annual interest payments are deposited into a second fund that pays 6.2% annually. What is his total accumulation at the end of 10 years?

A) 13,261 B) 13,585 C) 13,730 D) 13,805 E) 13,850

18. A perpetuity-immediate pays 100 a year for the first 10 years. Starting with year 11, each payment is 3% more than the previous one. At an annual effective rate of 4.5%, what is the present value of this perpetuity?

A) 5,213 B) 5,324 C) 5,375 D) 5,431 E) 5,486

19. Lucy deposits 1,000 into an account and makes an additional deposit of 2,000 two years later. The account accumulates at a constant force of interest. At the end of 4 years the accumulation is 3,431.75. Find the force of interest.

A) 0.035 B) 0.040 C) 0.045 D) 0.050 E) 0.055

20. A man takes out a thirty-year mortgage loan for 300,000. The loan interest rate is 6% convertible monthly. He also owns a fifteen-year zero-coupon bond which will mature for 100,000 on the same day as he makes the last payment of the fifteenth year of the mortgage. He plans to make larger monthly payments during the first fifteen years of the mortgage so that at the end of year fifteen he can use the 100,000 from the bond to retire the loan. What should his monthly payment for the first fifteen years be?

A) 1,798.65 B) 1,995.83 C) 2,187.71 D) 2,297.81 E) 2,798.65

21. Sue Pine invests 600 in an account that accumulates at an annual effective interest rate equal to x. After 4 years, Sue withdraws the entire balance of this account and re-deposits it into an account that accumulates at an annual effective rate of <u>discount</u> equal to x.

Eight years later (at t=12), the balance in Sue's account is 1,000.
What is the value of x?

A) 3.9% B) 4.0% C) 4.1% D) 4.2% E) 4.3%

22. A monthly-payment mortgage has an initial principal amount of 200,000, an initial term of 20 years, and an initial interest rate of 6.6% per annum (convertible monthly).

Immediately after the borrower makes the 100th monthly payment on this loan, the loan is refinanced at an interest rate of 5.4% convertible monthly. The new term of the loan is 15 years (beginning on the date of refinancing).

The borrower receives a payment of X from the new lender on the date of refinancing, which increases the outstanding balance of the loan. As a result, the monthly mortgage payment remains unchanged.

What is the amount of X?

A) 37,820 B) 37,990 C) 38,210 D) 38,450 E) 38,670

23. A newly-issued 20-year bond with a face amount of 1,000 pays semi-annual coupons at a rate of 6% per year. The bond is purchased to yield 4%, compounded semi-annually.

What is the amount of premium that is amortized in the 32nd coupon payment?

A) 8.01 B) 8.20 C) 8.37 D) 8.53 E) 8.87

24. Consider an investment portfolio consisting of two zero-coupon bonds:

- a 4-year zero-coupon bond with maturity value 4,000 and a yield of 4%
- a 7-year zero-coupon bond with maturity value 7,000 and a yield of 7%

What is the modified duration of this 2-bond portfolio?
(All of the interest rates in this problem are annual effective rates.)

A) 5.31 B) 5.36 C) 5.46 D) 5.56 E) 5.91

25. How many of the following factors have an influence on the interest rate charged for a loan?

- the lender's deferred consumption
- the risk that the borrower will default
- the collateral or guarantee protecting the loan
- the expected rate of inflation
- the term of the loan

A) 1 B) 2 C) 3 D) 4 E) All 5

26. A nominal interest rate convertible semi-annually, $i^{(2)}$, is equivalent to an annual effective interest rate of i. That is, the two rates represent the same rate of growth. Numerically, $i^{(2)} = 0.98\, i$.
Given that $i \neq 0$, what is the value of i?

A) 7.75% B) 8.00% C) 8.33% D) 8.67% E) 9.01%

27. A 20-year bond with a face amount (and maturity value) of 1,000 pays semi-annual coupons. The Bondo Corporation purchased this bond on its issue date at a price such that its yield to maturity is 5.2% (a nominal rate, convertible semi-annually).

The Bondo Corporation adjusts the bond's book value at each coupon date to maintain a constant rate of return over the life of the bond. During the bond's 5th year, its book value is increased by a total of 5.35.

If the Bondo Corporation sells the bond at par (i.e., for its face amount) on the bond's 10th anniversary, what yield, convertible semi-annually, will Bondo have earned during the 10 years that it owned the bond?

A) 2.4% B) 2.6% C) 2.8% D) 3.0% E) Another Value

28. An insurance company is examining its liabilities and the assets that support them to determine whether it is immunized against a small parallel shift in interest rates.

The company finds that its assets and liabilities have the following characteristics:

	Modified Duration	Modified Convexity
Assets	6	40
Liabilities	5.5	42

Assuming the assets have the same present value as the liabilities, which of the following adjustments to the company's asset portfolio will be required to achieve Redington immunization?

A) No changes are required.

B) Keep duration greater than 5.5; increase convexity to 42.

C) Decrease duration to 5.5; increase convexity to 42.

D) Decrease duration to 5.5; increase convexity to more than 42.

E) Decrease duration to less than 5.5; increase convexity to more than 42.

For Problems 29. and 30., use the following table of zero-coupon bond prices (per dollar of face amount):

Quarters	1	2	3
Zero-coupon bond price	0.985	0.971	0.954

29. The guaranteed rate for a 4-quarter interest rate swap with a level notional amount is 1.74%. Find the zero-coupon bond price for the fourth quarter.

A) 0.929 B) 0.933 C) 0.935 D) 0.937 E) 0.939

30. What is the swap rate for a 2-quarter interest rate swap that is deferred one quarter? (The swap has a level notional amount and applies to the 2^{nd} and 3^{rd} quarters only.)

A) 1.61% B) 1.63% C) 1.65% D) 1.67% E) 1.69%

31. A project requires initial funding of 2,500 and will return the following amounts at the end of each of the next 6 years: 1,000; 700; 2*X*; *X*; *X*; and 100. The project's internal rate of return is 4.9%.

Calculate the project's net present value at an annual effective interest rate of 10%.

A) 443 B) 255 C) 250 D) -255 E) -250

32. The following equation represents the yield curve of spot rates for a term of *k* years.

$$s_k = 0.08 + 0.003k - 0.003k^2$$

A four-year annuity-due has annual payments of 300. Using the yield curve, calculate the present value of this annuity

A) 1,005 B) 1,041 C) 1,088 D) 1,123 E) 1,140

33. Fifteen annual payments are made to an account earning an annual effective rate of 3%. The first payment is 200 and each subsequent payment increases by 5%.

How much larger is the present value if the payments are made at the beginning of the year rather than at the end of the year?

A) 100 B) 117 C) 212 D) 334 E) 344

34. The current price of a stock is 30. It pays an annual dividend of 5 that is expected to remain level in the future. The next dividend is payable 6 months from today.

If an investor using the dividend discount model considers this stock to be underpriced by 1.25, what valuation interest rate is that investor using?

A) 16.0% B) 16.3% C) 16.7% D) 17.0% E) 17.3%

35. Which theory of interest rates predicts that longer-term loans will have higher interest rates?

A) The market segmentation theory
B) The liquidity preference theory
C) The expectations theory
D) The preferred habitat theory
E) None of A, B, C, or D

Solutions

1. We may assume the payments are 1. Relating the present values we get:
$(1 - v^{30})/i = 4v^{15}(1 - v^{15})/i$.
Dividing by $(1 - v^{15})/i$ yields:
$1 + v^{15} = 4v^{15} \Rightarrow (1 + i)^{15} = 3 \Rightarrow i = 0.076$

Answer: D

2. The price of this bond is 1,035. To get the yield using the BA II Plus calculator, set N = 20, PV = -1,035, PMT = 30, and FV = 1,000. CPT I/Y = 2.770. The semi-annual effective rate is 2.77%. The yield is 5.54% convertible semi-annually.

Answer: B

3. We have $2,000/1.056 + 5,000/(1 + i)^3 = 6,068.36$, where i is the unknown yield.
$(1 + i)^3 = 5,000/(6,068.36 - 2,000/1.056) = 1.1978 \Rightarrow 1 + i = 1.062 \Rightarrow i = 0.062$

Answer: E

4. The present values of the payment is:
$2,000/1.062 + 4,000/1.062^3 + 6,000/1.062^5$
$= 1,883.24 + 3,339.54 + 4,441.49 = 9,664.27$

The weights for the Macaulay duration are
$w_1 = 1,883.24/9,664.27 = 0.1949$
$w_2 = 3,339.54/9,664.27 = 0.3456$
$w_3 = 4,441.49/9,664.27 = 0.4596$
$D_{mac} = 0.1949(1) + 0.3456(3) + 0.4596(5) = 3.5297$

Answer: D

5. Much of the information included in this problem is not needed (the bond's term, face amount, and yield). The only important numbers are the amounts of premium that were amortized in period 7 and period 12.

Since these two amounts occur 5 periods apart, their ratio is $(1+j/2)^5$. (We use $j/2$ as the rate per period because j is convertible semi-annually.)

Thus: $2.346\cdot(1+j/2)^5 = 2.706$ $\quad j = 2\cdot\left[(2.706/2.346)^{1/5}-1\right] = 0.05793$

Answer: A

6. The accumulation factor for the 10-year period is $(1.005)^{48}e^{0.05(6)} = 1.7150$.
Then $(1 - d^{(4)}/4)^{-40} = 1.7150$, and $1 - d^{(4)}/4 = 1.7150^{-1/40} = 0.9866$
So $d^{(4)} = 4\ (1\text{-}0.9866) = 0.05358$

Answer: C

7. The total accumulation is given by the future value version of the *PQ* formula (Formula (2.63) in Module 2 of this study guide).

$PV = 100\, s_{\overline{360}|0.6\%} + (s_{\overline{360}|0.6\%} - 360)/i$

$= 100(1{,}269.225) + (1{,}269.225 - 360)/0.006$
$= 126{,}922.50 + 151{,}537.50 = 278{,}460$

For the withdrawals, use the BA II Plus calculator and set I/Y = 0.6, PV = 278,460, PMT = -2,000, and FV = 0. Then CPT N = 301.59.

Answer: B

8. $(1 + s_3)^3 = (1 + i_{0,1})(1 + i_{1,2})(1 + i_{2,3})$
$(1 + i_{2,3}) = (1.04)^3/[(1.03)(1.032)] = 1.0582$

Answer: E

9. Using the time-weighted yield to get X we have
$(10{,}500/10{,}000)(12{,}800/12{,}500)(X/10{,}200) = 1.06466$
$X = 10{,}100$

The amount of interest earned is found from
$10{,}000 + 2{,}000 - 2{,}600 + I = 10{,}100 \quad \Rightarrow \quad I = 700$

For the dollar-weighted yield, we have:
$j = 700/[10{,}000 + (9/12)2{,}000 - (4/12)2{,}600] = 0.0658$

Answer: A

10. First find the initial monthly payments using the calculator.
Set N = 360, I/Y = 0.45, PV = -200,000, and FV =0. Then CPT PMT = 1,123.06.

Reset N = 180 (the number of remaining payments after 15 years.
Then CPT PV = -138,344.43. This is the balance after 15 years.
Reset PMT = 1,623.06 and CPT N = 107.75

Answer: B

11. The accumulation of deposits is:
$35\, s_{\overline{10}|0.033}\ (1.029)^{10} + 35\, s_{\overline{10}|0.029} = (406.82)(1.3309) + 399.39 = 940.83$

Including the bond's maturity value, she has 1,940.83 at time 10.
The annual effective yield is $(1{,}940.83/1{,}000)^{1/10} - 1 = 0.069$

Answer: C

12. We know the amount of principal repaid in the 10^{th} and 20^{th} payments, so we can find the loan interest rate based on the fact that the principal repaid increases by a factor of $(1+i)$ each payment period:

$$1 + i = \left(\frac{766.10}{408.12}\right)^{\frac{1}{10}} = 1.065 \qquad\qquad i = 0.065$$

The principal repaid in the 15^{th} payment (5 years after the 10^{th} payment) is:

$$408.12 \cdot 1.065^5 = 559.16$$

Answer: E

13. The present value of Tom's annuity is:
$100(Ia)_{\overline{10|}} = 100[(\ddot{a}_{\overline{10|}} - 10v^{10})/i] = 3{,}582.84$

The present value of Jerry's annuity is
$(X/10)(Da)_{\overline{10|}} = (X/10)[(10 - a_{\overline{10|}})/i] = 4.325X$

$X = 3{,}582.84/4.325 = 828.40$

Answer: B

14. Linus made the same annual deposits of 100 as Sally, plus an additional 100 per year in years 11-20. His balance at time 20 was 1/3 greater than Sally's, so we can conclude that the accumulated value of those additional amounts $(100s_{\overline{10|}})$ has a value equal to 1/3 of the value of level deposits of 100 for 20 years $(100s_{\overline{20|}})$:

$$(100s_{\overline{10|}}) = (100s_{\overline{20|}})/3$$
$$3 \cdot \frac{(1+i)^{10}-1}{i} = \frac{(1+i)^{20}-1}{i}$$
$$3 = (1+i)^{10} + 1$$
$$i = (3-1)^{1/10} - 1 = 0.07177$$

(In the next-to-last step, both sides were divided by $\left((1+i)^{10}-1\right)/i$, and we used the fact that $\left((1+i)^{10}+1\right)\cdot\left((1+i)^{10}-1\right) = \left((1+i)^{20}-1\right)$.)

Answer: D

15. The price of the bond is
$P = 50/1.045 + 50/1.040^2 + 1{,}050/1.038^3 = 1{,}032.93$

Answer: A

16. To get the price of the bond using the calculator,
set N = 3, I/Y = 6.4, PMT = 58 and FV = 1,000.
Then CPT PV = -984.08. The price is 984.08

The Macaulay duration is
$D_{mac} = [(58/1.064) + (58)(2)/1.064^2 + (1{,}058)(3)/1.064^3]/984.08$
$= 2.837$

Answer: C

17. The amount of interest earned in the first fund during year k is:

$$1{,}000k(0.056) = 56k$$

The accumulation of these payments with interest in the second fund is:

$$56(Is)_{\overline{10}|} = 3{,}730.48.$$

The total accumulation is 13,730.48.

Answer: C

18. This can be visualized as a 10-year annuity immediate plus a 10-year-deferred geometric perpetuity-immediate.

The present value is:
$100a_{\overline{10}|} + (1.045)^{-10}[103/(0.045 - 0.03)] = 791.27 + (0.6439)6{,}866.67 = 5{,}212.72$

Answer: A

19. The accumulation is $1{,}000e^{4\delta} + 2{,}000e^{2\delta} = 3{,}431.75$.
Let $x = e^{2\delta}$. Then we have
$1{,}000x^2 + 2{,}000x - 3{,}431.75 = 0$
The positive root of this equation is $x = 1.1052$.
$e^{2\delta} = x = 1.1052 \qquad \delta = (\ln 1.1052)/2 = 0.05$

Answer: D

20. The man wants to make a monthly payment that will leave a balance of 100,000 at the end of fifteen years so that he can use the 100,000 maturity value of the bond to retire the loan. Using the BA II Plus to represent this situation, we have:

N=180, I/Y=0.5, PV=300,000, and FV=-100,000. CPT PMT = -2,187.71.

(*Note that FV is <u>negative</u> 100,000 to reflect that this is an amount the borrower has to <u>pay</u>, just as PMT is negative for the same reason.)*

The payment is 2,187.71.

(Note: The normal monthly payment would be 1,798.65. (N=360, I/Y=0.5, PV=300,000, and FV=0. CPT PMT = -1,798.65) This is Answer Choice A, but it is not a value we need in order to solve the problem.)

Answer: C

21. The equation of value (using a valuation date of t=12) is:

$$600 \cdot (1+x)^4 (1-x)^{-8} = 1{,}000$$

Applying algebra, we have:

$$\left[(1+x)^4 (1-x)^{-8}\right]^{1/4} = (1{,}000/600)^{1/4} = 1.13622$$

$$(1+x)/(1-x)^2 = 1.13622$$

$$1 + x = 1.13622 \cdot (x^2 - 2x + 1)$$

$$1.13622x^2 - 3.27244x + 0.13622 = 0$$

$$x = 0.04225 \text{ or } 2.8379$$

The solution of 0.04225 (=4.225%) is reasonable. The other solution (283.79%) is not. While i can theoretically take on a value of 2.8379, d cannot be greater than 1.

Answer: D

22. This problem is designed for solution using a financial calculator.
First find the initial payment for the mortgage:
N=240, I/Y=0.55, PV=200,000, and FV=0. CPT PMT = -1,502.94.

Next find the outstanding balance after 100 payments:
N=140. CPT PV = 146,471.21 (the present value of the remaining 140 pmts.)
Put this value into a calculator memory (e.g., STO 4).

Then calculate the loan amount that can be repaid by continuing to make payments at the same level (1,502.94) for another 15 years at the new interest rate of 5.4% (a monthly effective rate of 0.45%):
N=180, I/Y=0.45. CPT PV = 185,140.30

Finally calculate the difference between the outstanding balance after 100 payments (146,471.21) and the amount that will be repaid over the next 15 years. That difference is the X, the amount by which the loan balance was increased when the borrower refinanced the loan.

$$\text{X} = 185{,}140.30 - 146{,}471.21 = 38{,}669.09$$

Answer: E

23. There are many possible approaches for solving this problem.
One method is to use the formula for amortization of bond premium or discount: $F(r-i)v^{n-k+1}$. For this problem, we have F=1,000, r=0.03 (half of the 6% annual coupon rate), i=0.02, n=40, and k=32.

$$F(r-i)v^{n-k+1} = 1{,}000(.03-0.02)1.02^{-(40-32+1)} = 8.37$$

Another approach is to calculate the value of the bond after 31 payments and after 32 payments. (The difference is the amount of premium amortized in the 32nd payment.)

A third approach is to calculate the price of the bond using the financial calculator's TVM worksheet, then use the Amortization worksheet (with P1 and P2 both equal to 32) to find the amount of premium (Principal, PRN) amortized in the 32nd payment.

Another method is to calculate the initial price of the bond, use that to calculate the interest earned during the <u>first</u> coupon period and the amount of premium amortized with the first coupon payment, and then increase that amortization amount by the 31-period accumulation factor. This last method is demonstrated here:

N=40, I/Y=2, PMT=30, FV=1,000. CPT PV = -1.273.55

With an initial value of 1,273.55, the bond earns interest of 25.4711 during the first coupon period. The coupon payment of 30 is 4.5289 greater than the interest earned, so it repays 4.5289 of principal. The 32nd coupon payment occurs 31 periods later, and it repays $4.5289 \cdot 1.02^{31} = 8.3676$ of principal.

Answer: C

24. The modified duration of the portfolio equals a weighted average of the modified durations of the assets in the portfolio, using the values of the assets as the weights. Therefore, we need to find the modified duration and value of each of the bonds.

The first bond has a value of $4{,}000 / 1.04^4 = 3{,}419.22$ and a modified duration of $4 / 1.04 = 3.846$.
The second bond has a value of $7{,}000 / 1.07^7 = 4{,}359.25$ and a modified duration of $7 / 1.07 = 6.542$.

The modified duration of this 2-bond portfolio is:

$$\frac{3{,}419.22 \cdot 3.846 + 4{,}359.25 \cdot 6.542}{3{,}419.22 + 4{,}359.25} = 5.357$$

Answer: B

25. All 5 of the factors affect the interest rate charged.
The lender's willingness to **defer consumption** is the underlying driver of an interest rate.

The **risk of default** affects the likelihood that the lender will be repaid. The loan rate has to be large enough to cover the lender's default losses from those borrowers who default.

Collateral (e.g., a house or a car) or a guarantee (e.g., by the government) decreases the lender's expected default losses and makes a lower interest rate possible.

Inflation reduces the value of the repayments the lender will receive, so higher inflation (actual or expected) results in higher interest rates.

Lenders charge higher interest rates for **longer-term** loans, perhaps because of the loss of liquidity and the loss of opportunities to invest in other opportunities during the term of the loan. Borrowers are likely willing to pay higher rates for longer-term loans because it provides them security that they will have the funds for a longer period.

Answer: E

26. This is essentially an algebra problem:

$$\left(1+\frac{i^{(2)}}{2}\right)^2 = \left(1+\frac{0.98i}{2}\right)^2 = 1+i$$

$$1+0.98i+0.49^2\cdot i^2 = 1+i$$

$$0.2401\cdot i^2 = 0.02i$$

$$i = \frac{0.02}{0.2401} = 0.0833$$

Answer: C

27. If a bond is purchased at a premium or a discount to its maturity value, the difference between the purchase price and the maturity value is eliminated over the remaining life of the bond by a series of adjustments to the bond's book value. ("Book value" is the value of the bond asset as it appears on the bond owner's balance sheet). The amount by which the bond's book value is adjusted increases on each successive coupon date. The adjustment (whether it is positive or negative) increases by a factor of $1+i$ from one period to the next, where i is the effective interest rate per coupon period.

In this problem, we know that the bond's book value increased by 5.35 during its 5th year (which includes the bond's 9th and 10th coupon payments), and that the effective interest rate per coupon period is 2.6%. Letting Adj_9 and Adj_{10} represent the book value adjustments on these dates, we have:
$Adj_9 + Adj_{10} = 5.35$ and $Adj_9 + 1.026 \cdot Adj_9 = 5.35$. So $Adj_9 = 2.64$

The adjustment in book value on each coupon date represents the difference between the coupon payment the bondholder received and the amount of interest the bondholder earned during the coupon period, based on the bond's yield. In this problem, the bond earned 2.64 more interest during the 9th coupon period than the amount of the coupon.

We can use the adjustment of 2.64 on the 9th coupon date to calculate the book value adjustment on any other coupon date. In particular, we can calculate what the adjustment would <u>theoretically</u> be on the 41st coupon date (the next coupon date after the bond matures) if the bond could continue beyond its maturity date. This is significant, because we have the information to calculate the amount of interest earned during the 41st coupon period.
Interest earned$_{41}$ $= BV_{40} \cdot \text{yield} = 1{,}000 \cdot 0.026 = 26.00$
Book Value adjustment$_{41}$ $= 2.64 \cdot 1.026^{(41-9)} = 6.00$
The amount of interest earned would be 26.00, and the book value adjustment would be 6.00, so the amount of the semi-annual coupon must be 26 – 6 = 20, which is 2% of the bond's face amount.

Now we know that the corporation bought a 20-year bond that pays semi-annual coupons at a 4% annual rate, and the price was such that the bond's yield to maturity is 5.2%. The price was, therefore: $20 \cdot a_{\overline{40}|2.6\%} + 1{,}000 \cdot 1.026^{-40} = 851.89$
The problem asks what yield the corporation will earn if it sells the bond for its face amount at the end of 10 years. We can find the yield using the BA II Plus: Set N=20, PV=-851.89, PMT=20, and FV=1,000. CPT I/Y = 2.995.
The semi-annual effective rate is 2.995%, so the bond's yield $\left(i^{(2)}\right)$ is 5.990%.

Answer: E (another value)

28. In order for the portfolio to be immunized, the following 3 conditions must be met:

1. PV(assets) = PV(liabilities)
2. Duration(assets) = Duration(liabilities)
3. Convexity(assets) > Convexity (liabilities)

The problem states that the first condition is met.
To meet Condition 2, the assets' duration must be decreased to exactly 5.5.
To meet Condition 3, the assets' convexity must increase to more than 42.

Answer: D

29. The formula for the fixed rate in this swap is:

$$R = \frac{1 - P_4}{\sum_{t=1}^{4} P_t}$$

In this formula, the P's are present value factors, and the subscripts measure time in quarter-years.

Since the zero-coupon bond prices (per dollar of face) are present value factors, we have all the values we need to calculate a swap rate for a 4-quarter swap, with the exception of P_4. However, the problem gives us the value of R, so we can solve for P_4:

$$0.0174 = \frac{1 - P_4}{0.985 + 0.971 + 0.954 + P_4}$$

$$0.0174 \cdot (0.985 + 0.971 + 0.954) + 0.0174 \cdot P_4 = 1 - P_4$$

$$1.0174 \cdot P_4 = 1 - 0.0174 \cdot (0.985 + 0.971 + 0.954) = 0.949366$$

$$P_4 = 0.949366 / 1.0174 = 0.93313$$

P_4 is the 4-quarter present value factor, and also the 4-quarter (1-year) zero-coupon bond price.

Answer: B

30. Because this deferred swap has a level notional amount, we can use the following formula for the swap rate:

$$R = \frac{P_1 - P_3}{\sum_{t=2}^{3} P_t}$$

Using the zero-coupon bond prices from the table as present value factors, we have:

$$R = \frac{0.985 - 0.954}{0.971 + 0.954} = \frac{0.031}{1.925} = 0.0161$$

Answer: A

31. First, calculate the value of X using the information given about the internal rate of return.

$$0 = -2500 + 1000v_{0.049} + 700v_{0.049}^2 + 2Xv_{0.049}^3 + Xv_{0.049}^4 + Xv_{0.049}^5 + 100v_{0.049}^6$$
$$835.5300 = X(2v_{0.049}^3 + v_{0.049}^4 + v_{0.049}^5)$$
$$X = 249.7301$$

Next, calculate the present value of the cash flows at i=0.10.

$$NPV = -2{,}500 + 1{,}000\left(1.1^{-1}\right) + 700\left(1.1^{-2}\right) + 2(249.73)\left(1.1^{-3}\right)$$
$$+249.73\left(1.1^{-4}\right) + 249.73\left(1.1^{-5}\right) + 100\left(1.1^{-6}\right)$$
$$NPV = -255.0660$$

Answer: D

32. Calculate the present value of each payment using the spot rate calculated from the given formula.

$$PV = 300 + \frac{300}{1+s_1} + \frac{300}{(1+s_2)^2} + \frac{300}{(1+s_3)^3}$$
$$= 300 + \frac{300}{1.08} + \frac{300}{1.074^2} + \frac{300}{1.062^3} = 1{,}088.3266$$

Answer: C

33. If the payments are made at the end of each year, then

$$PV = 200v + 200(1.05)v^2 + 200(1.05)^2 v^3 + ... + 200(1.05)^{14} v^{15}$$

$$= 200v \left[\frac{1 - \left(\frac{1.05}{1.03} \right)^{15}}{1 - \left(\frac{1.05}{1.03} \right)} \right] = 3,343.85$$

If the payments are made at the beginning of each year, then just multiply the previous PV by $1+i = 1.03$, resulting in a present value of 3,444.17.

Therefore, the present value if payments are made at the beginning of each year is 100.32 larger than if they are made at the end of each year.

Answer: A

34. Let i be the annual effective interest rate that the investor is using to discount future dividends. If the next dividend of amount 5 were due in one year, the calculated value of the stock would be $5/i$. Because the calculation is being done 6 months later (and the dividend is 6 months closer), the current value is $(1+i)^{1/2} \cdot 5/i$.

At a price of 30, the investor considers the stock to be underpriced by 1.25, indicating that the investor's calculation produces a value of 31.25. Thus we have an equation for i:

$$(1+i)^{1/2} \cdot 5/i = 31.25$$
$$(1+i) \cdot 25 = 31.25^2 \cdot i^2$$
$$976.5625i^2 - 25i - 25 = 0$$

Solving by the quadratic formula yields $i = 17.33\%$.
Note: If you have time during the exam, it is good to check your answer on a problem like this to see whether it satisfies the conditions of the problem:

$$1.1733^{0.5} \cdot 5/0.1733 = 31.25$$

Answer: E

35. All of the theories allow for the possibility that longer-term rates will be greater than short-term rates. However, only the "liquidity preference" theory predicts that longer-term rates will be higher.

The market segmentation and preferred habitat theories simply state that rates will vary by term based on supply and demand, and that different people will be supplying or demanding loans for different terms.

The expectations theory simply says that forward rates are an indicator of what future rates will be, but it does not require that longer-term rates be higher or lower than short-term rates.

The liquidity preference theory states that lenders must be compensated for giving up their liquidity, and must be compensated more for giving it up for a longer period. This implies that longer-term loans will have higher interest rates.

Answer: B

Practice Exam 4

Exam FM

Questions

1. If $\delta = 0.08$, what is the value of $i^{(6)} + d^{(4)}$?

A) 0.1591 B) 0.1597 C) 0.1608 D) 0.1615 E) 0.1621

2. A man has children aged 15, 18 and 20. He purchases an annuity for each one that pay 5,000 a year, with the first payment occurring now and payments continuing as long as the recipient is under age 30. The annuities are priced based on an annual effective interest rate of 5.5%. To the nearest 1,000, what is the total cost of these annuities?

A) 131,000 B) 135,000 C) 138,000 D) 142,000 E) 147,000

3. A company has liability payments of 5,000 and 2,000, due at the end of years 2 and 4 respectively. It purchases zero-coupon bonds maturing in 2 and 4 years, both earning the same interest rate. The cost of matching the liability payments exactly is 6,000. What is the interest rate for these bonds?

A) 5.4% B) 5.6% C) 5.8% D) 6.0% E) 6.2%

4. A man takes out a 30-year 6.4% monthly-payment mortgage for 150,000. After 12 years he refinances the mortgage at a new rate of 5.8% for a new term of 10 years. What are his new monthly payments?

A) 1,322 B) 1,330 C) 1,337 D) 1,342 E) 1,349

5. A woman buys a 20-year 1,000 par bond with 6% semi-annual coupons. The bond is priced to yield 5.6% convertible semi-annually. The coupon payments are deposited into a fund that earns 5% convertible semi-annually for the first 10 years and 5.4% convertible semi-annually for the last 10 years. What is her annual effective yield on this investment?

A) 5.2% B) 5.4% C) 5.6% D) 5.8% E) 6.0%

6. Given a 2-year spot rate of 0.046 and the forward rates $i_{2,3}$ = 0.037 and $i_{3,4}$ = 0.039, find the 4-year spot rate, s_4.

A) 0.036 B) 0.038 C) 0.040 D) 0.042 E) 0.044

7. A man purchases a 30-year annuity-immediate that makes annual payments. The first 10 payments are 200, the next 10 are 400 and the last 10 are 300. Based on a 6.5% annual effective rate, what is the annuity's present value?

A) 3,582 B) 3,617 C) 3,675 D) 3,713 E) 3,753

8. A man has a 30-year 6.6% home mortgage with monthly end-of-month payments of 766.39. What is the first period in which the principal repaid is more than 500?

A) 269 B) 274 C) 278 D) 281 E) 284

9. An annual-coupon corporate bond is priced to yield 6.7% annually at a price of 1,023.68. Its Macaulay duration is D_{mac} = 8.2135. Using the 1st-order Macaulay approximation method, estimate the change in price if interest rates decrease by 0.10%.

A) 7.880 B) 7.893 C) 7.914 D) 8.010 E) 8.018

10. A woman has an annual-payment 20-year annuity-immediate. The annuity pays 1,000 the first year. Subsequent payments decrease by 100 per year until they reach 100. The remaining payments stay at 100 per year. Find the present value of this annuity at an annual effective rate of 6.3%.

A) 4,695 B) 4,723 C) 4,749 D) 4,801 E) 4,862

11. A man borrows 50,000 for 10 years at a 7.6% annual effective rate. For the first 3 years he makes annual payments of only 3,000. What will his annual payments need to be for the final 7 years?

A) 9,876 B) 9,915 C) 9,963 D) 10,020 E) 10,088

12. An investor has a portfolio consisting of 20,000 worth of a 2-year bond with a modified duration of 1.92; 35,000 worth of a 3-year bond with a modified duration of 2.84; and 45,000 worth of a 5-year bond with a modified duration of 4.79. Find the modified duration for the entire portfolio.

A) 3.49 B) 3.53 C) 3.57 D) 3.61 E) 3.65

13. A woman invests 12,000. She is promised returns of 4,000 in 2 years, 6,000 in 3 years, and 8,000 in 4 years. Find the IRR for this investment.

A) 13.05% B) 13.27% C) 13.44% D) 13.59% E) 13.71%

14. A three-year 1,000 par value bond with 4.5% annual coupons is priced using the spot rates implied by the forward rates $i_{0,1} = 0.051$, $i_{1,2} = 0.047$ and $i_{2,3} = 0.043$. Find the price of the bond.

A) 964 B) 974 C) 984 D) 994 E) 1,004

15. A woman deposits 1,000 into a savings account. For the first 5 years the money accumulates with a force of interest of $\delta = 0.04$. For the next 3 years the money accumulates at a nominal discount rate of 0.06 convertible semi-annually. At the end of 10 years the money has accumulated at a rate equivalent to a level annual effective rate of 5.2%. What was the nominal annual interest rate convertible quarterly for the last 2 years?

A) 5.5% B) 5.7% C) 5.9% D) 6.1% E) 6.3%

16. You begin the year with 8,000 in an account. You make deposits of 2,000 on March 1 and 1,000 on November 1. You withdraw 500 on July 1. Your dollar-weighted yield for the year is 8.87%. How much interest did you earn?

A) 850 B) 861 C) 869 D) 873 E) 882

17. A 20-year annual-payment annuity-immediate pays 100 the first year and payments increase by 100 each year through year 10. Starting in year 11 each yearly payment is 5% greater than the previous payment. At a 6.8% annual effective rate, what is the present value of this annuity?

A) 8,175 B) 8,240 C) 8,290 D) 8,344 E) 8,395

18. A 3-year 1,000 par bond with 4.8% annual coupons is priced to yield 5.5%. What is the Macaulay duration of the bond?

A) 2.836 B) 2.844 C) 2.851 D) 2.863 E) 2.875

19. A man borrows 65,000 for 20 years and makes annual interest payments to the lender. He also makes annual contributions to a sinking fund to accumulate 65,000 to repay the principal. He makes sinking fund deposits of X for the first 10 years and $2X$ for the last 10 years. The sinking fund earns a 6.5% annual effective rate. Find X.

A) 1,232 B) 1,237 C) 1,242 D) 1,247 E) 1,252

20. You are given the yield curve $s_k = 0.068 + 0.002k - 0.001k^2$. Find the 3-year forward rate implied by this yield curve.

A) 4.3% B) 4.5% C) 4.9% D) 5.5% E) 5.9%

21. A 20-year bond with semi-annual coupons and a 1,000 par value (and 1,000 redemption value) is purchased on its issue date at a discount such that its yield to maturity is 6.4% convertible semi-annually.

The corporation owning the bond adjusts its book value on each coupon date based on the bond's 6.4% yield. During the 10^{th} year, the book value is adjusted by 6.10. What is the bond's annual coupon rate?

A) 4.9% B) 5.0% C) 5.1% D) 5.2% E) 5.3%

22. Account A earns simple interest at an annual rate x (where x is greater than 0). Account B earns compound interest based on an annual effective rate of discount equal to x.

1,000 is deposited into each of these accounts at t=0. No other deposits or withdrawals occur. At t=1, the balance in one account is 0.5% larger than the balance in the other account.

What will be the difference between the balances in the two accounts at t=2?

A) 10.46 B) 11.96 C) 13.46 D) 14.96 E) 16.46

23. A corporation is obligated to make three annual payments of 6,000; 8,000; and 10,000 at the end of the next 3 years (i.e., 6,000 at $t = 1$, 8,000 at $t = 2$, and 10,000 at $t = 3$).

To fund these liabilities, the corporation intends to purchase appropriate amounts of the following three annual-coupon bonds so that the bonds' coupon payments and maturity values will provide exactly the amounts needed at $t = 1$, 2, and 3:

Term (years)	Coupon Rate
1	4%
2	4.5%
3	5%

(Note: Each bond's redemption value is equal to its face amount.)

To the nearest 100, what is the total face amount of the three bonds that the corporation should purchase to provide the funds for the three payments?

A) 21,500 B) 21,700 C) 21,900 D) 22,100 E) 22,300

24. Company X's stock pays annual dividends. The next dividend is due one year from today and is expected to be \$1.50 per share. Dividends are expected to increase by 5% per year in all future years.

Based on the dividend growth model of stock valuation, and using a valuation interest rate of 12% (an annual effective rate), what is a fair price per share for this stock?

A) 21.43 B) 22.86 C) 23.57 D) 24.29 E) 25.00

25. A U.S. Treasury bill and a Canadian Treasury bill will both mature in 180 days. Both of these securities have a quoted rate of 3.2%. If their rates of return were quoted as annual effective rates, what would be the difference between the two rates?

A) 0.10% B) 0.09% C) 0.08% D) 0.07% E) 0.06%

26. An account accumulates interest at a force of interest that varies according to the following formula:

$$\delta_t = \frac{0.10}{(t+1)} \qquad 0 \le t \le 5$$

1,000 is deposited into the account at t=0.

What will be the balance in the account at t=2?

A) 1,116 B) 1,131 C) 1,163 D) 1,188 E) 1,213

27. A 25-year bond has a face amount (and maturity value) of 1,000. It pays semi-annual coupons at an 8% (annual) coupon rate. The bond is callable on any coupon date on or after its 10th anniversary, with a 5% call premium. An investor purchases this bond on its issue date at a price of 1,100.

If y is the yield that the investor will earn on the bond if it is held to maturity, what is the earliest coupon date on which the bond could be called and the investor would earn a rate of return greater than y?

A) 33rd B) 34th C) 35th D) 36th E) 37th

28. A company has an obligation to pay 1,000 in 2 years.

It purchases a 1-year zero-coupon bond and a 3-year zero-coupon bond, each with a 5% (annual effective) yield to maturity. The bonds' face amounts are such that the bonds' combined present value and their combined duration match the present value and duration (respectively) of the obligation described above. (Assume the liability is also valued at 5%.)

What is the sum of the maturity values of these two bonds?

A) 998 B) 1,000 C) 1,001 D) 1,003 E) 1,006

29. Which of the following statements are true?

I. For loans that do not provide inflation protection, an increase in the rate of inflation tends to favor the lender.
II. For inflation-protected loans, the cost of inflation protection can cause the contractual interest rate to be negative.
III. When there is greater uncertainty about the rate of inflation, the supply curve for money tends to shift upward.

A) I only B) II only C) III only
D) I and II only E) II and III only

30. Zero-coupon bond yields for terms of 1, 2, and 3 years are as follows:

Term (in years)	1	2	3
Zero-coupon bond yield	5%	6%	7%

What is the level swap rate for a 3-year interest swap with level notional amount?

A) 5.8% B) 6.1% C) 6.7% D) 6.8% E) 6.9%

31. A 12-year callable bond has semi-annual coupons. The annual coupon rate is 7%, and the bond is priced to yield at least 5% convertible semi-annually.

The bond is callable at its face value of 1,000 on its n^{th} coupon date or any coupon date thereafter. If the bond's price is 1,088, calculate n.

A) 5 B) 10 C) 12 D) 18 E) 24

32. Eric finances 15,000 at a nominal rate of 6.25% convertible monthly. The term of Eric's loan term is five years, with payments at the end of each month. Each month he pays 50 more than the required monthly payment in order to pay off the loan early.

What is the amount of Eric's final payment?

A) 357 B) 349 C) 342 D) 330 E) 328

33. Tony deposits 1,000 into an account at the end of each quarter. After 17 years, he has accumulated 150,000. If he continues to make quarterly deposits of 1,000, how many more years will it take Tony to double his 17-year accumulated value of 150,000?

A) 5 B) 7 C) 8 D) 10 E) 13

34. A 20-year bond was issued on August 1, 2008. The bond has a face amount of 1,000 and pays semi-annual coupons at a 6% (annual) rate.

On September 20, 2017, the bond is trading at a price that will provide a yield of 5.4% convertible semi-annually. What is the market price of the bond? (Assume 30-day months.)

A) 1,040 B) 1,049 C) 1,057 D) 1,066 E) 1,075

35. The spot rate yield curve for terms of 1, 2, and 3 years is as follows:

Term (in years)	1	2	3
Spot Rate	0.035	?	0.047

The swap rate for a 3-year interest rate swap is 4.6656%. What is the 2-year spot rate?

A) 0.0410 B) 0.0415 C) 0.0420 D) 0.0425 E) 0.0430

Solutions

1. The relations are $e^{\delta} = (1 + i^{(6)}/6)^6 = (1 - d^{(4)}/4)^{-4}$.
So $i^{(6)} = (e^{\delta/6} - 1)\times 6 = 0.0805$ and $d^{(4)} = (1 - e^{-\delta/4})\times 4 = 0.0792$.
Thus $i^{(6)} + d^{(4)} = 0.1597$

Answer: B

2. The children will receive 15, 12 and 10 payments, respectively. The annuities are annuities-due. The sum of the present values is

$5{,}000(\ddot{a}_{\overline{15}|} + \ddot{a}_{\overline{12}|} + \ddot{a}_{\overline{10}|}) = 5{,}000(10.590 + 9.093 + 7.952) = 138{,}175$.

Answer: C

3. We have $5{,}000/(1 + i)^2 + 2{,}000/(1 + i)^4 = 6{,}000$.
Letting $x = (1 + i)^2$, we have $5{,}000/x + 2{,}000/x^2 = 6{,}000$.
This reduces to $6x^2 - 5x - 2 = 0$.
The positive root is $x = (1 + i)^2 = 1.1287 \Rightarrow i = 0.062$

This problem can also be solved using the Cash Flow worksheet:
CF_0=-6,000, C01=0, C02=5,000, C03=0, C04=2,000. IRR CPT gives 6.24%.

Answer: E

4. First, to find the payment using the calculator set N =360, I/Y = 6.4/12, PV = 150,000 and FV = 0. Then CPT PMT = -938.26.

Second, find the balance at time 12: reset N = 144 and CPT FV = -120,160.54.
Third, reset I/Y = 5.8/12, N= 120, PV= 120,160.54 and FV = 0.
Then CPT PMT = -1,321.99.
(Suggested calculator technique: In the second step, set N=216 (number of payments remaining) and CPT PV=120,160.54. (This uses the prospective method instead of the retrospective method used above.) That way you already have 120,160.54 stored in PV and 0 in FV for the third step. Simply enter N = 120 and I/Y = 5.8/12; then CPT PMT = -1,321.99.)

Answer: A

5. To find the price of the bond with the calculator, set N = 40, I/Y = 2.8, PMT = 30, and FV = 1,000. Then CPT PV = -1,047.76.

The accumulation of the coupon payments with interest is:

$$30[s_{\overline{20}|0.025}(1.027)^{20} + s_{\overline{20}|0.027}] = 2{,}087.66$$

Total accumulation is 3,087.66.
Annual effective yield = $(3{,}087.66/1{,}047.76)^{1/20} - 1 = 0.056$

Answer: C

6. The basic formula is $(1 + s_n)^n = (1 + s_{n-1})^{n-1}(1 + i_{n-1,n})$.
$(1 + s_3)^3 = (1.046)^2(1.037) = 1.1346$
$(1 + s_4)^4 = 1.1346(1.039) = 1.1788$
Then $s_4 = ((1.1788)^{1/4}-1) = 0.042$

Answer: D

7. The present value of this annuity is:
$300a_{\overline{30}|} + 100a_{\overline{20}|} - 200a_{\overline{10}|} = 300(13.059) + 100(11.019) - 200(7.189) = 3{,}581.80$
(Explanation: All 3 annuity functions ($a_{\overline{30}|}$, $a_{\overline{20}|}$, and $a_{\overline{10}|}$) and are contributing (positively or negatively) during the first 10 years, so the payment amount equals 300+100-200=200. During the second 10 years, only the first 2 annuity functions ($a_{\overline{30}|}$ and $a_{\overline{20}|}$) are still making payments, so the total payment amount is 300+100=400. In the last 10 years, only the first annuity ($a_{\overline{30}|}$) is still making payments, so the payment amount during that period is 300.)

Answer: A

8. The amount of principal repaid in period k is PMT$v^{360-k+1}$.
Set $766.39v^{361-k} = 500$.
Then $(1 + i)^{361-k} = 1.0055^{361-k} = 766.39/500 = 1.5328$.
$361 - k = \ln(1.5328)/\ln(1.0055) = 77.867 \Rightarrow k = 361 - 77.867 = 283.135$
The first period in which the principal repaid is <u>more than</u> 500 is the 284th.

Answer: E

9. By the Macaulay approximation method:

$$P \approx 1{,}023.68 \cdot (1.067 / 1.066)^{8.2135} = 1{,}031.594$$

$$\Delta P \approx 1{,}031.59 - 1{,}023.68 = 7.914$$

By the modified approximation method:

$\Delta P \approx -(D_{\text{mod}})P(i)(\Delta i)$

$= -(8.2135/1.067)(1023.68)(-0.001) = 7.880$

Since the problem specified Macaulay approximation, the answer is 7.914.

Answer: C

10. The present value of this annuity is

$$100(Da)_{\overline{10|}} + 100v^{10}a_{\overline{10|}} = 100(10 - a_{\overline{10|}})/i + 100(1.063)^{-10}a_{\overline{10|}}$$

$a_{\overline{10|}} = 7.2566$, so we have:

$$100(10 - 7.2566)/0.063 + 100(1.063)^{-10}(7.2566) = 4{,}748.52$$

Answer: C

11. We first need to find the principal balance due after the third payment.
Set N= 3, I/Y = 7.6, PV = 50,000, and PMT = -3,000. Then CPT FV = -52,587.02.
The new balance is 52,587.02.
For the new payment amount, set N = 7, I/Y = 7.6, PV = -52,587.02 and FV = 0.
Then CPT PMT = 9,962.76.

Answer: C

12. The weights for the individual bonds are w_1 = 20,000/100,000 = 0.2, w_2 = 35,000/100,000 = 0.35, and w_3 = 45,000/100/000 = 0.45.

The modified duration of the portfolio is:

$$0.2(1.92) + 0.35(2.84) + 0.45(4.79) = 3.534$$

Answer: B

13. Using the CF worksheet, set CF_0 = -12,000, C01 = 0, C02 = 4,000, C03 = 6,000, and C04 = 8,000. Then press IRR CPT. The yield is 13.59%.

Answer: D

14. We need to find $(1 + s_n)^n$ for $n = 1$, 2 and 3.
$(1 + s_1) = 1 + i_{0,1} = 1.051$
$(1 + s_2)^2 = (1 + s_1)(1 + i_{1,2}) = (1.051)(1.047) = 1.1004$
$(1 + s_3)^3 = (1+ s_2)^2(1 + i_{2,3}) = (1.1004)(1.043) = 1.1477$
$P = 45/1.051 + 45/1.1004 + 1{,}045/1.1477 = 994.23$

Answer: D

15. The accumulation is $e^{0.04(5)}(1 - 0.06/2)^{-6}(1 + i^{(4)}/4)^8 = (1.052)^{10} = 1.6602$.
$(1 + i^{(4)}/4)^8 = 1.6602(e^{-0.2})(0.97)^6 = 1.1322$
Then $i^{(4)} = 4(1.1322^{1/8}-1) = 0.063$

Answer: E

16. The amount of interest can be found using the formula:
$I/[8{,}000 + 2{,}000(10/12) + 1{,}000(2/12) – 500(6/12)] = 0.0887$
$I = 0.0887(9{,}583.33) = 850.04$

Answer: A

17. The present value of the annuity is:

$$\text{PV} = 100(Ia)_{\overline{10}|} + 1{,}050v^{10} \cdot a^{g}_{\overline{10}|} = 100(Ia)_{\overline{10}|} + 1{,}050v^{10} \cdot \frac{1-\left(\frac{1+g}{1+i}\right)^{10}}{i-g}$$

$i = 0.068$, $g = 0.05$, $v^{10} = 0.5179$ $\qquad$ $(Ia)_{\overline{10}|} = (\ddot{a}_{\overline{10}|} - 10v^{10})/i = 35.170$

$$a^{g}_{\overline{10}|} = \left[1-\left(\frac{1+g}{1+i}\right)^{10}\right]/(i-g) = \left[1-\left(\frac{1.05}{1.068}\right)^{10}\right]/(0.068-0.05) = 8.684$$

PV = 100(35.17) + 1,050(0.5179)(8.684) = 8,239.81

Answer: B

18. The Macaulay duration is:
$D_{\text{mac}} = [48/1.055 + 2(48)/1.055^2 + 3(1{,}048)/1.055^3]/(\text{Bond Price})$
To find the bond price set N = 3, I/Y= 5.5, PMT = 48, and FV = 1,000.
Then CPT PV = -981.11. The bond's price is 981.11.
$D_{\text{mac}} = 2{,}809.22/981.11 = 2.863$

Answer: D

19. The accumulation in the sinking fund is
$Xs_{\overline{20}|} + Xs_{\overline{10}|} = X(38.825 + 13.494) = 52.319X$
Hence $X = 65{,}000/52.319 = 1{,}242.38$

Answer: C

20. The 3-year forward rate is $i_{3,4} = (1+s_4)^4/(1+s_3)^3-1$.
$s_4 = 0.068 + 0.002(4) - 0.001(16) = 0.060$
$s_3 = 0.068 + 0.002(3) - 0.001(9) = 0.065$
$i_{3,4} = 1.060^4/1.065^3 - 1 = 0.045$

Answer: B

21. There are two coupon dates during the 10th year, and therefore two adjustments to the bond's book value. The bond's yield is 6.4% convertible semi-annually, which is equivalent to a 3.2% effective rate per coupon period. If x is the amount of the first adjustment in the 10th year, then the second adjustment is $1.032x$ and we have:

$$x + 1.032x = 6.10 \qquad x = 6.10 / 2.032 = 3.00$$

The book value adjustment at the kth coupon payment payment is equal to $F \cdot (r-i)v^{n-k+1}$. This is the formula for premium amortization. Since this bond was purchased at a discount, r (the coupon rate) will be less than i, and the formula will give us the negative of the book value adjustment. Since we know that the book value adjustment for k=19 was 3.00, we can solve for the coupon rate:

$$F \cdot (r-i)v^{n-k+1} = 1{,}000 \cdot (r - 0.032)1.032^{-(40-19+1)} = -3.00$$

$$r - 0.032 = \frac{3.00 \cdot 1.032^{22}}{1{,}000} = -0.006$$

$$r = 0.032 - 0.006 = 0.026$$

The coupon rate is 0.026 per period, or 5.2% per year.

Answer: D

22. If an investment earns simple interest at a rate x, it will earn the same amount of interest during its first year as it would have earned at an annual effective rate of compound interest equal to x. An annual effective rate of discount represents a higher rate of compound growth than a compound interest rate of x. Therefore, the account accumulating at a rate of discount x will have a greater value at the end of 1 year than the account that earns simple interest. Thus the equation of value is:

$$1{,}000 \cdot (1-x)^{-1} = 1.005 \cdot \left(1{,}000\right) \cdot (1+x)$$
$$1 = 1.005 \cdot (1-x) \cdot (1+x)$$
$$1 - x^2 = \frac{1}{1.005}$$
$$x = 7.053\%$$

The difference between the balances in the two accounts at t=2 is:
$1{,}000 \cdot (1-0.070535)^{-2} - 1{,}000 \cdot (1+2 \cdot 0.070535) = 16.46$

Answer: E

23. At t = 3: Payment = 10,000, all of which must come from the 3-year bond.
Therefore $face_3$ = 10,000 / 1.05 = 9,523.81.
(At t = 3, the 3-year bond provides a redemption value of 9,523.81 plus a coupon of 476.19 (= 5% of 9,523.81), for a total of 10,000.)

At t = 2: Payment = 8,000, consisting of 476.19 coupon from the 3-year bond, plus the redemption value and a coupon from 2-year bond.
Therefore, $face_2$ = (8,000 – 476.19) / 1.045 = 7,199.82.
(At t = 2, the 3-year bond provides a 476.19 coupon, and the 2-year bond provides a redemption value of 7,199.82. plus a coupon of 323.99 (= 4.5% of 7,199.82.), for a total of 8,000.)

At t = 1: Payment = 6,000, consisting of 323.99 and 476.19 coupons from the 2- and 3-year bonds, plus the redemption value and a coupon from the 1-year bond.
Therefore, $face_1$ = (6,000 – 323.99 – 476.19) / 1.04 = 4,999.82.
(At t = 1, the 2- and 3-year bonds provide coupons of 323.99 and 476.19, and the 1-year bond provides a redemption value of 4,999.82. plus a coupon of 199.99 (= 4% of 4,999.82.), for a total of 6,000.)

Total face equals $face_1$ + $face_2$ + $face_3$ = 9,523.81 + 7,199.82 + 4,999.82 = 21,723.45

Answer: B

24. Based on the dividend growth model, we can write:

$$\text{Stock value} = \frac{1.50}{0.12 - 0.05} = 21.43$$

Answer: A

25. First we need to find the annual effective rates for the two securities. We will assume that each has a maturity value of 1,000 and will use $i^{\text{U.S.}}$ and i^{Can} to represent their annual effective rates. In each set of calculations, I stands for the amount of interest earned.

U.S. Treasury bill:

$$0.032 = \frac{360}{180} \cdot \frac{I}{1{,}000}$$

$$I = 0.032 \cdot \frac{180}{360} \cdot 1{,}000 = 16.00$$

$$i^{\text{U.S.}} = \left(1 + \frac{16}{1{,}000 - 16}\right)^{365/180} - 1 = 0.0332475$$

Canadian T-bill:

$$0.032 = \frac{365}{180} \cdot \frac{I}{1{,}000 - I}$$

$$I \cdot \left(1 + 0.032 \cdot \frac{180}{365}\right) = 0.032 \cdot \frac{180}{365} \cdot 1{,}000 = 15.78082$$

$$I = 15.78082 \div \left(1 + 0.032 \cdot \frac{180}{365}\right) = 15.53566$$

$$i^{\text{Can}} = \left(1 + \frac{15.53566}{1{,}000 - 15.53566}\right)^{365/180} - 1 = 0.0322595$$

The difference between the two rates is $0.0332475 - 0.0322595 = 0.0009880$.

Answer: A

26. The accumulated value at time 2 is:

$$1{,}000 \cdot e^{\int_0^2 \delta_t dt} = e^{\int_0^2 \frac{0.10}{t+1} dt}$$

We start by evaluating the integral:

$$\int_0^2 \frac{0.10}{t+1} \cdot dt = 0.10 \cdot \ln(t+1)\Big|_0^2 = 0.10 \cdot (\ln 3 - \ln 1) = 0.10986$$

Then the ending balance can be calculated:

$$1{,}000 \cdot e^{0.10986} = 1{,}116.12$$

(Note: This is a good problem for checking the reasonableness of your answer. The continuously compounded interest rate starts at 10% at time 0 $(0.10/(0+1))$ and decreases to 5% and 3.333% at time 1 and time 2. So the average growth rate over the period is probably a bit higher than 5%. On that basis, a 2-year accumulation factor of 1.116 seems reasonable.)

Answer: A

27. First calculate the value of y, the bond's yield to maturity.
Set N=50, PV=-1,100, PMT=40, and FV=1,000. CPT I/Y=3.5684.

The semi-annual effective yield is 3.5684%, so if y is expressed as a nominal rate convertible semi-annually, it would be 7.1368%. However, we don't need to be concerned with that value. We will work with the semi-annual effective rate.

We want to determine the <u>first</u> coupon date on which the bond could be called and the investor would earn a yield greater than y. The bond has a 5% call premium, so we set FV=1,050 and CPT N=34.78. Since the bond was purchased at a premium of 100, which is greater than the call premium (of 50), we know that a <u>later call</u> results in a <u>higher yield</u>. Therefore, the calculated value of 34.78 for N tells us that the bond will yield less than 3.5684% if called on the 34^{th} coupon date, and more than 3.5684% if called on the 35^{th} coupon date.

(Note: If we track the bond's book value based on holding the bond to maturity (and earning a 3.5684% semi-annual effective rate), the book value begins at 1,100 and decreases to 1,000 at the end of 25 years. As long as the book value calculated this way is greater than 1,050, a call is to the investor's disadvantage (reducing the yield below 3.5684%). But when the book value falls below 1,050, a call increases the investor's return. And a later call is better for the investor than an early call. The most advantageous (but least likely) outcome would be for the 25-year bond to be called after 24.5 years, when its book value has decreased to nearly 1,000.)

Answer: C

28. Let: B_1 = maturity value of the 1-year bond
B_3 = maturity value of the 3-year bond

$B_1 v + B_3 v^3 = 1,000 v^2$ PV(assets) = PV(liability)

$1 \cdot B_1 v + 3 \cdot B_3 v^3 = 2 \cdot 1,000 v^2$ D_{mac}(assets) = D_{mac} (liability)[1]

Subtracting the first equation from the second, we have:

$$2 \cdot B_3 v^3 = 1000 v^2$$

$$B_3 = 500 \div v = 500(1+i) = 500(1.05) = 525$$

Substituting B_3 into the first equation and solving for B_1 produces:

$B_1 = 500\, v = 500 / (1+i) = 500 / 1.05 = 476.19$

Total maturity value = $B_1 + B_3 = 525 + 476.19 = 1,001.19$

[1] *Actually, the values calculated in this equation are not the "durations" of the assets (left side) and the liability (right side). Each side of the equation should be divided by the present value of the respective cash flows. However, the two present values are equal, so each side of the equation would be divided by the same value. Therefore, we can express this equality without writing the denominators. Also, we equated the Macaulay durations (not modified durations). But the interest rate is 5% for each security, so the modified durations will also be equal.*

Answer: C

29. Statement I is false. Inflation is harmful to the lender, because the repayment is made in dollars that have less purchasing power than the dollars that were lent. In fact, inflation can help the borrower, as his/her ability to repay the loan improves if inflation causes the borrower's income to rise.

Statement II is true. The lender can expect to "pay a price" for inflation protection in the form of a lower interest rate. (This "price" is the variable c in Module 8 and in the study note on interest rates.) If current interest rates are relatively low (e.g., on a short-term loan) and inflation is either high or unpredictable, the cost of inflation protection could very well be larger than the compensation for deferred consumption, resulting in a negative real rate of return for the lender.

Statement III is true. When the rate of inflation is unpredictable, lenders demand a higher rate of return, which means that the supply curve shifts upward (toward higher interest rates).

Answer: E

30. The guaranteed interest rate (the swap rate) is the 3-year par coupon bond rate:

$$R = c = \frac{1 - P_3}{P_1 + P_2 + P_3}$$

The required zero-coupon bond prices are

$$P_1 = \frac{1}{1.05} = 0.952, \quad P_2 = \frac{1}{1.06^2} = 0.890, \quad P_3 = \frac{1}{1.07^3} = 0.816$$

The swap rate is:

$$\frac{1 - 0.816}{0.952 + 0.890 + 0.816} = 0.069$$

Answer: E

31. Since the callable bond is a premium bond, the bond is priced assuming it will be called at the earliest possible call date, *n*. We can use the financial calculator to solve for *n*.

Set I/Y = 2.5, PV = -1,088, PMT = 35, and FV = 1,000. CPT N = 10.062.

n=10.062

Answer: B

32. This problem can be solved with the financial calculator. First figure out the amount of Eric's original loan payment.

Set PV = 15,000, N=5(12)=60, I/Y=6.25/12=0.520833, and FV = 0.
Then CPT PMT=-291.7389.

The monthly payment amount is 291.7389.
Add 50 to the payment to get the amount of Eric's actual monthly payment.
291.7389 + 50 = 341.7389.

Keeping the old values in the calculator, enter the new payment amount and calculate N.

PMT = -341.7389, CPT N = 49.9655.

The calculated loan term is 49.9655.

There will be 50 payments, but the 50th payment will be a smaller amount. We need to calculate the outstanding loan balance immediately after the 49th payment. Keeping the same values in the calculator, change N to 49 and calculate the future value (the loan balance after 49 payments).

Set N = 49 and CPT FV = -328.2569.
The outstanding balance after 49 payments is 328.2569.

The last payment will be this amount (at time 49) plus one month's interest:

$$328.2569(1+0.0625/12)=329.9665$$

(Suggested calculator technique: Instead of checking the outstanding balance after 49 periods and accumulating it with interest for the 50th month, set N=50 to calculate the outstanding balance after 50 periods. The result is 11.7724 (a positive number, meaning an amount Eric would receive, because he overpaid by making 50 full payments). Add this positive amount to the negative value of PMT to find the final payment amount. The keystrokes are:
50 N CPT FV + RCL PMT =
The result is -329.9665, the (negative of the) net amount due at time 50.)

(Note: In the above calculation, we used a monthly payment amount to 4 decimal places (341.7389). If you round this to the nearest cent to reflect that Eric's actual payments were 341.74, the last payment changes by a few cents (to 329.91), but it doesn't change the answer choice.)

Answer: D

33. Using the 17-year accumulated value, we can solve for Tony's quarterly effective interest rate, since we know that $1{,}000 s_{\overline{68}|i} = 150{,}000$.

Set N = 68, PV = 0, PMT = -1,000, and FV = 150,000. CPT I/Y = 2.13228.
i = 2.13228%

Now, keeping the same values in the calculator, set FV = 300,000 and CPT N = 94.8427. It will take 94.8427 quarters to reach 300,000, or about 23.71 years, which means about 7 more years after reaching 150,000 at 17 years.

Alternatively, you could set PV = -150,000 (representing the 150,000 that has been accumulated in 17 years) and FV = 300,000 and CPT N = 26.8427 (the number of quarter-years for the account balance to grow from 150,000 to 300,000). So the answer is $26.8427 \div 4 = 6.7107$, or about 7 years.

Answer: B

34. This problem can be solved using the Bond worksheet or by formulas (with or without the TVM worksheet).

By formulas (and using the TVM worksheet):
First calculate the price of the bond as of the last coupon date before September 20, 2017. This is August 1, 2017, when there are 11 years (22 coupon periods) remaining. Calculate the price on this date at a 5.4% yield by setting N=22, I/Y=2.7, PMT=30, and FV=1,000. CPT PV=1,049.28.

Assuming 30-day months, there are 49 days between August 1 and September 20, and 180 days in the 6-month coupon period.
The <u>total</u> price on September 20 is:

$$1{,}049.28 \cdot 1.027^{49/180} = 1{,}056.92$$

This price includes accrued coupon of:

$$30 \cdot (49/180) = 8.17$$

The market price is 1,056.92 – 8.17 = 1,048.75.

Using the Bond worksheet:
SDT=9.2017, CPN=60 (per year), RDT=8.0128, RV=1,000, 360, 2/Y, YLD=5.4
PRI CPT = 1,048.75

Answer: B

35. We will first calculate the zero-coupon bond prices:

$$P_1 = \frac{1}{1.035} = 0.9662 \qquad P_2 = x \qquad P_3 = \frac{1}{1.047^3} = 0.8713$$

The swap rate is the par bond coupon rate, given by:

$$R = c = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 0.8713}{0.9662 + x + 0.8713} = 0.046656$$

Solving for x:

$$x = \frac{1 - 0.8713}{0.046656} - 0.9662 - 0.8713 = 0.92099$$

$$P_2 = x = .92099 = \frac{1}{(1 + r_2)^2}$$

$$r_2 = 0.92099^{-0.5} - 1 = 0.0420$$

Answer: C

Practice Exam 5

Exam FM

Questions

1. A man wishes to accumulate 100,000 by making monthly end-of-month deposits for 30 years into an account that earns 5.4% interest convertible monthly. After 10 years the interest rate increases to 6.6% convertible monthly. What should his new contributions be if he still wishes to accumulate 100,000 at the end of 30 years?

A) 60.70 B) 63.85 C) 68.50 D) 71.25 E) 74.65

2. Charlie deposits 5,000 into an account that earns an annual rate of i. Lucy buys a 25-year annuity-immediate for 5,000. The annuity has annual payments and is priced based on a 7% annual effective rate. She deposits her annual payments into a fund that pays 6.5% annually. After 25 years Charlie and Lucy have accumulated the same amount. Find i.

A) 6.7% B) 6.9% C) 7.1% D) 7.3% E) 7.5%

3. An investor buys a 10-year 1,000 par bond that has 7.5% semi-annual coupons and is priced to yield 6.8% convertible semi-annually. The bond is called at the end of 6 years with a redemption value of X. The yield to the investor is still 6.8% convertible semi-annually. Find X.

A) 1,009 B) 1,014 C) 1,019 D) 1,024 (E) 1,029

4. A man borrows 10,000 to be paid back over 30 years with level end-of-year payments at an annual effective interest rate of i. The sum of the principal repayments during year 5 and year 10 is equal to the principal repaid during year 15. Find i.

A) 8.9% B) 9.2% C) 9.5% D) 9.8% E) 10.1%

5. A man has two 20-year annuities-immediate. Each has a present value of 1,000. The first has annual payments and is valued at a 5.8% annual effective rate. The second has semi-annual payments and is valued at 5.4% convertible semi-annually. All payments are deposited into a fund that pays an annual effective rate of 6%. What is his accumulation at the end of 20 years?

A) 6,170 B) 6,190 C) 6,210 D) 6,230 E) 6,250

6. A woman has 10,000 in an account on January 1. She makes withdrawals of 400 on April 1 and 600 on November 1. Her dollar-weighted return for the year is 12.77%. What is her balance on December 31?

A) 10,123 B) 10,226 C) 10,317 D) 10,437 E) 10,501

7. Given $s_2 = 0.053$, $s_4 = 0.0575$ and $i_{2,3} = 0.058$, find $i_{3,4}$.

A) 0.066 B) 0.067 C) 0.068 D) 0.069 E) 0.070

8. A 3-year 1,000 par bond has 4.5% annual coupons. The forward rates implied by the yield curve are $i_{0,1} = 0.033$, $i_{1,2} = 0.038$ and $i_{2,3} = 0.042$. What is the price of the bond?

A) 1,011 B) 1,016 C) 1,021 D) 1,026 E) 1,031

9. Given $d^{(4)} = 0.064$, find $\delta + i^{(6)}$.

A) 0.123 B) 0.125 C) 0.127 D) 0.129 E) 0.131

10. A company has liabilities of 3,000, 5,000, and 2,000 due at the end of years 1, 2, and 3 respectively. It can purchase zero-coupon bonds to match its liabilities. Each bond has a par value of 1,000. The first bonds mature in one year and yield 5%, the second set of bonds will mature in two years and yield i, and the third set will mature in three years yielding 6%. The cost of matching the company's liabilities is 9,028.64. Find i.

A) 5.1% B) 5.3% C) 5.5% D) 5.7% E) 5.9%

11. A woman deposits 10,000 into an account that earns an interest rate of 5.4% convertible monthly. She makes level monthly end-of-month withdrawals at a rate that will exhaust the account at the end of 30 years. After 15 years, the interest rate for the account is increased to 6.3% convertible monthly. What new monthly withdrawal amount will exhaust the account at the end of the original 30-year period?

A) 57.50 B) 58.00 C) 58.50 D) 59.00 E) 59.50

12. Jack deposits 1,000 into an account on 01/01/13. Jill deposits 500 into an account on 01/01/14, and another 600 into the account on 01/01/15. On 01/01/17 the accounts have the same amount in them. The accounts earned the same annual effective rate. What was the interest rate?

A) 6.2% B) 6.4% C) 6.6% D) 6.8% E) 7.0%

13. An investment pays 2,000 at the end of year 1, 2,500 at the end of year 2, and X at the end of year 3. The investment earns 8% annually. The present value of the investment is 6,773.60. What is the Macaulay duration of the investment?

A) 2.137 B) 2.175 C) 2.204 D) 2.229 E) 2.253

14. A man wants to accumulate 250,000 in 25 years by making monthly end-of-month payments into a fund that earns 6.3% convertible monthly. His first payment is 100 and each subsequent payment is increased by X over the previous one. What must X be to achieve his goal?

A) 2.04 B) 2.09 C) 2.14 D) 2.19 E) 2.24

15. A 10-year 1,000 par bond with 6% semi-annual coupons is purchased to yield 5.6% convertible semi-annually. How much of the premium is amortized in the seventh period?

A) 1.33 B) 1.36 C) 1.39 D) 1.42 E) 1.45

16. A 10-year 1,000 par bond with 6% annual coupons is priced to yield 5.5%. Find the Macaulay duration of this bond.

A) 7.847 B) 7.898 C) 7.937 D) 7.962 E) 7.995

17. Money in an account earns 7% simple interest per year. What is the effective rate of interest during the 4th year (for the time interval [3,4])?

A) 5.61% B) 5.66% C) 5.71% D) 5.75% E) 5.79%

18. A man wants to retire in 25 years. He sets up an account by making monthly end-of-month payments of X. The account earns 6% convertible monthly. When he retires he wants to be able to make annual end-of-year withdrawals for 25 years. He wants the first withdrawal to be 10,000 and each subsequent one to be 3% more than the previous one. What should X be if interest rates stay the same?

A) 236 B) 239 C) 242 D) 245 E) 248

19. A man has a 30-year loan with level end-of-year payments. The principal repaid in year 5 is 159.68 and in year 10 it is 213.73. What is the amount of the annual loan payment?

A) 706 B) 711 C) 716 D) 721 E) 726

20. A man receives an inheritance of X, which he uses to buy a 30-year annuity-immediate. The annuity will have monthly payments of 100 for the first 10 years, 300 for the next ten years and 1,000 for the final 10 years. The annuity's price is based on an interest rate of 5.7% convertible monthly. Find X.

A) 53,750 B) 53,925 C) 54,175 D) 54,350 E) 54,525

21. An account accumulates interest at a force of interest that varies according to the following formula:

$$\delta_t = \frac{0.18}{(t+2)} \qquad 0 \le t \le 10$$

If 100 is deposited into this account at t=0 and an additional 100 is deposited at t=2, what is the balance in the account at t=4?

A) 227 B) 229 C) 231 D) 233 E) 235

22. A corporation takes out a 20-year loan. Under the terms of the loan, the corporation will make level quarterly interest payments based on a nominal annual interest rate of 10% convertible quarterly. At the end of 20 years, the corporation will repay the entire principal balance of the loan in a lump sum.

During the period of the loan, the corporation will make quarterly deposits to a sinking fund in order to accumulate a sum of money at the end of 20 years equal to the principal of the loan. The sinking fund earns interest at a nominal annual rate of $x\%$, convertible monthly.

If the total amount the corporation pays each quarter (the loan interest payment plus the sinking fund deposit) equals the level payment for a 20-year loan of the same amount at an interest rate of 12% convertible quarterly, what is the value of x?

A) 2.96% B) 3.06% C) 4.07% D) 4.10% E) 4.12%

23. Investor A purchased a 20-year bond on its issue date at a price equal to the face amount of the bond. The bond pays semi-annual coupons based on a 5% (annual) coupon rate.

After 5 years Investor A sold this bond to Investor B. Investor A earned a yield of 5.6% (a nominal annual rate, convertible semi-annually) over the 5 years he owned the bond.

Five years after purchasing the bond, Investor B sold the bond to Investor C. Based on Investor C's purchase price, the bond's yield to maturity is 6.5% convertible semi-annually.

What yield rate (expressed as a nominal rate, convertible semi-annually) did Investor B earn on the bond during the 5 years he owned it?

A) 2.2% B) 3.0% C) 3.7% D) 4.2% E) 4.8%

24. The following table shows the history of account balances, deposits, and withdrawals for a fund:

Date	Value Before Dep/Wdl	Deposit (+)/ Withdrawal (-)
January 1, 2016	1,000	0
March 11, 2016	X	500
January 1, 2017	1,400	

The dollar-weighted rate of return (based on simple interest) and the time-weighted rate of return for this fund during 2016 were equal.

What is the value of X?
(Assume 30-day months.)

A) 1,005 B) 1,000 C) 995 D) 990 E) 985

25. A corporation has liabilities that require annual payments of 10,000 at the end of each of the next 3 years (i.e., it must pay 10,000 at t=1, t=2, and t=3).

The corporation has purchased a 2-year bond and a 3-year bond, with the following characteristics:

Term (years)	Coupon Rate
2	5%
3	6%

The face amounts of these 2 bonds have been chosen so that they will generate total payments of 10,000 at t=2 and 10,000 at t=3.
(Note: Each bond's maturity value is equal to its face amount.)

The corporation now wishes to purchase a 1-year zero-coupon bond so that the total amount received at t=1 will also be 10,000. To the nearest dollar, what must the face amount of that 1-year bond be?

A) 8,985 B) 9,143 C) 9,282 D) 9,434 E) 9,551

26. A stock will pay quarterly dividends of 1.00 per share during the next year (a total of 4.00 per share during the year). The first quarterly dividend is due 3 months from now.

Dividends are expected to increase by 5% annually. (The first 4 dividends will be 1.00 each; the next 4 will be 1.05 each; the next 4 will be 1.1025 each; etc.).

To the nearest whole number, what is a fair price for this stock, based on a 10% required annual effective yield?

A) 80 B) 82 C) 83 D) 84 E) 85

27. Which of the following does <u>not</u> affect the interest rate that a bank pays on amounts in a savings account?

A) The bank's overhead expenses
B) The outlook for inflation
C) The bank's credit rating
D) The depositor's creditworthiness
E) The demand for loans in the region where the bank operates

28. An interest rate swap consisting of 4 annual swap payments has a notional principal amount of 1,000,000 and a fixed interest rate of 5.11% (an annual effective rate). The variable interest rate will be based on the price for a 1-year zero-coupon bond.

At t=2 (two years after the inception date), the price of a 1-year zero coupon bond with maturity value 100 is 95.30.

To the nearest dollar, what is the net settlement amount that the payer must pay at the end of the third year?

A) 1,649 B) 1,698 C) 1,749 D) 1,782 E) A different amount

29. A 15-year bond pays semi-annual coupons at a 6% (annual) coupon rate. Its par value and redemption value are both 10,000. Two years after its issue date, the bond is purchased at a discount to yield 8.00% convertible semi-annually.

What is the adjustment to the bond's book value at the time of the 9th semi-annual coupon payment?

A) 42.20 B) 43.88 C) 60.06 D) 82.77 E) 116.70

30. Zero coupon bond yields are as follows:

Term (in years)	1	2	3
Zero-coupon bond yield	5.2%	?	7.1%

The level swap rate for a 3-year interest rate swap with level notional amount is 7% What is the yield for a 2-year zero coupon bond?

A) 5.9% B) 6.0% C) 6.1% D) 6.2% E) 6.3%

31. Laura has an outstanding balance of 180,000 on a mortgage at a nominal rate of 5.5% convertible monthly.

She has 20 years of monthly (end-of-month) payments remaining on the loan. She has the opportunity to refinance her mortgage at a nominal rate of 4.5% convertible monthly. However, the new mortgage would have a term of 30 years.

How much more (or less) interest will Laura pay if she decides to refinance her mortgage?

A) 31,165 more B) 70,760 more C) 23,805 more
D) 23,800 less E) 31,165 less

32. A 10-year annual-coupon bond has a duration of 8.2 years, based on its 4% annual effective yield to maturity. The bond's redemption value is the same as its face value of 1,000.

If the bond's coupon rate were increased by 1%, what would be the new purchase price, assuming the same yield rate of 4%?

A) 1,053 B) 1,112 C) 1,162 D) 1,243 E) 1,300

33. A perpetuity-immediate makes a payment of 3 at the end of the first year, 2 at the end of the second year, and 1 at the end of the third year. This pattern of payments of 3, 2, 1 repeats indefinitely.

Calculate the present value of the perpetuity.

A) 176.5 B) 172.6 C) 59.5 D) 56.9 E) 55.6

34. You have liabilities that require a payment of 5,000 due in 3 years and another payment of 5,000 due in 5 years. You intend to purchase zero-coupon bonds to create an immunized portfolio in support of these liabilities. The zero-coupon bonds available to you are a 2-year bond and an 8-year bond, each of which is available for any face amount that you desire. *(Note: You are not attempting to create an "exact match" of the asset and liability cash flows.)*

Assuming that an annual effective interest rate of 6% applies to each of the assets and liabilities, and that the bonds you purchase result in having assets with the same present value and modified duration as the liabilities, what is the total maturity value of the bonds that are purchased (to the nearest 100)?

A) 9,800 B) 9,900 C) 10,000 D) 10,100 E) 10,200

35. The responsibilities assigned to a central bank vary from one country to another. Which of the following is sometimes not a responsibility of the central bank?

A) Facilitating the operation of the country's payment system

B) Supervision of banks that are authorized to operate in that country

C) Acting as "lender of last resort"

D) Issuing the country's currency

E) More than one of A, B, C, and D is sometimes not the central bank's responsibility

Solutions

1. To get the original payments set N = 360, I/Y = 0.45, FV = 100,000 and PV = 0. Then CPT PMT = -111.53.

To get the accumulation after 10 years reset N = 120 and CPT FV = 17,694.47.
(Note: It is also possible to set N=240 (the number of payments remaining) and CPT PV = -17,694.47. This simplifies the entries for the next step.)

Now set N = 240, I/Y = 0.55, PV = -17,694.47 and FV = 100,000.
Then CPT PMT = -68.50

Answer: C

2. To get Lucy's payments set N = 25, I/Y = 7, PV = -5,000 and FV = 0. Then CPT PMT = 429.05. Her accumulation in the fund is $429.05s_{\overline{25}|6.5\%}$ = 25,265.91. So Charlie's accumulation is $5{,}000(1+i)^{25}$ = 25,265.91. Solving for i, we have:

$$i = \left(\frac{25{,}265.91}{5{,}000}\right)^{1/25} - 1 = 0.067$$

Answer: A

3. To find the price of the bond set N = 20, I/Y = 3.4, PMT = 37.5 and FV = 1,000. Then CPT PV = -1,050.20. To find X reset N = 12 and CPT FV = 1,024.16.

Answer: D

4. The principal repaid in year k is $\text{PMT}v^{30-k+1}$. So we have $\text{PMT}v^{26} + \text{PMT}v^{21} = \text{PMT}v^{16}$, or $v^{10} + v^5 = 1$. If we let $x = v^5$ we have $x^2 + x = 1$.

The quadratic formula gives us $x = 0.6180$ (and an extraneous root).
So $i = (0.6180)^{-1/5} - 1 = 0.101$.

Answer: E

5. The annual payments from the first annuity are:

$1{,}000 / a_{\overline{20|}5.8\%} = 85.77$

The semi-annual payments from the second annuity are:

$1{,}000 / a_{\overline{40|}2.7\%} = 41.19$

For each year, the accumulated value of the deposits as of the end of the year is $85.77 + 41.19(1.06)^{1/2} + 41.19 = 169.37$. Thus the deposits represent a 20-year annuity-immediate with payments of 169.37. Its future value is $169.37 s_{\overline{20|}.06} = 6{,}230.42$.

Answer: D

6. The denominator for the dollar-weighted interest rate is
$10{,}000 - 400(9/12) - 600(2/12) = 9{,}600$.

The interest amount I is $9{,}600(0.1277) = 1{,}225.92$.

The balance on December 31 is $10{,}000 - 1{,}000 + 1{,}225.92 = 10{,}225.92$.

Answer: B

7. $(1 + s_3)^3 = (1 + s_2)^2(1 + i_{2,3}) = (1.053)^2(1.058) = 1.1731$
$1 + i_{3,4} = (1 + s_4)^4/(1 + s_3)^3 = 1.0575^4/1.1731 = 1.066$

Answer: A

8. The bond price is $P = 45/(1 + s_1) + 45/(1 + s_2)^2 + 1045/(1 + s_3)^3$.
$1 + s_1 = 1 + i_{0,1} = 1.033$. $(1 + s_2)^2 = (1 + s_1)(1 + i_{1,2}) = 1.0723$
$(1 + s_3)^3 = (1 + s_2)^2(1 + i_{2,3}) = (1.0723)(1.042) = 1.1173$
$P = 45/1.033 + 45/1.0723 + 1{,}045/1.1173 = 1{,}020.82$.

Answer: C

9. The fundamental relations are: $e^{\delta} = (1 + i^{(6)}/6)^6 = (1 - d^{(4)}/4)^{-4}$
$(1 - d^{(4)}/4)^{-4} = (1 - 0.064/4)^{-4} = 0.984^{-4} = 1.0666$.
Hence, $\delta = \ln(1.0666) = 0.0645$ and $i^{(6)} = (1.0666^{1/6} - 1)(6) = 0.0648$.
Thus $\delta + i^{(6)} = 0.0645 + 0.0648 = 0.1293$

Answer: D

10. The cost to match the liabilities is:
$3{,}000/1.05 + 5{,}000/(1 + i)^2 + 2{,}000/1.06^3 = 9{,}028.64$
$5{,}000/(1 + i)^2 = 4{,}492.26 \Rightarrow 1 + i = (5{,}000/4{,}492.26)^{1/2} = 1.055$

Answer: C

11. To get the original payments, set N = 360, I/Y = 0.45, PV = 10,000 and FV = 0. Then CPT PMT = 56.15. Then reset N = 180 and CPT FV = 6,917.22, which is the balance after 15 years. Then reset I/Y = 0.525. PV = -6917.77 and FV = 0. Then CPT PMT = 59.50.

Answer: E

12. Let i be the rate earned. The total in Jack's account is $1{,}000(1 + i)^4$. The amount in Jill's account is $500(1 + i)^3 + 600(1 + i)^2$.
Equating these values and setting $1 + i = x$, the equation reduces to:
$10x^2 - 5x - 6 = 0$.
The positive root is $x = 1 + i = 1.064$.

Alternatively, use the Cash Flow worksheet. The present value of Jack's deposit equals the present value of Jill's two deposits, so the NPV of his deposit minus hers is 0.
Set CF_0=-1,000, C01=500, and C02=600. Then press IRR CPT. Result: 6.394%.

Answer: B

13. The present values of the first two payments are 2,000/1.08 = 1,851.85 and $2{,}500/1.08^2 = 2{,}143.35$. The corresponding weights are:
$w_1 = 1{,}851.85/6.773.6 = 0.2734$ and $w_2 = 2{,}143.35/6{,}773.6 = 0.3164$.
Then $w_3 = 1 – 0.2734 – 0.3164 = 0.4102$. The Macaulay duration is
$D_{\text{mac}} = 0.2734(1) + 0.3164(2) + 0.4102(3) = 2.1368$

Answer: A

14. Using the future-value form of the *PQ* formula, we have:
$100s_{\overline{300|}} + X(s_{\overline{300|}} - 300) / i = 250{,}000$, and $i = 6.3\% / 12 = 0.525\%$
$250{,}000 = 100(725.88) + X(725.88 – 300)/0.00525$
$X = 177{,}412/81{,}120 = 2.187$

Answer: D

15. The amount of the premium amortized in the k^{th} period is
$(r - i)(1{,}000)v^{20-k+1} = 1{,}000(0.030 - 0.028)(1.028)^{-(21-k)}$
For the 7th period the amount is $2(1.028)^{-14} = 1.359$

Answer: B

16. The price of the bond is $P = 1{,}037.69$.
(N=10, I/Y=5.5, PMT=60, and FV=1,000. CPT PV = -1,037.69.)
The Macaulay duration is $[60(Ia)_{\overline{10|}} + 10(1{,}000)v^{10}]/P$ for $i = 5.5\%$.
$(Ia)_{\overline{10|}} = 38.143$, so $D_{\text{mac}} = (2{,}288.60 + 5{,}854.30)/1{,}037.69 = 7.847$

Answer: A

17. The effective interest rate during the 4th year is:
$[a(4) - a(3)]/a(3) = (1.28 - 1.21)/1.21 = 0.0579$

Answer: E

18. To determine the amount to deposit, we need to know the cost of the retirement benefits, which is the present value of a geometric annuity: $10{,}000 \cdot a^{3\%}_{\overline{25|}i}$.

The value of i, based on interest rates remaining the same (which is 6% convertible monthly), is $1.005^{12} - 1 = 0.061678$, so we have:

$$10{,}000 \cdot a^{3\%}_{\overline{25|}0.061678} = 10{,}000 \cdot \frac{1 - \left(\frac{1.03}{1.061678}\right)^{25}}{0.061678 - 0.03} = 167{,}645.92$$

To find the monthly deposits, set N = 300, I/Y = 0.5, PV = 0, and FV = 167,642. Then CPT PMT = -241.92.

Answer: C

19. The principal repaid in year k is $\text{PMT}v^{30-k+1}$.
For year 5, $\text{PMT}v^{26} = 159.68$. For year 10, $\text{PMT}\underline{v}^{21} = 213.73$.
Then $v^{-5} = (1 + i)^5 = 213.73/159.68 = 1.3385 \Rightarrow 1 + i = 1.060$.
Then $\text{PMT} = 159.68(1.06)^{26} = 726.45$

Answer: E

20. This can be viewed as the sum of 3 annuities-immediate. The first is a 30-year annuity with monthly payments of 100, the second a 10-year-deferred 20-year annuity with monthly payments of 200, and the third a 20-year-deferred 10-year annuity with monthly payments of 700. The present value of this combined annuity is:

$$\begin{aligned} PV &= 100a_{\overline{360}|} + 200v^{120}a_{\overline{240}|} + 700v^{240}a_{\overline{120}|} \\ &= 100(172.295) + 200(0.5663)(143.014) + 700(0.3207)(91.308) \\ &= 53{,}925 \end{aligned}$$

Answer: B

21. In this problem, we need to calculate an accumulation factor for the period from time 0 to time 4 (for the deposit of 100 at time 0), and another accumulation factor for the period from time 2 to time 4 (for the second deposit of 100, which occurs at time 2). We can write an expression for the balance in the account at time 4 as follows: $100 \cdot a(4) + 100 \cdot a(4) / a(2)$.

The formula for finding these accumulation functions when the force of interest varies continuously is:

$$a(t) = e^{\int_0^t \delta(u)du}$$

We need values for $a(2)$ and $a(4)$, so we will evaluate the integral for t=2 and t=4:

$$\int_0^t \delta(u)\,du = \int_0^t \frac{0.18}{u+2}\,du = 0.18 \cdot \ln(u+2)\Big|_0^t = 0.18 \cdot \ln\left(\frac{t+2}{2}\right)$$

Applying this to the formula for $a(2)$ and $a(4)$, we have:

$$a(2) = e^{\int_0^2 \frac{0.18}{u+2}du} = e^{0.18 \cdot \ln\left(\frac{2+2}{2}\right)} = 2^{0.18} = 1.13288$$

$$a(4) = e^{\int_0^4 \frac{0.18}{u+2}du} = e^{0.18 \cdot \ln\left(\frac{4+2}{2}\right)} = 3^{0.18} = 1.21866$$

Now we have the values we need to calculate the balance at time 4:

$$100 \cdot a(4) + 100 \cdot a(4) / a(2) = 100 \cdot (1.21866) + 100 \cdot (1.21866 / 1.13288) = 229.44$$

Answer: B

22. The problem does not give the amount of the loan. But since we are calculating a rate (an interest rate) rather than a dollar amount, we can choose any amount for the loan. All the other dollar amounts will be in proportion, and the calculated rate will not be affected by the amount we choose.

Let the loan amount be 100,000
Then the quarterly interest payment on the loan $= 100{,}000 \cdot 10\% / 4 = 2{,}500$

The problem states that the <u>total</u> amount paid each quarter (interest on the loan, plus deposit to the sinking fund) is equal to the quarterly payment on a level-payment loan at a rate of $i^{(4)} = 12\%$. So the total quarterly payment is:

$$\frac{100{,}000}{a_{\overline{80}|i=3\%}} = \frac{100{,}000}{\left(1-(1.03)^{-80}\right)/0.03} = 3{,}311.17$$

(This value can also be calculated using the BA II Plus's TVM functions.)

The sinking fund deposit is the difference between the interest payment and the total quarterly payment = 3,311.17 – 2,500 = 8,11.17. To solve for the sinking fund interest rate, use the TVM functions:
Set N = 80, PV = 0, PMT = 811.17, and FV = 100,000.
CPT I/Y = 1.0295 (The quarterly effective interest rate is 1.0295%.)

Convert this rate to a monthly effective rate, and then to a nominal rate convertible monthly:

$1.010295^{\frac{1}{3}} = 1.003420$ (monthly effective rate is 0.3420%)
$12 \times 0.003420 = 0.04104$ (nominal rate convertible monthly is 4.104%)

Answer: D

23. The problem does not state the face amount of the bond. But (as in Problem 22.) the answer is an interest rate and all of the information in the problem involves time periods, interest rates, and coupon rates, so we can select any face amount for our calculations, and the resulting yield rate will be the same. Let the face amount be 1,000.

We will first calculate the price at which Investor A sold the bond to Investor B. Investor A held the bond for 5 years after purchasing it for its face amount (1,000), and the yield over those 5 years was 5.6%, so we have:
N = 10, I/Y = 2.8, PV = -1,000, and PMT = 25. CPT FV = 1,034.08.

Five years later (on the 10th anniversary of this 20-year bond), B sold the bond to C at a price such that the yield to maturity for C was 6.5%. We can calculate that sale price as follows:
N = 20, I/Y = 3.25, PMT = 25, and FV = 1,000. CPT PV = -890.95.

To find B's yield over the 5 years the bond was owned:
N = 10, PV = -1,034.18, PMT = 25, and FV = 890.95. CPT I/Y = 1.1007.
Yield $= 2 \times 1.1007\% = 2.2014\%$

Answer: A

24. Dollar-weighted rate of return:

$$\frac{1,400-(1,000+500)}{1,000\cdot 1+500\cdot(1-70/360)}=\frac{-100}{1,402.\overline{7}}=-0.071287$$

Time-weighted rate of return:

$$\frac{X}{1,000}\cdot\frac{1,400}{X+500}$$

Since the two rates are equal, we have:

$$\frac{X}{1,000}\cdot\frac{1,400}{X+500}-1=-0.071287$$
$$1,400X=0.928713\cdot(1,000X+500,000)$$
$$1,400X-928.713X=464,356.44$$
$$X=985.29$$

Answer: E

25. The payment at t=3 is supplied by the 3-year bond only. That bond pays 1.06 per unit of face amount at t=3 (1.00 of maturity value plus 0.06 of coupon).

Face amount of 3-yr. bond = $\dfrac{10,000}{1.06} = 9,433.96$

At t=1 and t=2, the 3-year bond will pay coupons of $9,433.96 \times 0.06 = 566.04$

Remaining payment at t=2 must be supplied by the 2-year bond, which pays 1.05 per unit. at t=2.

$10,000 - 566.04 = 9,433.96$

Face amount of 2-year bond = $\dfrac{9,433.96}{1.05} = 8,984.73$

At t=1, the 2-year bond will pay a coupon of $8,984.73 \times 0.05 = 449.24$

Total payments at t=1 from 2-year and 3-year bonds = 566.04 + 449.24 = 1,015.27.

Remaining amount needed at t=1 is 10,000 – 1,015.27 = 8,984.73.

Face amount of 1-year zero-coupon bond is 8,984.73.

Answer: A

26. If the stock paid only annual dividends (a dividend of 1 at the end of the first year, 1.05 at the end of the second year, etc.), the present value of the dividends, and therefore the fair value of the stock would be:

$$1 \cdot a_{\overline{\infty}|i=10\%}^{g=0.05} = \frac{1}{0.10 - 0.05} = 20$$

However, there are 4 payments of 1 during the first year, one occurring at the end of each quarter. Their accumulated value at the end of the year can be determined by addition:

$$1.05^{0.75} + 1.05^{0.50} + 1.05^{0.25} + 1 = 4.147$$

This value can also be found by evaluating $4 \cdot s_{\overline{1}|}^{(4)}$:

$$4 \cdot s_{\overline{1}|}^{(4)} = 4 \cdot \frac{(1+i)^1 - 1}{i^{(4)}} = \frac{4i}{i^{(4)}} = \frac{4i}{4[(1+i)^{1/4} - 1]} = 4.147$$

The payments in each year after the first have values that increase by a factor of 1.05 annually. We can analyze the present value of all the payments by using the first equation above and replacing the 1 (representing a payment of 1 at the end of the first year) by 4.147 (representing the total value at the end of the first year of all 4 payments):

$$4.147 \cdot a_{\overline{\infty}|i=10\%}^{g=0.05} = \frac{1}{0.10 - 0.05} = 4.147 \cdot 20 = 82.94$$

Answer: C

27. The only factor in this list that does not affect the interest rate a bank pays on savings accounts is the **depositor's creditworthiness**. Since the bank is the borrower, it is not concerned about the credit rating of a savings customer.

The other items are all relevant in setting the rate paid on savings accounts. The bank must cover its **overhead expenses** with the margin between the rates it pays and the rates it collects, so they affect the rate paid. **Inflation** affects interest rates in the environment, including the rates that customers demand and the rates that competitor banks are paying. The **bank's credit rating** can influence the rates paid, as a poorly-rated bank may need to pay higher rates in order to encourage customers to entrust their savings to that bank. And if there is a high **demand for loans**, the bank may pay higher rates on savings in order to attract the deposits needed to make more loans.

Answer: D

28. The payment under an interest rate swap is equal to the difference between the two interest rates (the fixed rate and the variable rate) multiplied by the notional principal amount.

In this case, the fixed rate is 5.11%, the variable rate can be determined from the price of the 1-year zero-coupon bond $\left(0.9530^{-1}-1=0.0493179\right)$, and the notional amount is 1,000,000. The difference between the two rates, multiplied by 1,000,000 is $(0.0511-0.0493179)\cdot 1{,}000{,}000=1{,}782.06$. This is the difference in the amount of interest that would be earned during the 3rd year at 5.11% and at 4.93179%. Since the fixed rate is higher than the variable rate, it is an amount that the payer (the fixed rate payer) would owe to the receiver (the variable rate payer) at the end of the third year of the swap (at t=3).

Answer: D

29. -Note that it doesn't matter <u>when</u> the bond was purchased at a price to yield 8%. The only significant fact is that, at the time of the 9th coupon, the owner is tracking the bond's book value on the basis of an 8% yield to maturity.

If you recall the formula for calculating the amortization of premium or discount on a bond, you can simply calculate the answer as follows:

$$F \cdot (r-i) \cdot v^{n-t+1} = 10{,}000 \cdot \left(0.03 - 0.04\right) \cdot 1.04^{30-9+1} = 42.196$$

Or you can reason it out based on the fact that the amount of discount being amortized is the change in the bond's book value from just after the 8th coupon to just after the 9th coupon. So we can calculate the bond's book value as of those two dates and determine the difference. Since there are 30 coupon dates for this 15-year bond, the 8th and 9th coupons correspond to the dates when 22 and 21 payments remain, respectively. The book values on those dates are determined as follows:

N = 21. I/Y = 4, PMT = -300, and FV = -10,000. CPT PV = 8,597.084

N = 22, I/Y = 4, PMT = -300, and FV = -10,000. CPT PV = 8,554.888.

The book value adjustment at the time of the 9th coupon payment is:
8,597.08 – 8,554.89 = 42.196

Answer: A

30. The guaranteed interest rate (the swap rate) is the 3-year par coupon bond rate:

$$R = c = \frac{1-P_3}{P_1+P_2+P_3}$$

The known zero-coupon bond prices are:

$$P_1 = \frac{1}{1.052} = 0.951, \quad P_3 = \frac{1}{1.071^3} = 0.814$$

Thus: $$0.07 = \frac{1-.814}{0.951+P_2+0.814} \rightarrow P_2 = 0.892 \rightarrow r_2 = 0.892^{0.5} - 1 = 5.9\%$$

Answer: A

31. If Laura keeps the original loan, she has 240 monthly payments remaining on an 180,000 loan at an effective monthly rate of *i=0.055/12*= 0.0045833. Her monthly payments are:

$$P = 180{,}000 / a_{\overline{240}|0.0045833} = 1{,}238.1972$$

If she makes 240 payments of this amount, she will have paid a total amount of 297,167.32, of which 180,000 is principal repayment and 117,167.317 is interest.

If Laura refinances, she has 360 payments remaining on the 180,000 loan at an effective monthly rate of *j*=0.045/12= 0.00375. Her payments are:

$$P = 180{,}000 / a_{\overline{360}|0.00375} = 912.0336$$

If she makes 360 payments of this amount, she will have paid 328,332.08, of which 180,000 is principal and 148,332.08 is interest.

If Laura refinances, she will pay a total of 31,164.76 more in interest than if she continues to repay her mortgage on the original schedule.

Answer: A

32. Using the formula for the duration of a coupon bond to solve for the coupon rate, we have:

$$D_{\text{mac}} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{\text{Bond} \quad \text{Price}} = \frac{1{,}000r(Ia)_{\overline{10}|} + 10(1{,}000)v^{10}}{1{,}000r \cdot a_{\overline{10}|} + (1{,}000)v^{10}} = 8.2$$

$$8.2 \cdot \left(1{,}000r \cdot a_{\overline{10}|} + (1{,}000)v^{10}\right) = 1{,}000r(Ia)_{\overline{10}|} + 10(1{,}000)v^{10}$$

$$r \cdot \left(8.2 \cdot 1{,}000 \cdot a_{\overline{10}|} - 1{,}000(Ia)_{\overline{10}|}\right) = 10(1{,}000)v^{10} - 8.2 \cdot (1{,}000)v^{10}$$

$$r \cdot \left(8{,}200 \cdot 8.11090 - 1{,}000 \cdot 41.99225\right) = 1{,}216.0155$$

$$r = 0.0496$$

The bond's coupon rate is 5%. If it were increased to 6%, then the price of the bond (at a yield of 4%) would be:

$$P = 1{,}000(0.06)a_{\overline{10}|0.04} + 1{,}000v_{0.04}^{10} = 1{,}162.2179$$

Answer: C

(Note: If you used the exact calculated coupon rate of 4.96% and increased it to a coupon rate of 5.96%, the calculated price of 1,159 still leads to the same answer choice.)

33. One way to solve this is to break up the perpetuity into three perpetuities-due, each of which makes payments every 3 years. The first perpetuity-due is deferred 1 year and makes payments of 3 every third year. The second is deferred 2 years and makes payments of 2, and the third is deferred 3 years and makes payments of 1.

The annual effective rate is i = 0.034. However, the effective rate per payment period for these annuities is the 3-year effective rate, which we will call j. Its value is $j=(1.034)^3-1=0.10551$.

The present value of the combined perpetuity is:

$$PV = 3\ddot{a}_{\overline{\infty}|j}v_i + 2\ddot{a}_{\overline{\infty}|j}v_i^2 + 1\ddot{a}_{\overline{\infty}|j}v_i^3$$

$$= \frac{3}{\frac{0.10551}{1.10551}(1.034)} + \frac{2}{\frac{0.10551}{1.10551}(1.034)^2} + \frac{1}{\frac{0.10551}{1.10551}(1.034)^3} = 59.4790$$

Alternatively, we can calculate the future value of the first 3 payments as of the date of the 3rd payment, and then treat the total payments as being a perpetuity with a payment of that amount every third year:

Equivalent payment amount: $3\cdot 1.034^2 + 2\cdot 1.034 + 1 = 6.27547$

Present value of perpetuity: $6.27547 / 0.10551 = 59.4790$

Answer: C

34. First, analyzing the liabilities, we have:

$$PV(L_3) = \frac{5{,}000}{1.06^3} = 4{,}198.10$$
$$PV(L_5) = \frac{5{,}000}{1.06^5} = 3{,}736.29$$
$$Total\,PV = 7{,}934.38$$
$$PV(L_3) = 52.91\% \left(= \frac{4{,}198.10}{7{,}934.38} \right) \text{ of total.}$$
$$PV(L_5) = 47.09\% (= 1 - 52.91\%) \text{ of total.}$$

Since there is one interest rate (6%) for all of the assets and liabilities, each security's D_{mod} is equal to its D_{mac} divided by 1.06. Therefore, if the combined assets' D_{mac} equals the combined liabilities' D_{mac}, then their D_{mod}'s will also match.

D_{mac} for liabilities $= 0.5291 \cdot 3 + 0.4709 \cdot 5 = 3.9418$

So the D_{mac} for the assets must be 3.9418.
Let x be the value of the 2-year bond as a proportion (percentage) of the total asset value. Then:

$$2 \cdot x + 8 \cdot (1 - x) = 3.9418$$
$$8 - 3.9418 = 6x$$
$$x = 0.6764 \text{ (2-year bond's proportion of total present value)}$$
$$1 - x = 0.3236 \text{ (8-year bond's proportion of total PV)}$$
$$PV \text{ of bonds: } \quad 0.6764 \cdot 7{,}934.38 = 5{,}366.56 \text{ PV for 2-year bond}$$
$$0.3236 \cdot 7{,}934.38 = 2{,}567.83 \text{ PV for 8-year bond}$$
$$\text{Face amounts: } \quad 5{,}366.56 \cdot (1.06^2) = 6{,}029.87 \text{ for 2-year}$$
$$2{,}567.83 \cdot (1.06^8) = 4{,}092.73 \text{ for 8-year}$$
$$\text{Total face amount} = 6{,}029.87 + 4{,}092.73 = 10{,}122.59$$

Answer: D

35. Every central bank facilitates the operation of its country's **payment system** (answer choice A), and also acts as **lender of last resort** (answer choice B). The other two answer choices (**supervision of banks**, and **issuing currency**) may or may not be functions of a country's central bank. So the answer is "more than one of A, B, C, and D."

Answer: E

Practice Exam 6

Exam FM

Questions

1. An 8 year bond that pays semi-annual coupons at a 6% annual rate has a price of 1,050. The bond can be called at its par value of X on any coupon date starting at the end of year 6. The price guarantees that a purchaser will receive a yield of at least 5% convertible semi-annually. Calculate X.

   A) 986 B) 721 C) 999 D) 944 E) 1,276

2. The rate at which an investment grows is greatest under which of the following interest scenarios?

   A) $d = 0.056$ B) $\delta = 0.057$ C) $d^{(2)} = 0.058$
   D) $i^{(4)} = 0.059$ E) $i = 0.060$

3. Terry purchases an annuity that makes payments at the beginning of each month for 36 months. The monthly payments are a constant amount of 15 for the first 24 payments. Then the 25^{th} payment is 20, the 26^{th} payment is 25, and the payments continue to increase by 5 each period until the 36^{th} and final payment. At a nominal interest rate of 6% convertible monthly, what is the present value of this annuity?

   A) 823.1 B) 764.0 C) 829.1 D) 827.5 E) 871.6

4. A stock is currently trading at 39.35. The next dividend, payable one year from today, is expected to be 1.00. An investor is analyzing the stock's current price using the dividend growth model and assuming a future growth rate of 6% for the dividends. He determines that the stock's current price is consistent with a valuation interest rate of i. What is the value of i?

   A) 6.00% B) 6.41% C) 7.01% D) 7.71% E) 8.54%

5. Paul pays 100,000 today for a 4-year investment that returns cash flows of 60,000 at the end of each of years 3 and 4. Suppose that, at 15%, the net present value of Paul's cash flows is equal to the net present value of Kelly's cash flows, where Kelly makes an investment of X one year from today that returns cash flows of 60,000 at the end of each of years 4 and 5. Calculate X.

   A) 94,316 B) 98,503 C) 105,380 D) 103,937 E) 90,379

6. An appliance store offers to sell a television for 5,000. The current market loan rate is a nominal rate of 10% convertible monthly. As an inducement, the store offers 100% financing at a nominal rate of 6% convertible monthly. The loan is to be repaid in equal installments at the end of each month for a 3-year period.

   If the appliance store has to make monthly payments on a loan at market rates, but finances its customer with the inducement loan, how much more interest is the store paying over the 36-month period than it is charging to the customer who bought the television?

   A) 311 B) 420 C) 175 D) 332 E) 308

7. A fund earned investment income of 8,000 during 2014. The beginning and ending balances of the fund were 95,000 and 120,000 respectively. A deposit was made at time K during the year. No other deposits or withdrawals were made. The fund's dollar-weighted rate of return during 2014 was 7.5235%. Determine K.

   A) March 1 B) April 1 C) May 1 D) July 1 E) September 1

8. Andy purchases a 16 year annuity-immediate that pays 100 the first year and has payments that increase by 4% each year. Rick purchases a 16-year annuity-immediate that pays X the first year, with payments decreasing by 2% each year thereafter. At an annual effective rate of 5%, both annuities have the same present value. Calculate X.

   A) 148.7 B) 145.2 C) 124.5 D) 123.2 E) 120.0

9. Katie purchases a 15-year bond that pays semi-annual coupons at a 5% annual coupon rate. She pays 2,345 for the bond, which can be called at its par value X on any coupon date starting at the end of year 10. The price guarantees that Katie will receive a yield of at least 4% convertible semi-annually. Mark purchases a 15-year bond identical to Katie's except it is not callable. Assuming the same yield, what is the price of Mark's bond?

   A) 2,168 B) 2,170 C) 2,405 D) 2,300 E) 2,411

10. A monthly-payment car loan has an interest rate of 9%, convertible monthly. The outstanding loan balances immediately after the 12th and 13th payments are 8,608.11 and 8,398.93, respectively.

    To the nearest dollar, what will the outstanding loan balance be after the 15th payment?

    A) 7,996 B) 7,991 C) 7,986 D) 7,981 E) 7,976

11. The balance in an account 1.5 years from today will be 100 (assuming no deposits or withdrawals during that time). Find the current balance if the account earns interest based on a nominal rate of discount of 5% convertible quarterly.

    A) 86.8 B) 96.4 C) 92.7 D) 92.9 E) 92.2

12. An annuity-due pays an initial benefit of 1 per year, with the benefit increasing by 10.25% every four years. The annuity is payable for 40 annual payments. Using an annual effective rate of 2%, calculate the future value of this annuity (i.e., its value as of 40 years from now).

    A) 42 B) 69 C) 83 D) 59 E) 93

13. A company plans to raise capital by selling 30-year bonds with semi-annual coupons and a 6% (annual) coupon rate. They anticipate that investors will buy these bonds at a price to yield a 6.5% nominal annual rate, convertible semi-annually.

    Before the bonds are offered to the public, market interest rates for similar bonds rise to 7.6% (nominal annual rate, convertible semi-annually). By what percentage will the sale price of the bonds be reduced as a result of this increase in market interest rates?

    A) 12.1% B) 13.1% C) 14.1% D) 15.1% E) 16.1%

14. The following table shows the history of balances, deposits, and withdrawals for a fund:

Date	Balance Before Dep/Wdl	Deposit (+)/ Withdrawal (-)
January 1, 2016	2,000	0
June 1, 2016	2,150	-1,000
January 1, 2017	?	

If this account has a dollar-weighted return for 2016 that is equal to its time-weighted return for the same period, then the account balance on January 1, 2017, must be in which of the following ranges?
(Assume 30-day months in your calculations.)

A) 1,100 to 1,150

B) 1,150 to 1,200

C) 1,200 to 1,250

D) 1,250 to 1,300

E) A value less than 1,100 or greater than 1,300

15. A 20 year 5,000 par value bond that pays 4% annual coupons matures at par. It is purchased to yield 5% annually for the first 12 years and 6% annually thereafter. What is the amount of discount amortized with the 8th coupon payment?

A) 9 B) 15 C) 25 D) 58 E) 160

16. Todd borrows X for nine years at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding balance immediately after the fifth payment is 4,506.74. Calculate the principal repaid in the first payment.

A) 551 B) 565 C) 681 D) 574 E) 384

17. Suppose a yield curve for spot rates is given by the following equation:
$s_t = 0.08 - 0.001t + 0.002t^2$

What would be the annual effective forward interest rate for a loan originating at time t=4, with a term of 3 years?

A) 8.1% B) 10.8% C) 17.1% D) 26.1% E) 34.2%

18. Ken takes out a mortgage for 200,000. Mortgage payments are to be made monthly for 30 years with the first payment occurring one month after the loan date. The annual effective rate of interest is 5%. Starting with the 100th payment, Ken increases the amount of his monthly payments by 400 in order to repay the mortgage more quickly.

Calculate the total amount of interest paid during the duration of the loan.

A) 136,216 B) 135,215 C) 134,615 D) 131,516 E) 125,651

19. A bond is currently selling for 11,747, and at that price it provides a yield-to-maturity of 7.00% (an annual effective interest rate). The bond's Macaulay duration is 3.56.

Using a first-order Macaulay approximation, estimate what the bond's price will be if interest rates decrease by 20 basis points.

A) 11,669 B) 11,825 C) 11,830 D) 11,836 E) 11,914

20. An insurance company is examining its liabilities and the assets that support them to determine whether it is immunized against a small parallel shift in interest rates.

The company finds that its assets and liabilities have the following characteristics:

	Present Value (in Millions)	Modified Duration	Modified Convexity
Assets	600	5.7	45
Liabilities	600	5.5	42

Which of the following parallel shifts in interest rates would cause the value of these assets to be less than the value of the liabilities?

A) A small increase

B) A small decrease

C) Either an increase or a decrease

D) Neither an increase nor a decrease (i.e., the portfolio is immunized)

E) Cannot be determined from the information given

21. An annuity-immediate has 32 initial quarterly payments of 20 followed by a perpetuity of quarterly payments of 25 starting in the 9^{th} year. Find its present value at a nominal rate of 16% convertible quarterly.

A) 510 B) 165 C) 814 D) 536 E) 506

22. Which of the following statements are true?

I. The Federal Open Market Committee (FOMC) sets a target for the federal funds rate.

II. Banks that have inadequate reserves may take an overnight loan from the Federal Reserve at the federal funds rate.

III. Banks generally prefer to borrow money from the Federal Reserve rather than from other banks.

A) I only B) II only C) III only
D) I and II only E) None of A, B, C, or D

23. Brent would like to accumulate 100,000 at the end of 17 years to pay college expenses for his daughter. The annual effective rate is 6% and Brent will be making monthly payments. How much does he need to deposit each month if his first payment is today and he makes a total of 204 payments?

A) 288 B) 286 C) 285 D) 283 E) 282

24. Below is a 4-year yield curve with one missing entry.

Years to maturity	1	2	3	4
Zero Coupon Bond Yield	3.0%	4.0%		5%

The theoretically correct swap rate for a 4-year interest rate swap with a level notional amount is 4.94%. The spot rate that is missing in the above table lies in which of the following ranges?

A) 4.0% - 4.15% B) 4.15% - 4.3% C) 4.30% - 4.45%
D) 4.45% - 4.6% E) 4.60% - 4.75%

25. Suppose Chris takes out a loan of amount X that will be repaid by payments of 2,000 at the end of each year for 15 years. The total amount of interest paid on the loan is 6,124. Calculate the interest paid in the first payment.

A) 408 B) 60 C) 716 D) 672 E) 464

26. You are given an annuity-immediate paying 10 annually for twenty years. After the twenty years, the payments decrease by one per year until they reach a payment of 1. The payments of 1 continue forever. At an annual effective rate of 6%, what is the present value of this annuity?

A) 129 B) 133 C) 132 D) 131 E) 134

27. Consider the following two zero-coupon bonds:

	Remaining Term to Maturity	Maturity Value	Current Market Price
Bond 1	5 years	6,000	4,359
Bond 2	10 years	8,000	3,442

What is the modified duration (in years) of this two-bond portfolio?

A) 6.3 B) 6.7 C) 7.2 D) 7.6 E) 7.9

28. What is the modified duration of a 5-year 2,000 par value bond with 8% annual coupons and an annual effective yield of 7%?

A) 4.327 B) 4.044 C) 3.550 D) 3.802 E) 3.287

29. The present value of a perpetuity of 6,000 paid at the end of each year, plus the present value of a perpetuity of 8,000 paid at the end of every 4 years is equal to the present value of an annuity of X paid at the end of each year for 30 years. The valuation interest rate is 6% convertible quarterly.

Calculate the value of X.

A) 9,479 B) 9,400 C) 9,475 D) 9,410 E) 9,264

30. The following are prices of zero-coupon bonds with a maturity value of 100.

Term (years)	Price
1	96.23
2	94.12
3	89.23
4	84.59
5	82.48

Determine the 4-year forward rate.

A) 2.55% B) 5.20% C) 5.49% D) 12.10% E) 13.76%

31. Suppose that the following is the term structure of yields for zero coupon bonds.

Term	Zero Coupon Bond Rate
1 year	5%
2 years	7%
3 years	8%
4 years	9%

A four-year 1,000 face value bond with annual coupons has a price of 895. Calculate the annual coupon rate using the yield rates given above.

A) 7.3% B) 7.0% C) 6.5% D) 6.0% E) 5.6%

32. A 1,000 par-value bond is bought to yield a nominal rate of 7% convertible semi-annually. The bond will be redeemed for 1,100 at maturity and pays semi-annual coupons at a coupon rate of 6% convertible semi-annually. The present value of the coupons is 426.50. What is the price of the bond?

A) 905 B) 913 C) 929 D) 979 E) 984

33. Mary has a 6,000 loan and is being charged an interest rate of 8% convertible monthly for a term of 4 years. At the end of each month, Mary pays only half of the interest due. However, with the 10^{th} interest payment, Mary pays an additional 3,000. What is her outstanding balance immediately after the 10^{th} payment?

A) 3,000 B) 3,102 C) 3,203 D) 4,257 E) 5,881

34. An investor's required real rate of return is 3% continuously compounded. The continuously compounded inflation rate for the next 2 years is known to be 2%. The investor makes a 2-year loan that has a 10% probability of default, with an expected recovery of 25% in the event of default.

    What is the minimum continuously compounded interest rate that could be acceptable to this investor for this loan?

    A) 5.9%   B) 6.7%   C) 7.8%   D) 8.9%   E) 9.5%

35. A 4-year interest rate swap has a notional principal amount of 1,000,000. The swap has annual settlement periods, with payments to be made at the end of each year for four years. The fixed payments are based on a swap rate of 5.0%. The variable payments are based on the spot price for 1-year zero-coupon Treasury securities.

    At t=2 (two years after the swap is initiated), the prices (per 1,000 of maturity value) for 1-year and 2-year zero-coupon Treasury securities are 951.00 and 902.00, respectively.

    Based on these prices for Treasury securities, what is the market value of the swap for the payer?

    A) 585   B) 535   C) -535   D) -585   E) A different value

Solutions

1. A bond's price is calculated assuming the worst case scenario for the investor. Since this is a callable bond (with no call premium) that selling at a premium, the price is based on the assumption that it will be called at the earliest call date, i.e., at the time of the 12th coupon payment. So we can write:

$$1{,}050 = X(0.03)a_{\overline{12}|0.025} + Xv^{12}$$

$$X = \frac{1{,}050}{(0.03)a_{\overline{12}|0.025} + v^{12}} = 998.7741$$

Answer: C

2. We will calculate the equivalent annual effective rate for each variable.

A) $d = 0.056$ $\quad 1+i = \dfrac{1}{1-d} = \dfrac{1}{1-0.056} = 1.0593$

$i = 5.93\%$

B) $\delta = 0.057$ $\quad 1+i = e^{\delta} = e^{0.057} = 1.0587$

$i = 5.87\%$

C) $d^{(2)} = 0.058$ $\quad v = (1-\dfrac{d^{(2)}}{2})^2 = (1-0.029)^2 = 0.942841 \qquad 1+i = 1/v = 1.0606$

$\underline{i = 6.06\%}$ This is the largest value of i.

D) $i^{(4)} = 0.059$ $\quad 1+i = \left(1+\dfrac{i^{(4)}}{4}\right)^4 = \left(1+\dfrac{0.059}{4}\right)^4 = 1.0603$

$i = 6.03\%$

E) $i = 0.060$

$i = 6.00\%$

Answer: C

3. We can think of this as a 36-month annuity-due with payments of 15, plus a deferred arithmetic increasing annuity-due with payments of 5, 10, 15, etc., starting with the 25th payment.

$$i = \frac{0.06}{12} = 0.005$$

$$PV = 15\ddot{a}_{\overline{36}|i} + v^{24}5(I\ddot{a})_{\overline{12}|i} = 495.5305698 + 333.6115032 = 829.1421$$

Answer: C

4. To price a stock using the dividend growth model:

$P = \dfrac{Div}{(i-g)}$, where g is the constant percentage growth rate.

$$39.35 = \frac{1}{(i-0.06)} \qquad \rightarrow \qquad i = 0.085413$$

Answer: E

5. First calculate the net present value of Paul's cash flows. There is a cash flow of −100,000 at time 0, and cash flows of 60,000 at time 3 and 4. So,

$$NPV = -100,000 + \frac{60,000}{1.15^3} + \frac{60,000}{1.15^4} = -26,243.83132$$

Now set this net present value equal to the present value of Kelly's cash flows, which consist of a value of −X at time 1, and cash flows of 60,000 at time 4 and 5.

$$-26,243.83132 = \frac{-X}{1.15} + \frac{60,000}{1.15^4} + \frac{60,000}{1.15^5} \quad \rightarrow \quad X = 103,936.5747$$

(Note: All of these calculations can be done in the Cash Flow worksheet. (In what follows, all of the values are divided by 1,000 for convenience.) To find the present value of Paul's investment, set CF_0=-100, C01=0, F01=2, C02=60, and F02=2. Then press NPV, set I=15, press the down arrow and CPT NPV, which is -26.2438. Save this value in a memory (e.g., STO 4). Then change the values in the Cash Flow worksheet to CF_0=0, C01=0, F01=3, C02=60, and F02=2. Calculate the NPV as before, this time getting a value of 64.1358. This is the present value of Kelly's returns, ignoring her investment of X. With her investment of X at time 1, her NPV of 64.1358 will be reduced to the level of Paul's NPV: -26,2438. That is, X has an NPV at time 0 of 64.1358-(-26.2438)=90.3796. At time 1 (when cash flow X occurs), this value is 90.3796 (1.15)=103.9365, so X is 103,936.57.)

Answer: D

6. First calculate the monthly payment for each of the loans. Then find the total cost for each loan over the 36 months and calculate the difference.

   For the market rate loan, you can use the TVM worksheet on the BAII Plus or apply the logic below to get the monthly payment and total payments:

   $5{,}000 = (Pmt)a_{\overline{36}|0.10/12}$ $\qquad Pmt = 161.34$

   Total payments = 161.34*36=5,808.24

   For the inducement loan, again you can use BAII Plus or the logic below to get the payment and total payments:

   $5{,}000 = (Pmt)a_{\overline{36}|0.06/12}$ $\qquad Pmt = 152.11$

   Total payments = 152.11*36=5,475.96

   The cost to the dealer for the inducement, in terms of net payments (the difference in interest paid), is: 5,808.24 − 5,475.96 = 332.28

   Answer: D

7. The beginning and ending balances were 95,000 and 120,000. We also know that the investment income was 8,000, so the deposit must have been 17,000. (The balance increased by 25,000, of which 8,000 was investment income, so the amount deposited was 17,000.) The dollar-weighted rate of return is:

   $$\frac{8{,}000}{95{,}000 + 17{,}000(1-K)} = 0.075235$$

   This leads to K=0.3333, so the deposit is made after 4 months. The answer is May 1.

   Answer: C

8. First calculate the present value of Andy's annuity:

   $$PV = 100 \cdot a_{\overline{16}|0.05}^{4\%} = 100 \cdot \frac{1-(1.04/1.05)^{16}}{0.05-0.04} = 1{,}419.6571$$

   Then, set this present value equal to the present value of Rick's annuity:

   $$1{,}419.6571 = X \cdot a_{\overline{16}|0.05}^{-2\%} = X \cdot \frac{1-(0.98/1.05)^{16}}{0.05-(-0.02)} = 9.548856X$$

   $$X = 1{,}419.6571 / 9.548856 = 148.67$$

   Answer: A

9. Since the coupon rate on Katie's bond is higher than the yield rate, it is a premium bond and therefore should be priced assuming it will be called at the earliest possible call date, i.e., at 10 years. The price of the bond is given as 2,345, so we have:

$$2{,}345 = X(0.025)a_{\overline{20}|0.02} + Xv_{0.02}^{20}$$
$$2{,}345 = X[0.025a_{\overline{20}|0.02} + v_{0.02}^{20}]$$
$$2{,}345 = X(1.081757)$$
$$X = 2{,}167.7693$$

Mark's bond is not callable, so it is priced based on the full term of 15 years:

$$P = 2{,}167.77(0.025)a_{\overline{30}|0.02} + 2{,}167.77v_{0.02}^{30}$$
$$P = 2{,}410.52$$

(Note: This demonstrates that a non-callable bond can have a significantly higher value than an otherwise similar callable bond.)

Answer: E

10. The monthly effective interest rate for this loan is $\dfrac{9\%}{12} = 0.75\%$.

$\left(i^{(12)} = 9\%;\ i^{(12)}/12 = 0.75\% \text{ is the monthly effective rate.}\right)$

We can determine the amount of the monthly payment as follows:
Bal_{12} (1+i) – Pmt = Bal_{13} (i represents the <u>monthly</u> effective rate, 0.75%)
8,608.11 (1.0075) – Pmt = 8,398.93
Pmt = 273.74

We can then determine Bal_{15} by applying the above "successive value formula" two more times, to move from Bal_{13} to Bal_{15}:
Bal_{13} (1+i) – Pmt = 8,398.93(1.0075) – 273.74 = 8,188.18 = Bal_{14}
Bal_{14} (1+i) – Pmt = 8,188.18 (1.0075) – 273.74 = 7,975.76 = Bal_{15}

Alternatively, once we know that the monthly payments are 273.74, we can find the remaining term of the loan after 12 payments:
Set I/Y=0.75, PV=8,608.11, PMT=273.74, and FV=0. CPT N = 36
There are 36 payments remaining after the 12^{th} payment, so there will be 33 payments remaining after the 15^{th} payment.
Set N=33. CPT PV = 7,976.

Answer: E

11. We are given that $d^{(4)} = 0.05$. The 1.5-year (6-quarter) accumulation factor based on that rate is $\left(1-0.05/4\right)^{-6} = 1.07839$. So the current balance in the account is: $100/1.07839 = 92.73$.

Answer: C

12. One way to solve this problem is to treat the payments of each 4-year period as a single payment made at the beginning of that period and having the same present value as the 4 payments. We can then calculate the future value of an annuity-due that makes payments of those amounts every 4th year:

$$PV = \ddot{a}_{\overline{4}|0.02} + v^4(1.1025)\ddot{a}_{\overline{4}|0.02} + v^8(1.1025)^2\ddot{a}_{\overline{4}|0.02} + \cdots + v^{36}(1.1025)^9\ddot{a}_{\overline{4}|0.02}$$

$$PV = \ddot{a}_{\overline{4}|0.02}[1 + v^4(1.1025) + v^8(1.1025)^2 + \cdots + v^{36}(1.1025)^9]$$

$$PV = \ddot{a}_{\overline{4}|0.02}[\frac{1-(v^4(1.1025))^{10}}{1-v^4(1.1025)}] = 42.2448$$

$$FV = 42.2448(1.02)^{40} = 93.2782$$

Alternatively, we can observe that this situation is equivalent to a 10-year annuity-due with quarterly payments that increase once per year. In that case, the <u>annual</u> increase is 10.25% and the <u>quarterly</u> effective interest rate is 2%. We can find the future value using the formula for a geometric annuity:

$$FV = 4 \cdot s_{\overline{10}|i}^{10.25\%(4)} = 4 \cdot s_{\overline{10}|i}^{10.25\%} \cdot \frac{i}{d^{(4)}}, \text{ where } i = 1.02^4 - 1 = 8.2432\%$$

$$= 4 \cdot \frac{1.082432^{10} - 1.1025^{10}}{0.082432 - 0.1025} \cdot \frac{0.082432}{4 \cdot \left(1 - 1.082432^{-1/4}\right)} = 93.28$$

Answer: E

13. Using the BA II Plus, calculate the bond's price at 6.5% and at 7.6%:
Set N=60, I/Y=3.25, PMT=30, and FV=1,000. CPT PV = -934.37.
Store this value (e.g., STO 4).
Then reset I/Y=3.8 and CPT PV = -811.94.

Divide by the stored value (the original price at 6.5%) and calculate the percent reduction in value. The keystrokes are:
$\div$ RCL 4 = $+/-$ $+1$ = 0.13103
The percentage decrease in price is 13.1%.

Answer: B

14. First write expressions for the time-weighted and dollar-weighted rates of return:

$$i^{TW} = \frac{2{,}150}{2{,}000} \cdot \frac{x}{1{,}150} - 1 = 0.000934783 \cdot x - 1$$

$$i^{DW} = \frac{1{,}000 + x - 2{,}000}{2{,}000 \cdot 1 - 1{,}000 \cdot \frac{7}{12}} = \frac{x - 1{,}000}{1{,}416.6\overline{6}}$$

Then equate the two expressions and solve for x:

$$0.000934783 \cdot x - 1 = \frac{x - 1{,}000}{1{,}416.6\overline{6}}$$

$$1.32428 \cdot x - 1{,}416.6\overline{6} = x - 1{,}000$$

$$x = 1{,}284.92$$

Answer: D

15. The accumulation (or "accrual") of discount with each coupon payment is equal to the amount of interest earned during the coupon period, minus the amount of the coupon payment. If we think of the bond as a loan, then the amount that the borrower (the issuing corporation) owes to the lender (the bondholder) increases with interest (at the yield rate) during each coupon period, and then is reduced by the amount the borrower pays (the coupon). For this bond, the 4% coupon is less than either of its yield rates (5% for 12 years, then 6%), so the bondholder will be "capitalizing" the unpaid interest at the end of each coupon period.

To determine the amount of interest capitalized at time 8, i.e., the amount of discount "accrued" ("amortized") we can calculate the interest earned during the 8th year and subtract the 8th coupon payment. To do this, we start with the bond's book value at time 7. At that time, there are 5 remaining years at a 5% yield, followed by 8 years at a 6% yield, so the bond's value is:

$$200a_{\overline{5}|0.05} + v_{0.05}^{5}200a_{\overline{8}|0.06} + v_{0.05}^{5}v_{0.06}^{8}5{,}000 = 4{,}296.97$$

The interest in the 8th coupon is: 4,296.97*0.05=214.8486. Each coupon is 200, so the principal portion of the 8th coupon is: 200-214.8486=-14.8486. The amount of interest "capitalized" (the amount of discount amortized) is 14.85.

Answer: B

16. The outstanding balance after the 5th payment is 4,506.74. We can use this to find the payment amount by treating this balance as the amount of a 4-year loan:

$4,506.74 = Pmt(a_{\overline{4}|0.08})$ $\quad Pmt = 4,506.74 / \left[\left(1-1.08^{-4}\right)/0.08\right] = 1,360.6786$

Next, we can solve for the initial loan amount, when there were 9 payments remaining:

$$PV = 1,360.68(a_{\overline{9}|0.08}) = 1,360.68 \cdot \left[\left(1-1.08^{-9}\right)/0.08\right] = 8,500.02$$

Finally, calculate the interest portion of the first payment:

$$8,500.02 \cdot 0.08 = 680.00$$

The interest portion of the payment is 680, so the principal portion is:

$$1,360.68 - 680 = 680.68$$

(Note: the solutions for Pmt and PV above can be done entirely on the BA II Plus to save time.)

Answer: C

17. We need to calculate $i_{4,7}$, which can be found from $(1+i_{4,7})^3 = \dfrac{(1+s_7)^7}{(1+s_4)^4}$.

First, we need to calculate s_7 and s_4.

Using the equation for the spot rates of the yield curve, we have:

$$s_7 = 0.08 - 0.001 * 7 + 0.002 * 7^2 = 0.171$$

$$s_4 = 0.08 - 0.001 * 4 + 0.002 * 4^2 = 0.108$$

Then:

$$(1+i_{4,7})^3 = \frac{(1+s_7)^7}{(1+s_4)^4} = \frac{1.171^7}{1.108^4} = 2.0032688$$

$$(1+i_{4,7}) = 2.0032688^{1/3} = 1.2606071 \qquad i_{4,7} = 26.06\%$$

Answer: D

18. First calculate the monthly payment amount for a 30-year loan of 200,000 at a 5% annual effective interest rate using the BA II Plus or by applying formulas, as follows:

$$i = (1.05)^{1/12} - 1 = .004074 = 0.4074\%$$
$$200{,}000 = Pmt \cdot a_{\overline{360}|0.4074\%} \qquad Pmt = 1{,}060.11$$

Next calculate the outstanding balance after the 99th payment:

$$Bal_{99} = 200{,}000(1+i)^{99} - 1{,}060.11 s_{\overline{99}|} = 170{,}162.81$$

Then add 400 to the payment and see when the loan will be paid off:

$$170{,}162.81 = \left(1{,}060.11 + 400\right) a_{\overline{n}|i} = 1{,}460.11 \cdot \left(1 - 1.004074^{-n}\right) / 0.004074$$
$$\left(170{,}162.81 \cdot 0.004074\right) / 1{,}460.11 = 1 - 1.004074^{-n}$$
$$1.004074^{n} = 1 / \left[1 - \left(170{,}162.81 \cdot 0.004074\right) / 1{,}460.11\right] = 1.904046$$
$$n = -\ln 1.904046 / \ln 1.004074 = 158.3880$$

There will be 158 more payments, which will be 1,460.11 each, and then one last partial payment. To find the amount of the partial payment, we need the balance after the 158th payment (actually, the 257th payment, if we include the first 99):

$$Bal_{158} = 170{,}162.81(1+i)^{158} - 1{,}460.11 s_{\overline{158}|} = 564.89$$

The last payment includes this outstanding balance plus interest for one period. So the last payment is $564.98(1+i) = 567.28$.

In total, there are 99 payments of 1,060.11, then 158 payments of 1,460.11, and a final payment of 567.27. The total amount paid is 336,215.55, and the amount of interest paid is 336,215.55 – 200,000 = 136,215.55.

This problem can also be solved entirely on the BA II Plus:
N=360, I/Y=0.004074, PV=200,000, and FV=0. CPT PMT = -1,060.11.
N=261 (no. of pmts. remaining after 99 pmts.) CPT PV = 170,162.81.
PMT=-1,460.11, CPT N = 158.3880.
N=158, CPT FV = -564.98.

$$564.89 \times 1.004074 + 158 \times 1{,}460.11 + 99 \times 1{,}060.11 = 336{,}215.55$$
$$336{,}215.55 - 200{,}000 = 136{,}215.55$$

Answer: A

19. The effect on a bond's price of a change in the market interest rate can be estimated using either the bond's modified duration (D_{mod}) or its Macaulay duration (D_{mac}). In this case, we are to use the first-order Macaulay approximation method to estimate a bond's price after its yield decreases by 20bp (0.20%). The bond's initial price is 11,747 and its D_{mac} is 3.56.

    Applying the formula for a first-order Macaulay approximation, we have:

$$P = 11{,}747 \cdot \left(\frac{1.07}{1.068}\right)^{3.56} = 11{,}825.50$$

    Answer: B

20. Here is an analysis of each of the answer choices:

    A) Small increase: Assets will decrease in value more than liabilities, because assets have a larger duration.

    B) Small decrease: The assets' value will increase more than the liabilities' value due to their larger duration.

    C) Increase or decrease: This would be the answer if the durations were equal and the assets had a smaller convexity than the liabilities.

    D) Neither: This would be the answer if the durations were equal and the assets had a larger convexity than the liabilities. In that case, the portfolio is immunized against small parallel shifts in interest rates.

    Answer: A

21. This can be analyzed as a 32-payment annuity with a payment amount of 20, followed by deferred perpetuity with a payment amount of 25:

$$PV = 20a_{\overline{32}|0.04} + \frac{25}{0.04}v^{32} = 357.4710 + 178.1612 = 535.6322$$

    Alternatively, we can consider it as a perpetuity of 25 minus a 32-payment annuity with a payment amount of 5:

$$PV = \frac{25}{0.04} - 5a_{\overline{32}|0.04} = 625 - 89.3678 = 535.6322$$

    Answer: D

22. Statement I is true. The FOMC sets a target for the federal funds rate, but the actual rates charged are decided by the banks when they lend to each other.
Statement II is false. The interest rate for loans from the Fed is called the "discount rate." The federal funds rate is the rate that banks charge each other for interbank loans.
Statement III is false. A bank typically prefers to borrow from other banks, because it can borrow at a lower rate (the federal funds rate, rather than the discount rate), and also because it shows that the other banks considers it creditworthy.

Answer: A

23. Using formulas (and noting that this is an annuity-due), we have:

$$i = (1.06)^{1/12} - 1 = .004868$$

$$100,000 = Pmt(\ddot{s}_{\overline{204|}i})$$

$$Pmt = 100,000 / \left[\left(1.004868^{204} - 1\right) / \left(0.004868 / 1.004868\right)\right] = 286.1561$$

Or using the BA II Plus with its TVM worksheet set for END mode (even though this is an annuity-due), we can perform the calculations as follows:
Set N=204, I/Y=0.4868, PV=0, and FV=100,000. CPT PMT = -287.5490.
$287.5490 \div 1.004868 = 286.1561$

(The PMT that was calculated, -287.5490, is the amount that would be required if deposits were made at the end of each month. But since the first monthly payment is being made "today" (at time 0), this is an annuity-due and the payments are made one month earlier. So we can divide 287.5490 by the one-month accumulation factor, 1.004868, to find the monthly deposit.)

Answer: B

24. The swap rate for this swap is found by the formula:

$$R = \frac{1 - P_4}{P_1 + P_2 + P_3 + P_4}$$

Thus we have

$$0.0494 = \frac{1 - (1.05)^{-4}}{1.03^{-1} + 1.04^{-2} + (1 + r_3)^{-3} + 1.05^{-4}}$$

This gives
$(1 + r_3)^{-3} = 0.872534 \rightarrow r_3 \approx 4.65\%$

Answer: E
Note: Answers will vary somewhat according to the degree of precision used. That is why the choices were given in ranges.

25. First calculate the loan amount. We know that the total interest is 6,124 and that 15 payments of 2,000 were made. So, the loan amount, *X*, is the difference: $X = 15 * 2{,}000 - 6{,}124 = 23{,}876$

Now solve for the interest rate:
Set N=15, PV=23,876, PMT=-2,000, and FV=0. CPT I/Y = 2.999925.
The interest rate is 3%, and the amount of interest in the first payment is:
$23{,}876 \cdot 0.03 = 716.28$

Answer: C

26. This series of payments consists of a 20-year annuity-immediate paying 10, a 20-year-deferred 9-year decreasing annuity-immediate, and then a 29-year-deferred perpetuity paying 1.

$$PV = 10a_{\overline{20}|0.06} + v^{20}(Da)_{\overline{9}|0.06} + v^{29}\left(\frac{1}{0.06}\right)$$
$$PV = 114.6992 + 11.4240 + 3.0759 = 129.1992$$

Answer: A

27. Given the terms, maturity values, and prices of the bonds, we can calculate their current yields to maturity and their modified durations:

$$\text{Bond 1: } \left(\frac{6{,}000}{4{,}359}\right)^{1/5} - 1 = 0.06599 \qquad D_{\text{mod}} = \frac{5}{1.06599} = 4.6905$$

$$\text{Bond 2: } \left(\frac{8{,}000}{3{,}442}\right)^{1/10} - 1 = 0.08800 \qquad D_{\text{mod}} = \frac{10}{1.08800} = 9.1912$$

The modified duration of the portfolio is the weighted average of these two durations, using each bond's value as its weight:

$$D_{\text{mod}}^{portfolio} = \frac{4{,}359 \cdot 4.6905 + 3{,}442 \cdot 9.1912}{4{,}359 + 3{,}442} = 6.6763$$

Answer: B

28. We will use the formula for Macaulay duration to find D_{mac} for the bond, then convert from Macaulay duration to modified duration.
To find the price of the bond, set N=5, I/Y=7, PMT=160, and FV=2,000.
CPT PV = -2,082.0039.

$$D_{\text{mac}} = \frac{160(Ia)_{\overline{5}|0.07} + 5(2{,}000)v^5}{2{,}082.0039} = 4.327254$$

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{4.327254}{1.07} = 4.044163$$

Answer: B

29. The present value of the two perpetuities is

$$\frac{6{,}000}{1.015^4 - 1} + \frac{8{,}000}{1.015^{16} - 1} = 127{,}519.2892$$

This is also the present value of the 30-year annuity:

$$i = 1.015^4 - 1 = 0.06136$$

$$127{,}519.2892 = X \cdot a_{\overline{30}|0.06136}$$

$$X = \frac{127{,}519.2892}{\left(1 - 1.06136^{-30}\right) / 0.06136} = 9{,}399.70$$

Answer: B

30. The 4-year forward rate, $i_{4,5}$, can be found from:

$$i_{4,5} = \frac{P_4}{P_5} - 1$$

where P_4 and P_5 are the zero-coupon bond prices per unit.

Using the zero-coupon bond prices per 100 from the table, we can simply write:

$$i_{4,5} = \frac{84.59}{82.48} - 1 = 0.0255$$

Answer: A

31. The price of the bond will be the present values of the coupons plus the present value of the redemption value. Let X be the coupon amount.

$$PV = \frac{X}{1.05} + \frac{X}{1.07^2} + \frac{X}{1.08^3} + \frac{X}{1.09^4} + \frac{1{,}000}{1.09^4} = 895$$
$$895 - \frac{1000}{1.09^4} = X\left(\frac{1}{1.05} + \frac{1}{1.07^2} + \frac{1}{1.08^3} + \frac{1}{1.09^4}\right)$$
$$X = 56.0608$$

With a calculated coupon of 56.0608 and a face value of 1,000, the bond's annual coupon rate is 5.6%.

Answer: E

32. The coupons are 1,000(0.03) = 30, the effective yield per coupon period is 3.5%, and the present value of the coupons is 426.50. From this information, we can solve for n, the number of coupon periods:

$$426.50 = 30a_{\overline{n}|0.035} = 30 \cdot \frac{1-1.035^{-n}}{0.035}$$

$$1.035^{-n} = 1 - \frac{426.5 \cdot 0.035}{30} = 0.50242$$

$$n = -\frac{\ln 0.50242}{\ln 1.035} = 20.0086$$

Alternatively, I/Y=3.5, PV=-426.50, PMT=30, and FV=0. CPT N = 20.

Now price the bond with n = 20 and a redemption value of 1,100:

$$P = 426.50 + 1{,}100v_{0.035}^{20} = 979.16$$

Alternatively, change FV to 1,100 and CPT PV=-979.16

Answer: D

33. Each period Mary pays only half of the interest that has accrued at a monthly rate of 0.08/12=0.6667%. Since she pays only half of the interest, the other half (or 0.3333% of the loan balance) increases the outstanding balance of the loan, so the amount she owes increases by a factor of 1.003333 each period. (This is an example of negative amortization.)

After 10 payment periods, the loan balance has grown to:

$$6{,}000 \cdot 1.03333^{10} = 6{,}203.03$$

Since Mary pays 3,000 on this date (in addition to her usual half-of-the-interest payment), the loan balance reduces to 3,203.03.

Answer: C

34. For every dollar lent, the investor wants to receive a payment (2 years later) with buying power equal to $e^{2\cdot(0.03)} = 1.06184$ dollars.
Because the inflation rate is known to be 2% continuously compounded, the investor needs to receive $1.03184 \cdot e^{2\cdot(0.02)} = 1.10517$ dollars at time 2 for each dollar lent at time 0 in order to achieve the desired rate of return.

If 10% of the contractual repayments will be in default, and only 25% of the value of those defaults will be recovered, then 7.5% of the repayments will be lost, and the investor will receive only 92.5% of the total repayment amount. Therefore, the repayment amount for each dollar lent must be at least $1.10517 / 0.925 = 1.19478$.

For a 2-year loan, this corresponds to a continuously compounded annual rate of $(\ln 1.19478) / 2 = 0.08898$.

Answer: D

35. As of t=2, the payment to be made at t=3 can be calculated exactly. The payment at t=4 has to be valued based on the 1-year forward rate, which is not necessarily the rate that will apply when that payment is determined (based on the 1-year spot rate at t=3).

The payment at t=3 is:

$$\text{Pmt}_3 = \left[\left(\frac{1}{0.951} - 1\right) - 0.05\right] \cdot 100{,}000 = 152.47$$

The present value of this payment is $152.47 \cdot 0.951 = 145.00$

The projected payment at t=4, based on the 1-year forward rate as of t=2, is:

$$\text{Pmt}_4 = \left[\left(\frac{1}{0.902 / 0.951} - 1\right) - 0.05\right] \cdot 100{,}000 = 432.37$$

The present value of this payment is $432.37 \cdot 0.902 = 390.00$

Total PV = 145 + 390 = 535
Each of these payments is an amount that the fixed-rate payer (considered the “payer” under the swap) will receive from the variable-rate payer (considered the “receiver” under the swap). Therefore, the “payer” has a position with a market value of 535.

Alternative Solution:
PV(variable payments) = $100{,}000 \cdot (1 - 0.902) = 9{,}800$
PV(fixed payments) = $100{,}000 \cdot (0.05)(0.951 + 0.902) = 9{,}265$
PV of expected (net) payments by variable rate payer = 9,800 – 9,265 = 535

Answer: B

Practice Exam 7

Exam FM

Questions

1. A corporation receives an interest-only loan of 1,000,000 from a bank. The corporation agrees to make interest payments to the bank at the end of each quarter for 25 years (a total of 100 payments), plus a single principal payment of 1,000,000 at the end of 25 years. During the 25-year term of the loan, the corporation also makes level quarterly deposits to a sinking fund in order to accumulate 1,000,000 to repay the loan at the end of 25 years. Interest is credited to the balance in the sinking fund at a nominal rate of 5% per annum, compounded quarterly.

   During the 10^{th} year of the loan (i.e., between the 9^{th} and 10^{th} anniversaries of the date when the loan was initiated), the net interest paid by the corporation is 80,000. To the nearest 0.1%, what interest rate (convertible quarterly) is the bank charging the corporation on this interest-only loan? (Note: "Net interest paid" is the amount of interest the corporation pays to the bank on the loan, less the amount of interest the corporation earns on the balance in the sinking fund.)

   A) 6.8% B) 7.8% C) 8.4% D) 9.2% E) 11.2%

2. A 30 year 10,000 bond pays 3% annual coupons and matures at par. It is purchased to yield 5% for the first 15 years and 7% thereafter. Calculate the price of the bond.

   A) 5,848 B) 6,172 C) 5,637 D) 6,418 E) 4,862

3. George borrows X for 20 years at a nominal rate of 12% convertible monthly, to be repaid with equal payments at the end of each month. The outstanding balance immediately after the 10^{th} payment is 297,000. How much total interest will George pay for this loan?

   A) 793,243 B) 658,660 C) 493,069 D) 487,854 E) 300,175

4. Suppose a total of 30 semi-annual payments of amount 5 are made starting exactly six years from today. Assuming an annual effective rate of 6%, what is the future value of these payments at a time 30 years from today? (Assume that after the payments end, the investment is left in the same account earning interest.)

   A) 708 B) 411 C) 243 D) 399 E) 450

5. Joel buys a newly-issued 10-year bond at a price that assures him a yield of at least 7% per annum, convertible semi-annually. The bond has a face amount (and maturity value) of 1,000 and pays semi-annual coupons at an 8% (annual) rate. It is callable with a 5% call premium on any coupon date on or after its 5^{th} anniversary.

   What actual yield does Joel earn on this bond if it is called and redeemed for its face amount after 8 years?

   A) 6.8% B) 7.0% C) 7.2% D) 7.3% E) 7.5%

6. Andy deposits X into an account that earns a 10% annual effective interest rate for 3 years and then a nominal interest rate of 5% convertible semi-annually for the 3 years after that. If, after the 6 years, his future value is 200,000, how much interest did he earn during the 3^{rd} year?

   A) 15,678 B) 18,750 C) 129,571 D) 24,200 E) 56,130

7. An association had an initial balance of 200 on Jan 1 and also had deposits of 25 on March 31^{st}, June 30^{th}, and September 30^{th}. The association had a withdrawal of 30 on Feb 28^{th}, a withdrawal of 60 on June 30^{th}, and ended with a balance of 250 on December 31^{st}. Calculate their dollar-weighted rate of return.

   A) 23.34% B) 32.10% C) 35.62% D) 39.18% E) 42.99%

8. You are given the following data for zero-coupon Treasury bonds:

Years to maturity	Price per 100 maturity value
0.5	98.06
1.0	95.79
1.5	93.34
2.0	91.05

The price for a 2-year Treasury note with semi-annual coupons is 102.02 per 100 of par value.

What is the annual coupon rate for this 2-year note?

A) 5.7% B) 5.8% C) 5.9% D) 7.0% E) 7.6%

9. At time t=0, Mark puts a one-time deposit of 1,000 into a fund crediting interest at an annual effective rate i.

At time t=2, Louis puts a one-time deposit of 1,000 into a different fund crediting interest at a force $\delta_t = \dfrac{1}{3+t}$.

At time t=18, the amounts in each fund will be equal. Calculate i.

A) 2.9% B) 5.3% C) 8.3% D) 9.4% E) 10.5%

10. Money accumulates in a fund at an annual effective interest rate of i during the first 6 years and at an annual effective rate of $3i$ thereafter. A deposit of 1 is made into the fund at time zero. It accumulates to 1.84 at the end of 11 years and 2.83 at the end of 16 years. What is its accumulated value at the end of 4 years?

A) 1.09 B) 1.13 C) 1.22 D) 1.31 E) 1.55

11. An annuity-due has 40 quarterly payments of 50, followed immediately by a perpetuity with quarterly payments of X. If the present value at an annual effective rate of 16% is 2,000, what is the value of X?

A) 151 B) 157 C) 167 D) 179 E) 194

12. A perpetuity-immediate has annual payments of 100. At an annual effective interest rate i, the present value of this perpetuity is 1,250.

    What is the Macaulay Duration of this perpetuity based on this same interest rate i?

    A) 11.5 B) 12.0 C) 12.5 D) 13.0 E) 13.5

13. A 1,000 par value bond that pays 8% annual coupons has an annual effective yield of i, where $i > 0$. The bond's book value at the end of year 3 is 1,099.84, and its book value at the end of year 5 is 1,087.27. Calculate i.

    A) 7.3% B) 6.7% C) 6.2% D) 5.9% E) 5.5%

14. Consider the following 3 series of payments:

    a. $10 \cdot (Ia)_{\overline{11}|}$, an 11-year increasing annuity with payments of 10, 20, …, 110
    b. an 11-year annual-coupon bond with a face amount of 100 and a coupon rate of 10%
    c. $10 \cdot a_{\overline{11}|}^{27.1\%}$, an 11-year geometric annuity with an initial payment of 10 and an annual growth rate of 27.1%

    Each of these series consists of 11 payments, and in each case the first payment is 10 and the final payment is 110.

    Let $D_{\text{mod}}^{\text{a}}$, $D_{\text{mod}}^{\text{b}}$, and $D_{\text{mod}}^{\text{c}}$ be the modified durations for annuities a., b., and c., respectively, based on an annual effective interest rate of 10%.

    Which of the following correctly describes the relationship among the modified durations for these 3 annuities?

    A) $D_{\text{mod}}^{\text{a}} > D_{\text{mod}}^{\text{b}} > D_{\text{mod}}^{\text{c}}$

    B) $D_{\text{mod}}^{\text{a}} > D_{\text{mod}}^{\text{c}} > D_{\text{mod}}^{\text{b}}$

    C) $D_{\text{mod}}^{\text{b}} > D_{\text{mod}}^{\text{a}} > D_{\text{mod}}^{\text{c}}$

    D) $D_{\text{mod}}^{\text{b}} > D_{\text{mod}}^{\text{c}} > D_{\text{mod}}^{\text{a}}$

    E) $D_{\text{mod}}^{\text{c}} > D_{\text{mod}}^{\text{a}} > D_{\text{mod}}^{\text{b}}$

15. At time zero, Sal makes a deposit of 300 into an account earning a nominal annual interest rate of 3% compounded monthly. At the same time, Rick makes a deposit of 250 into another account earning an annual effective interest rate of i. During the 2nd year, both accounts earn the same amount of interest. What is the amount of interest that Rick's account will earn during the 6th year?

A) 8.9 B) 9.8 C) 10.8 D) 11.2 E) 14.5

16. The time-weighted rate of return for the fund with the transactions in the table below is 12%. What is the dollar-weighted rate of return?

Date	***Value before transaction***	***Deposits***	***Withdrawals***
1/1/2015	980		
6/1/2015	1,010	30	
10/1/2015	1,055		X
1/1/2016	1,060		

A) 11.78% B) 11.87% C) 12.00% D) 12.25% E) 12.52%

17. The common stock of Acme Corporation pays semi-annual dividends on April 1 and October 1. As of January 1, the next dividend is expected to be 1.50 per share. The company's dividend payout is expected to increase by 4% each April 1. (The October 1 dividend each year will be equal to the amount paid the prior April 1.)

Based on the dividend growth model of stock valuation, and using a valuation interest rate of 12% (an annual effective rate), what is a fair price per share for Acme stock as of January 1? (Assume 30-day months.)

A) 37.50 B) 38.59 C) 38.73 D) 39.70 E) 40.84

18. Julia finances a 315,000 mortgage for 25 years at a nominal rate of 6.5% convertible monthly. Julia will be making monthly payments, with her first payment due one month after receiving the loan. If she adds 125 to each loan payment to pay off the loan sooner, what is the dollar amount of Julia's last payment?

A) 918 B) 923 C) 936 D) 1,254 E) 2,252

19. Suppose a company has liabilities requiring payments of 100,000 in one year, 200,000 in two years, 300,000 in three years, and 400,000 in four years. Also, suppose that they want to fund those liabilities by an exact match of cash flows using the following zero-coupon and annual-coupon bonds. How many bond A's should they buy? (Assume that fractional bonds can be purchased.)

Bond	Annual Eff. Yield	Coupon Rate (annual coupons)	Par Value	Term (years)
A	4.5%	Zero Coupon	1,000	1
B	5.0%	6%	1,000	2
C	5.5%	6%	1,000	3
D	6.0%	6%	1,000	4

A) 40.7 B) 45.6 C) 52.5 D) 89.6 E) 100

20. At the same time, Dan and Darci deposit money into two different funds. Dan deposits 200 and Darci deposits 80. Both accounts earn the same rate of interest. The amount of interest earned in Dan's account during the 10^{th} year is the same as the amount of interest earned in Darci's account during the 20^{th} year. Determine the amount of interest earned in Dan's account during the 13^{th} year.

A) 23.1 B) 57.6 C) 49.1 D) 63.2 E) 52.6

21. How much should you pay today for an annuity with 30 payments where the initial payment of 500 is three years from today and each subsequent annual payment is 6% greater than the previous payment? Let the annual effective interest rate equal 8%.

A) 11,589 B) 9,731 C) 9,426 D) 9,200 E) 8,969

22. Which of the following factors affect the credit spread for a loan?

I. Expected inflation
II. Creditworthiness of the borrower
III. Term of the loan
IV. Cost of inflation protection

A) All but I. B) All but II. C) All but III.
D) All but IV. E) Another combination

23. You are buying a perpetuity with annual payments as follows:
    i) Payments of X at the end of the first year and every three years thereafter

    ii) Payments of $X+1$ at the end of the second year and every three years thereafter

    iii) Payments of $X+2$ at the end of the third year and every three years thereafter.

    The interest rate is 5% convertible semi-annually. If the present value is 38.86, calculate X.

    A) 0.98 B) 1.00 C) 1.02 D) 1.04 E) 1.06

24. Annual payments of 500 are made at the beginning of each year for 30 years to an account earning an annual effective rate of 7%. The interest earned in the account each year is reinvested into another fund earning a 4.5% annual effective rate.

    At the end of the 30 years, what is the accumulated value of the 30 payments and the reinvested interest?

    A) 36,325 B) 47,230 C) 26,300 D) 30,504 E) 41,252

25. Which of the following statements are true?

    I. An unsecured loan generally has a higher interest rate than a secured loan.

    II. A student loan that is guaranteed by the U.S. government is an example of a secured loan.

    III. A borrower's creditworthiness is affected by the borrower's assets and the variability of his/her income.

    A) I. and II. B) I. and III. C) II. and III.
    D) All E) Another combination

26. Zero-coupon bond prices are as follows:

Term (in years)	Zero-coupon Bond Price per 100
1	95.25
2	89.95
3	84.02
4	78.94

A 3-year accreting interest rate swap based on the above rates has notional amounts of 4 million, 5 million, and 6 million in years 1, 2, and 3, respectively.

To the nearest 100, what is the amount of the net settlement payment made by the payer at the end of the first year?

A) 38,400 B) 40,500 C) 43,500 D) 46,800 E) 49,200

27. A loan of 1,000 is repaid with equal payments at the end of each quarter for 10 years. The principal portion of the 13th payment is 1.5 times the principal portion of the 5th payment. Calculate the amount of the quarterly loan payment.

A) 60 B) 26 C) 57 D) 69 E) 131

28. Michael is the Chief Financial Officer for ABC Company. For tax purposes, he needs to determine the amount of interest that ABC will pay at the end of the 5th year of a bank loan. It is a 7-year loan for 1,000,000 at a 10% annual effective interest rate, with level annual payments at the end of each year. Which of the following is closest to the number that he needs?

A) 35,650 B) 51,080 C) 65,000 D) 89,450 E) 170,000

29. Using the following table of spot rates, calculate the total future value at time 5 of a payment of 3,000 made today and a payment of 3,000 made at time 3. Assume that the payment at time 3 will be invested at today's forward rates.

Term (years)	*Annual yield*
1	6.00%
2	6.10%
3	6.40%
4	6.80%
5	7.50%

A) 7,684 B) 7,411 C) 7,882 D) 7,566 E) 8,568

30. Amanda receives a 10-year increasing annuity-immediate that pays 30 at the end of the first year and has payments that increase by 5 each year thereafter. Kevin receives a 10-payment decreasing annuity that pays X at the end of the first year and its payments decrease by 2 each year thereafter. At an annual interest rate of 4%, both annuities have the same present value. Calculate X.

A) 61.60 B) 42.53 C) 28.60 D) 59.24 E) 47.99

31. Louise is going to pay off a loan of 30,000 with semi-annual payments at the end of each 6-month period. The loan bears interest at a nominal rate of 5% convertible semi-annually.

Her first payment is X and each subsequent payment increases by 1 until the loan is paid off.

If the loan term is 10 years, calculate X.

A) 1,750 B) 1,916 C) 2,399 D) 3,423 E) 3,881

32. Let $d^{(4)} = 0.02$. Calculate the total amount of interest earned at the end of 13 years on a deposit account at this rate if semi-annual deposits of 500 are made at the beginning of each 6-month period.

A) 1,977 B) 1,926 C) 1,845 D) 1,777 E) 1,521

33. At the beginning of the year, a student organization started an account with a deposit of 3,000.

    At the end of each month, the student organization deposited dues of 100. At the end of the 5^{th} and 7^{th} months, withdrawals of 500 were made. The amount in the account at the end of the year is 3,750.

    Calculate the dollar-weighted rate of return for the student organization's account during the year.

    A) 16.7% B) 17.2% C) 13.3% D) 18.0% E) 83.0%

34. A 1,000 par value bond pays semi-annual coupons at a 6% annual coupon rate and has a maturity date of November 1, 2023. On September 18, 2016, this bond was purchased to yield 8.00% convertible semi-annually.

    What was the market price (excluding accrued coupon) at which the bond was purchased on September 18, 2016?
    (Assume a 360-day year.)

    A) 870 B) 889 C) 893 D) 894 E) 915

35. Two zero-coupon bonds, A and B, will each mature in 8 years. Both bonds have the same current price, but Bond A has a yield to maturity of 8%, and Bond B has a yield to maturity of 7.5%.

    Of the following possible parallel shifts in interest rates, which would result in a higher current price for Bond A than Bond B?

    A) a small shift upward B) a small shift downward
    C) a small shift either up or down D) neither a shift up nor down
    E) cannot be determined from the information given

Solutions

1. First, determine the amount of the quarterly deposits to the sinking fund:
N=100 I/Y=1.25 (=5% / 4) PV=0 FV=1,000,000
CPT PMT→5,074.28

Next, determine the amount of interest earned during the 10th year. We find this by calculating the increase in the sinking fund balance during the 10th year (from immediately after the 36th payment until immediately after the 40th payment. If we determine the total amount by which the balance increased during the year and subtract the deposits made during the year (4 x 5,074.28 = 20,297.12), we will have the amount of interest credited during the year. Starting from the above values in TVM, we change N to 36 and then to 40, each time calculating the value of FV:

N=36 CPT FV = 228,928.65 N=40 CPT FV = 262,272.37

So the growth in the sinking fund balance is 32,343.71 (= 262,272.37 – 228,928.65).
Subtracting the amount of this increase that was due to new deposits (20,297.12), we find that earned interest accounted for 12,046.60.

The problem states that "net interest" during the 10th year was 80,000, so we have:

Interest paid – Interest earned = net interest = 80,000
Interest paid = 80,000 + Interest earned = 80,000 + 12,046.60 = 92,046.60

The interest paid during the 10th year (or any other year) is 92,046.60.

Finally, we can calculate the loan's nominal interest rate, convertible quarterly. Technically, we should first determine the interest paid each quarter and determine the quarterly effective rate:

$$\frac{i^{(4)}}{4} = \frac{92,046.60 / 4}{1,000,000} = 0.023012$$

Then the nominal rate convertible quarterly is $i^{(4)} = 4 \cdot (0.023012) = 9.2047\%$

Answer: D

2. The easiest way to solve this problem is to find the present value of the coupons plus the present value of the redemption value.

$$P = 300a_{\overline{15}|0.05} + v_{0.05}^{15}300a_{\overline{15}|0.07} + v_{0.05}^{15}v_{0.07}^{15}(10,000) = 6,171.64$$

Answer: B

3. First, calculate the payment amount by using the information given regarding the outstanding balance and applying the prospective method:

   Set N=230, I/Y=1, PV=297,000, and FV=0. CPT PMT = -3.305.1809.

   Next, calculate the original loan amount with a payment of 3,305.1809, 240 monthly payments, and a monthly effective rate of 1%.
   Set N=240 and CPT PV = 300,174.6003.

   Next, to calculate the total interest paid, calculate the difference between the total payments and the amount borrowed. The contribution is 240 times 3,305.1809, or 793,243.416. This gives us a total interest paid amount of 793,243.416 – 300,174.6003 = 493,068.8157.

Answer: C

4. This is a deferred annuity-due. We will think of the time in terms of semi-annual periods. It will take 15 years to make the semi-annual payments. The last payment will be made at time 20.5. However, calculating the future value of an annuity-due will give us the value as of time 21. Then the payments will be left in the fund for an additional 9 years.

$$i = (1.06)^{1/2} - 1 = 0.0295630141$$

$$FV = 5\ddot{s}_{\overline{30}|2.9563\%}(1.029563)^{18} = 410.85$$

Answer: B

5. First we need to find out how much Joel paid for the bond. We need to do the calculation 2 ways: with a call at 5 years and without a call. Joel paid whichever price is lower.

   (For example, if the price with a call at 5 years is lower than the price without a call, then he would pay the price with a call. If he had paid the higher price (based on no call), then he would earn less than 7% if the bond is called at 5 years. He must pay the lower price in order to <u>assure</u> a yield of at least 7%.)

   With a call at 5 years:
   N=10 I/Y=3.5 PMT=40 FV=1,050
   CPT PV→1,077.03

   With no call:
   N=20 I/Y=3.5 PMT=40 FV=1,000
   CPT PV→1,071.06

   The amount Joel paid was 1,071.06. His yield when the bond is called at 8 years is calculated as follows:

   N=16 PV=-1,071.06 PMT=40 FV=1,050
   CPT I/Y→3.642%

   The yield convertible semi-annually is $2\times 3.642\% = 7.284\%$.

   Answer: D

6. This is a lump sum deposit that earns different rates. First, calculate X, the lump sum deposit, by setting up a future value calculation:

$$200{,}000 = X(1+0.10)^3[(1+\frac{0.05}{2})^2]^3$$
$$X = 129{,}571.2796$$

   To calculate the amount of interest earned during the third year we need the difference between the future values at time 2 and time 3. (Or you could multiply the future value at time 2 by 10%.)

$$FV_3 - FV_2 = 129{,}571.28(1+0.10)^3 - 129{,}571.28(1+0.10)^2 = 15{,}678.13$$

   Answer: A

7. After organizing the information, we have the following summary:

Date	***Value before transaction***	***Deposits***	***Withdrawals***
1/1	200		
2/28			30
3/31		25	
6/30		25	60
9/30		25	
12/31	250		

Now, using the dollar-weighted equation:

$$i = \frac{250 - 200 - [-30 + 25 - 35 + 25]}{200 - 30(1 - 2/12) + 25(1 - 3/12) - 35(1 - 6/12) + 25(1 - 9/12)}$$
$$i = 0.3561644$$

Answer: C

8. Price of bond = PV of redemption value + PV of coupons
PV of coupons = Price of bond – PV of redemption value
= 102.02 – 91.05 = 10.97

PV of coupons = Coupon amount $\cdot$ Sum of PV factors on coupon dates
10.97 = Coupon amount $\cdot(0.9806 + 0.9579 + 0.9334 + 0.9105)$

Coupon amount = $\dfrac{10.97}{(0.9806 + 0.9579 + 0.9334 + 0.9105)} = 2.90$

Annual coupon rate = Total amount of coupon payments per year / par value

$$= \frac{2 \cdot 2.90}{100} = 5.80\%$$

Answer: B

9. The value of the accumulation function for Louis is:

$$a(t) = e^{\int_2^{18} (3+t)^{-1} dt}$$
$$= e^{\ln(3+t)\big|_2^{18}} = e^{\ln 21 - \ln 5} = 4.2$$

Now, set that amount equal to the value of Donald's accumulation function and solve for i:

$$(1+i)^{18} = 4.2 \qquad i = 0.08299128$$

Answer: C

10. Notice that the two accumulated values we are given (the values at time 11 and at time 16) are both during the period when the annual effective interest rate is $3i$. So we can immediately solve for $3i$:

$$1.84 \cdot (1+3i)^5 = 2.83 \qquad 3i = (2.83/1.84)^{1/5} - 1 = 0.089918$$

To find the value at time 4, we need to discount the time 11 value to time 6 using a rate of $3i$, and then to time 4 using a rate of i:

$$1.84 \cdot \left(1.089918^{-5}\right)(1 + 0.089918/3)^{-2} = 1.1277$$

Answer: B

11. This is a 40-payment annuity-due and a deferred perpetuity-due. Set up a present value calculation with the perpetuity payment as X and a quarterly effective interest rate of i, *where* $i = (1.16)^{1/4} - 1 = 0.037802$:

$$2{,}000 = 50\ddot{a}_{\overline{40}|i} + v^{40}X\ddot{a}_{\overline{\infty}|i}$$

$$2{,}000 = 50 \cdot \frac{1 - 1.037802^{-40}}{0.037802/1.037802} + 1.037802^{-40} \cdot X \cdot \frac{1.037802}{0.037802}$$

$$2{,}000 = 1{,}061.5174 + 6.22329 \cdot X$$

$$X = (2{,}000 - 1{,}061.5174)/6.22329 = 150.80$$

Answer: A

12. First solve for the annual effective interest rate, i. Then apply the formula for Macaulay duration.

$$1{,}250 = 100 \cdot a_{\overline{\infty}|} = 100 \cdot \frac{1}{i}$$

$$i = \frac{100}{1{,}250} = 8\%$$

$$D_{\text{mac}} = \frac{(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}} = \frac{\frac{\ddot{a}_{\overline{\infty}|}}{i}}{\frac{1}{i}} = \ddot{a}_{\overline{\infty}|} = \frac{1}{d} = \frac{1}{0.08/1.08} = 13.5$$

Answer: E

13. A bond's book value changes over time as a result of interest that is earned (at the yield rate) and coupons that are paid out. In this example, we are interested in how these two factors caused the bond's book value to change from 1,099.84 at time 3 to 1,087.27 at time 5.

Recognizing the effect of interest earned and coupons paid from time 3 to time 5, we can write:

$$BV_4 = BV_3 \cdot (1+i) - 80 = 1{,}099.84 \cdot (1+i) - 80$$

$$BV_5 = BV_4 \cdot (1+i) - 80 = \left[1{,}099.84 \cdot (1+i) - 80\right] \cdot (1+i) - 80 = 1{,}087.27$$

Writing this as a quadratic equation and applying the quadratic formula to solve for $(1+i)$, we have:

$$1{,}099.84 \cdot (1+i)^2 - 80 \cdot (1+i) - (80 + 1{,}087.27) = 0$$

$$1 + i = 1.06721,\ -0.99447$$

Only the positive result is valid, so $i = 6.72\%$.

This problem can also be solved using the BA II Plus, as follows:
Set N=2, PV=-1,099.84, PMT=80, and FV=1,087.27. CPT I/Y = 6.721.
These entries describe what happens during the 2-year period from time 3 to time 5. The initial value is 1,099.84, and we can think of this as the value of the bond (or the price paid to acquire it at time 3. Then the bond paid 80 at the end of each year for 2 year, and its value at the end of the 2 years (at time 5) was 1,087.27.

Answer: B

14. Since each series starts with a payment of 10 and ends with a payment of 110, their durations will differ based on the size and timing of the intervening payments.

The **bond** has the smallest payments in years 2-10 (level payments of 10), so its duration is heavily weighted toward the payment at time 11. As a result, it will have the longest duration.

The arithmetic annuity has payments that increase linearly from 10 to 110. The geometric annuity's payments grow more slowly in the early years, and then grow rapidly in the last few years in order to reach 110 at time 11. Consequently, the geometric annuity puts more weight on the later payments than the arithmetic annuity does (but not as much as the bond). So its duration will be greater than the arithmetic annuity's duration.

Conclusion: $D^{\mathrm{b}}_{\mathrm{mod}} > D^{\mathrm{c}}_{\mathrm{mod}} > D^{\mathrm{a}}_{\mathrm{mod}}$
(Note: The above analysis talks in terms of Macaulay duration, which simply applies weights to the <u>*time*</u> *at which each payment occurs. However, the conclusion applies to modified duration as well, since all of the series are valued at 10%, so* $D_{\mathrm{mod}} = D_{mac}\,/\,1.10$ in each case.)

Answer: D

15. First, we need to solve for i by finding the future value of each account after one year and multiplying that value by the respective annual effective interest rate of that account.

$$250(1+i)\cdot i = 300\left(1+\frac{0.03}{12}\right)^{12}\cdot\left[\left(1+\frac{0.03}{12}\right)^{12}-1\right] = 9.4023$$

$$250\cdot\left(i+i^2\right) = 9.4023$$

$$250i^2 + 250i - 9.4023 = 0$$

Using the quadratic formula: i= 0.036292, -1.03629.
Only the positive value is valid.
The amount of interest that Rick will earn during the 6th year is:

$$250\cdot(1.036292)^5\cdot(0.036292) = 10.8433$$

Answer: C

16. First, use the given time-weighted rate of return to solve for X. Once we know X, we can solve for the dollar-weighted rate of return.

$$(1+0.12)=\frac{1,010}{980}\cdot\frac{1,055}{1,010+30}\cdot\frac{1,060}{1,055-X} \qquad X=65.5308$$

The dollar-weighted equation is:

$$i=\frac{1,060-980-[30-65.5308]}{980+30(1-5/12)-65.5308(1-9/12)}=0.11775$$

Answer: A

17. The fair price of the stock is the present value of future dividends. As of the previous October 1 (6 months before the next dividend), this value can be calculated as the present value of an annuity with semi-annual payments and annual increases of 4%. Based on an initial (annual) payment amount of 3, this present value is:

$$3\cdot a_{\overline{\infty}|12\%}^{4\%\,(2)}=3\cdot\frac{1}{i-g}\cdot\frac{i}{i^{(2)}}=3\cdot\frac{1}{0.12-0.04}\cdot\frac{0.12}{0.1166}=38.59$$

The fair price as of January 1 is this value accumulated with interest for three months:

$$38.59\cdot(1.12)^{1/4}=39.70$$

Answer: D

18. First, calculate the payment amount without the extra money added.
Set N=300, I/Y=6.5/12=0.54167, PV=315,000, and FV=0. CPT PMT = -2,126.90.

Julia adds 125 to each payment, so she is making a monthly payment of 2,251.90. This will pay off the loan faster, so we need to solve for the new loan term.
Set PV=-2,251.90 and CPT N=262.41.

That means Julia will make 262 full payments and one partial payment. To calculate the amount of the last partial payment, set N=263 (the number of the final payment) and CPT FV = 1,328.60. This positive value is a "refund" Julia would receive at time 262 if she made a full payment at time 262. It represents the amount by which she overpaid the loan by paying 2,251.90 at time 262. So the correct value for the final payment is:

$$2,251.90-1,328.60=923.30$$

Answer: B

19. On this type of problem, work from the longer-term bonds to the shorter-term bonds.

Bond D provides a total payment of 1,000+60 per bond at maturity. To generate 400,000 at time 4 requires $400,000 / 1,060 = 377.3584$ of bond D. The 377.3584 D bonds produce 377.3584×60=22,641.50 in coupons at times 1, 2, and 3. The amount needed at time 3 is then 300,000-22,641.504 = 277,358.50.

Bond C also produces 1,000+60 per bond at maturity. To produce total payments of 277,358.50 at time 3 requires $277,358.496 / 1,060 = 261.6590$ of bond C. With 377.3584 D bonds and 261.6590 C bonds, there will be a total of 377.3584×60 + 261.6590×60=38,341.04 in coupon payments at time 2. The remaining amount needed at time 2 is then 200,000-38,341.04 = 161,658.96.

Each bond B will generate 1,000+60 at maturity. Since 161,658.96 is needed at time 2, it can be provided by $161,658.96 / 1,060 = 152.5084$ of bond B. With 377.3584 D bonds, 261.6590 C bonds, and 152.5084 B bonds, there will be 377.3584×60 + 261.6590×60 + 152.5084×60=47,491.55 in coupon payments at time 1. The additional amount needed at time 1 has then decreased to 100,000-47,491.55 = 52,508.45.

Bond A provides 1,000 per bond at maturity, so 52.50845 units of bond A are needed.

Answer: C

20. Since both accounts earn the same rate of interest, and we know that Dan's account earns as much in the 10th year as Darci's does in the 20th year, we can conclude that their balances at the beginning of those years (Dan at time 9, and Darci at time 19) were equal. Since Dan deposited 2.5 times as much as Darci (200/80 = 2.5), we can conclude that it takes 10 years (the difference between time 9 and time 19) for an account balance to grow by a factor of 2.5. Therefore:

$$(1+i)^{10} = 2.5 \qquad i = 2.5^{1/10} - 1 = 0.095958$$

To find the interest earned in Dan's account during the 13th year, we simply calculate his balance at time 12 and multiply by the interest rate:

$$200 \cdot 1.095958^{12} \cdot 0.095958 = 57.63$$

Answer: B

21. This is a 2-year-deferred geometric annuity-immediate. Its present value is:

$$500 \cdot {}_{2|}a_{\overline{30}|8\%}^{6\%} = 500 \cdot 1.08^{-2} \cdot \frac{1-\left(\frac{1.06}{1.08}\right)^{30}}{0.08-0.06} = 9{,}199.83$$

This annuity can also be treated as a 3-year-deferred geometric annuity-due, which lends itself to the artificial interest rate method:

$$500 \cdot {}_{3|}\ddot{a}_{\overline{30}|8\%}^{6\%} = 500 \cdot 1.08^{-3} \cdot \ddot{a}_{\overline{30}|j}$$

$$1+j = \frac{1.08}{1.06} = 1.018868$$

$$500 \cdot 1.08^{-3} \cdot \frac{1-(1.018868)^{-30}}{0.018868/(1.018868)} = 9{,}199.83$$

Answer: D

22. The only factor listed that does not affect the credit spread for a loan is the cost of inflation protection (IV.). The borrower's creditworthiness (II.) is clearly a consideration in the rate that is charged (and the spread above the risk-free rate). Expected inflation (I.) is also a consideration, since inflation typically makes it easier for the borrower to repay the loan, resulting in a lower default rate. And the term of the loan (III.) is a consideration, since a longer-term loan introduces the risk that changing conditions over a longer period could result in a default.

Answer: D

23. This perpetuity's payment pattern is: X, X +1, X+2, X, X +1, X +2, etc.

The X's in this stream of payments form a perpetuity-immediate with annual payments, so their present value is X / i.
The 1's in the payment stream can be viewed as a 2-year-deferred perpetuity-due with payments every third year. If j is the 3-year effective interest rate, the value of this series is $v_i^2 \cdot 1 / d_j$.
The 2's in the payment stream represent a 2-year-deferred perpetuity-due with payments every third year. The value of this series is $v_i^3 \cdot 2 / d_j$.

The interest rate is 5% convertible semi-annually, so we can compute i and j and the other needed interest functions as follows:

$$i = (1 + 0.05 / 2)^2 - 1 = 0.050625 \qquad j = (1 + i)^3 - 1 = 0.159693$$

$$v_i = 1.050625^{-1} = 0.9518144 \qquad d_j = 0.159693 / 1.159693 = 0.137703$$

The value of this perpetuity is given as 40, so we have:

$$X / i + v_i^2 \cdot 1 / d_j + v_i^3 \cdot 2 / d_j = 40$$

$$X / 0.050625 = 40 - \left(\frac{1.050625^{-2}}{0.137703} + \frac{1.050625^{-3}}{0.137703} \cdot 2 \right)$$

$$X = 0.050625 \cdot (40 - 19.10301) = 1.05791$$

Answer: E

24. The accumulated value of the payments without interest is 30*500=15,000.

The accumulated value of the interest is the future value of a 30-payment annuity with arithmetically increasing payments. The first payment is 35, the second is 70, etc., and the last payment is 1,050. The accumulated value of this annuity at 4.5% is:

$$35(Is)_{\overline{30}|0.045} = 35 \cdot \frac{\ddot{s}_{\overline{30}|} - 30}{0.045} = 26,251.86$$

The total, including the balance in the 7% account (the sum of the deposits) is:

$$15,000 + 26,251.86 = 41,251.86$$

Answer: E

25. Statement I is true. The risk of default on an unsecured loan (one with no collateral) is greater, so the interest rate charged is higher.
Statement II is false. A guaranteed loan is <u>not</u> a secured loan. There is no collateral "securing" the loan (such as a car or a house that the lender can repossess or foreclose on).
Statement III is true. A borrower's assets and the variability of his/her income are both considerations that affect the borrower's creditworthiness.

Answer: B

26. In order to calculate the amount of the period 1 net settlement payment, we need to determine the swap rate. Because the notional amount of this swap varies from period to period, the simplified formulas for swap rate cannot be used. Therefore, we will use the general formula for a swap rate:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

The present value factors (P) are simply the zero-coupon bond prices divided by 100. The forward rates (f) can be found from the present value factors:

$$f^*_{[0,1]} = \frac{1}{P_1} - 1 = \frac{1}{0.9525} - 1 = 0.04987$$

$$f^*_{[1,2]} = \frac{P_1}{P_2} - 1 = \frac{0.09525}{0.08995} - 1 = 0.05892$$

$$f^*_{[2,3]} = \frac{P_2}{P_3} - 1 = \frac{0.8995}{0.8402} - 1 = 0.07058$$

All the values have been filled in below. The notional amounts (Q) have been divided by 1,000,000 for convenience.

$$\begin{aligned} R &= \frac{Q_1 \cdot f^*_{[0,1]} \cdot P_1 + Q_2 \cdot f^*_{[1,2]} \cdot P_2 + Q_3 \cdot f^*_{[2,3]} \cdot P_3}{Q_1 \cdot P_1 + Q_2 \cdot P_2 + Q_3 \cdot P_3} \\ &= \frac{4 \cdot 0.04987 \cdot (0.9525) + 5 \cdot 0.05892 \cdot (0.8995) + 6 \cdot 0.07058 \cdot (0.8402)}{4 \cdot (0.9525) + 5 \cdot (0.8995) + 6 \cdot (0.8402)} \\ &= 0.06074 \end{aligned}$$

The net settlement payment for the first period equals the notional amount for the period (4 million) times the difference between the swap rate (6.074%) and the 1-year spot rate on the inception date:

$$4{,}000{,}000 \cdot (0.06074 - 0.04987) = 43{,}480$$

Answer: C

27. First, we need to solve for i, the quarterly effective rate. We know that the principal portion of the 13th payment is 1.5 times the principal portion of the 5th payment, so we have:

$$(1+i)^{13-5} = 1.5 \qquad i = 1.5^{1/8} - 1 = 0.05199$$

To find the quarterly payment using the BA II Plus,
set N=40, I/Y=5.199, PV=1,000, and FV=0. CPT PMT = 59.87.

Answer: A

28. First find the payment required for a 1,000,000 loan for 7 years with payments made at the end of each year at a 10% annual effective rate:

$$1{,}000{,}000 = Pmt \cdot a_{\overline{7}|0.10} \qquad Pmt = 205{,}405.50$$

We can find the amount of interest paid in the 5th payment, based on the outstanding balance after the 4th payment:

$$Bal_4 = 205{,}405.50 a_{\overline{7-4}|0.10} = 510{,}813.08$$

The amount of interest due in the 5th payment will be 10% of the outstanding balance after the 4th payment, or 51,081.31

This problem can also be done entirely on a financial calculator.
Set N=7, I/Y=10, PV=1,000,000, and FV=0. CPT PMT = 205,405.50.
Set N=3 (number of payments remaining at time 4) and CPT PV = 510,813.08.
Then the interest paid at the end of the 5th year is 10% of 510,813.08, resulting in the same answer as the above calculations: 51,081.31.

Answer: B

29. One approach to this problem is to find the 3-year-forward 2-year rate, $i_{3,5}$ (the rate for the period from time 3 to time 5), and use that rate to accumulate the second payment from time 3 to time 5. The first payment, of course, is accumulated at the 5-year spot rate.

$$(1+i_{3,5})^2 = \frac{(1+s_5)^5}{(1+s_3)^3} = \frac{(1.075)^5}{(1.064)^3} = 1.191838574$$

$$FV = 3{,}000(1+s_5)^5 + 3{,}000(1+i_{3,5})^2$$

$$FV = 3{,}000(1.075)^5 + 3{,}000(1.191838574) = 7{,}882.40$$

Another approach is to calculate the present value at t=0 of the second payment, and then accumulate that value plus the amount of the first payment to time 5 using the 5-year spot rate.

$$\left(3{,}000 + \frac{3{,}000}{1.064^3}\right) \cdot 1.075^5 = 7{,}882.40$$

Answer: C

30. For both Amanda and Kevin, we can use the formula for a present value of an annuity with payments of P, $P+Q$, $P+2Q$, …, $P+(n\text{-}1)Q$. The formula is:

$$Pa_{\overline{n}|i} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

Substituting the values for Amanda, her present value is:

$$30a_{\overline{10}|0.04} + 5\left(\frac{a_{\overline{10}|0.04} - 10v^{10}}{0.04}\right) = 412.7336$$

Substituting the values for Kevin, his present value is:

$$Xa_{\overline{10}|0.04} - 2\left(\frac{a_{\overline{10}|0.04} - 10v^{10}}{0.04}\right)$$

Setting the present value of Kevin's annuity equal to 412.7336 and solving for X, we have:

$$Xa_{\overline{10}|0.04} - 2\left(\frac{a_{\overline{10}|0.04} - 10v^{10}}{0.04}\right) = 412.7336$$

$$X = 59.2408$$

Answer: D

31. Set the loan amount of 30,000 equal to the present value of an increasing arithmetic annuity and solve for X. Using the PQ formula, we can write:

$$30{,}000 = Xa_{\overline{20}|0.025} + 1\left(\frac{a_{\overline{20}|0.025} - 20v_{0.025}^{20}}{0.025}\right)$$

$$X = \frac{30{,}000 - 1\left(\frac{a_{\overline{20}|0.025} - 20v_{0.025}^{20}}{0.025}\right)}{a_{\overline{20}|0.025}} = \frac{29{,}864.65}{15.5892} = 1{,}915.7269$$

Answer: B

32. The quarterly effective rate of discount is $\frac{0.02}{4} = 0.005$.

The 6-month effective interest rate is: $j = (1-0.005)^{-2} - 1 = 0.01007550$.

The future value of 26 payments of 500 made at the beginning of each 6-month period is: $500\ddot{s}_{\overline{26}|j} = 500 \cdot \frac{1.0100755^{26} - 1}{0.0100755 / 1.0100755} = 14,926.1247$

Subtract the 26 deposits of 500 from this FV to get the total interest earned:

$14,926.1247 - 26 \times 500 = 1,926.1247$

Answer: B

33. Let A=initial fund balance = 3,000. Let B=the final fund balance = 3,750. Let C=the net contributions = 12(100)-2(500)=200.
Then I=interest earned = B-A-C = 3,750-3,000-200=550,

Dollar-weighted rate of return is:

$$i = \frac{I}{A + \sum C_t(1-t)}$$

$$= \frac{550}{3,000 + 100 \cdot (1/12) \cdot \sum_{t=1}^{11} t - 500(7/12) - 500(5/12)}$$

$$= \frac{550}{3,050} = 0.18033$$

$$i = \frac{550}{3050} = 0.18033$$

Answer: D

34. Using the Bond workbook:
Set SDT=9.1816, CPN=60, RDT-11.0123, RV=1,000, 360, 2/Y, and YLD=8.
CPT PRI = 892.92
The calculator also provides the Accrued Interest (accrued coupon), which is AI = 22.83. We don't need the accrued interest for this problem, but note that the calculator's value for AI matches the figure calculated below.

Using formulas:
Price on prior coupon date (5/1/16) = $1{,}000 \cdot (1.04)^{-15} + 30 \cdot a_{\overline{15}|4\%} = 888.82$
Or calculate this value using the TVM worksheet:
N=15, I/Y=4, PMT=30, and FV=1,000. CPT PV = -888.82.

Total price on 9/18/16 = 888.82 $(1.04^{(137/180)})$ = 915.75, where 137 is the number of days from 5/1 to 9/18 (4 months + 17 days = 137 days).

Calculate accrued interest on 9/18/16: $30 \cdot (137/180) = 22.83$.

Calculate market price on 9/18/16: 915.75 – 22.83 = 892.92.

Answer: C

35. Bond A has a modified duration of 8/1.08 = 7.407.
Bond B's modified duration is 8/1.075 = 7.442.

Since B's duration is larger than A's, the price for B will change more when interest rates change. If rates rise, B's price will decrease more than A's, and if rates fall, B's price will increase more than A's.

Of the answer choices, only A) correctly describes a situation where A's price will be higher than B's.

Answer: A

Practice Exam 8

Exam FM

Questions

1. Alex invests 50,000 today in a 10-year project. At the end of the 5th, 7th, and 9th years he receives payments of 22,000. Rather than keeping these payouts, Alex reinvests them in a fund that earns a 6% annual effective rate and provides a lump-sum payout 10 years from today. What is the net present value of this project today valued at an annual effective interest rate of 5%?

   A) 1,523 B) 4,267 C) –5,907 D) –1,523 E) –4,267

2. The following are prices (per 100 face) for zero-coupon bonds:

Bond Term (in years)	Price (per 100)
1	95.87
2	90.88
3	85.70

   Find the annual effective yield for a 3-year annual-coupon par bond (i.e., a bond that is selling at par).

   A) 5.15% B) 5.25% C) 5.35% D) 5.40% E) 5.50%

3. Tim makes four payments of 200 at four-year intervals starting today. Interest is credited at a nominal interest rate of 5% compounded semi-annually for the first 9 years and at a 3% nominal rate of discount compounded monthly thereafter. Calculate the accumulated value of the four payments 25 years from today.

   A) 1,394 B) 1,554 C) 1,538 D) 1,311 E) 1,449

4. Based on an annual effective interest rate i, values are calculated for the interest functions v, d, and δ.

   If $v = 8 \cdot i$, what is the value of $\delta - d$?

   A) 0.00544 B) 0.00547 C) 0.00550 D) 0.00555 E) 0.00558

5. Brian borrows 40,000. The principal must be repaid in 20 years, but interest payments are due at the end of each year at an annual effective rate of 4%. Brian pays the interest each year but also deposits X into a fund at the end of each month to accumulate enough to repay the loan principal in 20 years. This fund earns an annual effective rate of 5%. What are Brian's total payments for this loan?

   A) 55,360 B) 59,800 C) 55,660 D) 56,190 E) 72,000

6. Ruth plans to deposit 50 into a fund at the beginning of every 6-month period for 28 years, starting today. At an annual effective rate of j ($j > 0$), the balance 28 years from today will be four times the balance in the fund 14 years from today (before the deposit on that date). Calculate the total amount of interest she will earn from this investment over the 28 years.

   A) 7,596 B) 3,900 C) 10,396 D) 7,196 E) 7,797

7. Tony buys two bonds. One bond is priced to yield a 4% annual effective rate and is a 3-year zero coupon bond that can be redeemed for 1,000. The second bond is a 1,000 par value 6% semi-annual coupon bond that matures in 6 years. Both of the bonds have the same purchase price. Calculate the annual effective yield of the coupon bond.

   A) 4.2% B) 6.4% C) 6.9% D) 8.6% E) 8.9%

8. Brenda just turned 20 and wants to set up a fund that will pay her X when she is 40 and 50 years old, and that will also pay her 1,000 when she is 45 and 55 years old. Brenda is setting up the fund with a lump sum deposit today of 875, which is just enough to provide her intended future payments. If the fund earns an annual effective rate of 4.5%, calculate X.

   A) 725 B) 672 C) 650 D) 481 E) 392

9. Austin bought an annual perpetuity-due in 2016 that pays 4 in "even" years (i.e. 2016, 2018, 2020, etc.) and 7 in "odd" years (i.e. 2017, 2019, etc.). Calculate the present value of this perpetuity-due on its purchase date at an annual effective rate of 5%.

   A) 104.1 B) 107.3 C) 111.3 D) 114.7 E) 200.0

10. Michael deposits 300 at time 4 and and 200 at time t into an account growing at a force of interest of:

$$\delta(t) = \frac{6}{(t+4)}$$

   If the total value of these two deposits at time 8 is 4,500, which of the following is closest to t?

   A) 4 B) 5 C) 6 D) 7 E) 8

11. A corporation borrows 100,000 and agrees to make an interest payment of 2,250 at the end of each quarter for 20 years, plus a single principal payment of 100,000 at the end of 20 years. During the 20-year term of the loan, the corporation also makes level semi-annual deposits to a sinking fund in order to accumulate 100,000 to repay the loan at the end of 20 years.

   If interest is credited to the balance in the sinking fund at a nominal rate of 6% per annum, compounded quarterly, what is the amount of "net interest" paid by the borrower during the 8^{th} year of the loan?

   A) 6,775 B) 7,105 C) 7,575 D) 8,125 E) 8,586

12. A 10,000 par-value bond has a maturity date of June 1, 2031. The bond is purchased on September 1, 2016, for a total price (including accrued coupon) of 9,552. Based on that purchase price, the purchaser's yield to maturity is 6% convertible semi-annually.

   What is the bond's annual coupon rate?
   (Note: The accrued coupon is calculated using a simple interest methodology. Assume 30-day months in your calculations.)

   A) 5.1% B) 5.2% C) 5.3% D) 5.4% E) 5.5%

13. Alex and Rudy are both going to buy bonds at the same time. Alex is buying a 15-year 1,000 par value bond redeemable at par with semi-annual coupons. Rudy is buying a 5-year 800 par value bond redeemable at par with semi-annual coupons. Alex's bond is sold to yield 4% convertible semi-annually and the yield rate of Rudy's bond is 10% convertible semi-annually. The price of Alex's bond is twice that of Rudy's bond and their coupon rates are equal. Calculate the bonds' nominal coupon rate payable semi-annually.

A) 8.75% B) 8.57% C) 7.85% D) 6.75% E) 4.28%

14. Jeanette takes out a 50-year loan of 30,000. Instead of making level payments, Jeanette's annual payments at the end of each year are all equal to 600 plus the amount of interest owed. The lender charges an annual effective interest rate of 7%. How much total interest will Jeanette pay for this loan?

A) 52,500 B) 53,550 C) 78,690 D) 52,000 E) 59,200

15. An annuity-immediate that will make 10 quarterly payments of 300 has a present value of 2,700. What would be the quoted nominal discount rate convertible monthly for this annuity?

A) 7.85% B) 7.28% C) 7.67% D) 7.80% E) 7.75%

16. You are given the following term structure of spot interest rates:

Term (in years)	Spot interest rate
1	4.00%
2	4.50%
3	5.25%
4	5.75%
5	6.00%

Calculate the future value at time 5 of a four-year annuity-due issued today with four annual payments of 3,500.

A) 16,964 B) 16,555 C) 16,733 D) 17,494 E) 15,920

17. Calculate the modified duration, at i = 0.045, for a series of payments consisting of 400 at the end of the 2^{nd} year, 300 at the end of the 3^{rd} year, 200 at the end of the 4^{th} year, and 100 at the end of the 6^{th} year.

A) 3.30 B) 3.22 C) 3.00 D) 2.91 E) 2.54

18. Given the following spot rates and forward rates, what is the accumulated value at $t = 6$ of a deposit of 1,000 made at $t = 2$?

$s_2 = 3.6\%$ $i_{4,5} = 5.3\%$
$s_4 = 4.4\%$ $i_{5,6} = 5.5\%$

A) 1,200 B) 1,210 C) 1,220 D) 1,230 E) 1,240

19. Seth borrows X for 9 years at an annual effective rate of 5.75%. He will be repaying the loan with level payments at the end of each 6-month period. The principal repaid in the 2^{nd} payment is 580. Calculate the outstanding balance immediately after the 7^{th} payment.

A) 7,770 B) 8,470 C) 8,710 D) 9,000 E) 9,380

20. John has a 10,000 loan that bears interest at an annual effective interest rate of 9% and requires 15 annual payments. Instead of level payments, each of John's payments will be 13% smaller than the previous payment, after an initial payment of X. Find X.

A) 1,240.59 B) 2,277.42 C) 2,297.86 D) 9,600.00 E) 3,853.97

21. A company's stock is valued at 50 per share assuming an annual effective rate of 12%. Annual dividends are expected to be paid at the end of each year forever. The first dividend is 5, payable one year from now. Each subsequent dividend is expected to be $X\%$ greater than the previous year's dividend. Calculate X.

A) 3.5 B) 2.8 C) 2.4 D) 2.0 E) Does Not Exist

22. A bond with annual coupons is currently selling for 10,800. At that price, the bond provides a yield-to-maturity of 6.00% (an annual effective interest rate). The bond has a Macaulay duration (not a modified duration) of 8.00

Suppose that an unwary purchaser pays 11,016 for this bond. Considering the effects of both duration and convexity, which of the following describes the yield to maturity that the purchaser will realize?

A) slightly less than 5.735%

B) slightly more than 5.735%

C) slightly less than 6.265%

D) exactly 6.265%

E) slightly more than 6.265%

23. Rafael bought a 10-year 1,000 par value bond for a price of 1,025. The bond is callable in six years at par. The purchase price guarantees a yield of at least 5% convertible semi-annually. Calculate the amount of each semi-annual coupon.

A) 54.93 B) 53.24 C) 52.82 D) 27.44 E) 26.60

24. Given the following forward rates, what would be the fixed interest rate for a 1-year-deferred, 2-year interest rate swap? (Assume that the notional amount is level.)

$$f^*_{[0,1]} = 0.03200$$

$$f^*_{[1,2]} = 0.04202$$

$$f^*_{[2,3]} = 0.04603$$

A) 3.98% B) 4.00% C) 4.03% D) 4.05% E) 4.07%

25. Dora borrows 5,000 at a nominal interest rate of 7% convertible semi-annually and will repay all interest and principal in 8 years.

Dora uses the borrowed amount to buy five 1,000 par value bonds with semi-annual coupons of 30 that are priced to yield 6% convertible semi-annually. The bonds are redeemed at par in 8 years immediately before Dora repays her original loan. Dora also invests all coupons as each one is received into a fund earning a nominal interest rate of 5% convertible semi-annually.

Calculate Dora's net gain or loss in 8 years.

A) 684 B) 763 C) -763 D) -684 E) -5,763

26. A 4-year loan accrues interest at an annual effective rate of 4.5% for the first two years, and at a force of interest of $\frac{1}{1+2t}$ after the first two years. If the loan is repaid by a single payment at the end of its 4-year term, what equivalent annual effective rate of discount did the borrower pay?

A) 9.1% B) 9.4% C) 9.7% D) 10.0% E) 10.4%

27. Sara invests 75 at the beginning of each quarter for 15 years into a fund earning a nominal interest rate of 6% convertible quarterly. At the end of each quarter Sara reinvests the interest earned in that fund into another fund earning an annual effective rate of 4.5%. At the end of the 15 years, Sara's total accumulated value is X. Calculate X.

A) 4,585 B) 5,585 C) 6,442 D) 6,994 E) 7,089

28. You are given the following information about the activity in an investment account:

Date	Fund Value Before Activity	Deposits	Withdrawals
January 1, 2016	1,000		
April 1, 2016	1,100		300
T	850	400	
December 31, 2016	1,300		

If the time-weighted yield during 2016 is 1% more than the dollar-weighted yield for this account, what is the Date T?

A) May 1, 2016 B) June 1, 2016 C) July 1, 2016
D) September 1, 2016 E) October 1, 2016

29. A specially designed 5-year bond provides for level coupon payments at the end of the 1st, 3rd, and 5th years, plus payment of its par value at the end of the 5th year. There are no other payments. Each coupon is equal to 5% of the bond's par value.

The bond is purchased on its issue date for a price that will produce a 10% annual effective yield to maturity. Based on that yield, what is the bond's Macaulay duration (measured in years) as of the purchase date?

A) 4.23 B) 4.41 C) 4.65 D) 4.70 E) 4.82

30. Adam took out a loan for 36,000 with level quarterly payments and financed it at an annual effective rate of 6.75% over 10 years. Adam sends an extra 1,000 with the 5th and 10th payments. What is the interest portion of the 11th payment?

A) 426.16 B) 438.60 C) 444.30 D) 456.16 E) 462.00

31. A portfolio consists of the following two zero-coupon bonds:

Term	Face Amount	Annual Effective Yield
4	10,000	2.5%
10	6,000	6.5%

These bonds have been purchased to support a liability that requires a single payment due in 6 years. This combination of assets provides a “fully immunized” portfolio.

What is the amount of the liability payment that is due in 6 years?

A) 9,050 B) 11,878 C) 16,000 D) 20,050 E) 24,828

32. A premium bond pays semi-annual coupons. The amount of premium amortized with the 4th coupon payment is 7.32. The amount of premium amortized in the 8th payment is 10.45. What amount of premium is amortized in the 11th payment?

A) 13.65 B) 13.11 C) 12.80 D) 12.50 E) 12.11

33. Amy wants to purchase an item on January 1, 2030, that will cost 15,000. Today is January 1, 2017, and to finance the item, she deposits X into an account each January 1 starting now, and $2X$ into the account every July 1st.

The account earns an annual effective interest rate of 7%, compounded semi-annually.

Calculate X. (No deposit will be made on 1/1/2030.)

A) 232 B) 237 C) 243 D) 251 E) 278

34. Which of the following statements are true?

I. Treasury bills are issued as zero-coupon bonds.

II. "Treasury strips" are zero-coupon bonds based on U.S. Treasury securities, but not issued by the U.S. Treasury.

III. Treasury notes and bonds are available either with or without inflation protection.

A) All but I. B) All but II. C) All but III.
D) All E) Another combination

35. A 1-year-deferred amortizing swap has notional amounts of 3 million, 2 million, and 1 million during the 2^{nd}, 3^{rd}, and 4^{th} years. (Because the swap is deferred one year, there is no notional amount in the 1^{st} year.)

The swap is based on the following spot rates:

Term (in years)	Spot Rate
1	5.1%
2	5.7%
3	6.1%
4	6.4%

If the 1-year spot rate at the end of the 3^{rd} year is 6.2%, what net settlement amount (to the nearest 100) will be paid by the payer at the end of the 4^{th} year?

A) 6,100 B) 4,500 C) 2,000 D) -100 E) -1,400

Solutions

1. There are only two cash flows in this NPV calculation, the investment of -50,000 at time zero and the payout from the fund at time 10. The second cash flow is the future value of the reinvested payouts from the project. The accumulated value of these payouts is:

   $$FV = 22{,}000(1.06)^5 + 22{,}000(1.06)^3 + 22{,}000(1.06) = 78{,}963.3147$$

   In the NPV calculation, there are the two cash flows: $CF_0 = -50{,}000$ and $CF_{10} = 78{,}963.3147$. Evaluate these cash flows at a rate of 5%.

   $$NPV = -50{,}000 + 78{,}963.3147 v_{0.05}^{10} = -1{,}523.3746$$

   Answer: D

2. We will work with a bond that has a face amount of 100 and pays an annual coupon at a rate r (so the annual coupon payment equals $100 \cdot r$). Since it is a par bond, the present value of its payments is 100. Using present value factors based on the zero-coupon bond prices given in the problem, we have:

   $$100 \cdot r \cdot (0.9597 + 0.9088 + 0.8570) + 100 \cdot 0.8570 = 100$$

   $$r = \frac{1 - 0.8570}{0.9597 + 0.9088 + 0.8570} = 0.052467$$

   Answer: B

3. First find the annual effective interest rates for the first nine years, and for the remaining 16 years.

   For the first 9 years, $i = \left(1 + \frac{0.05}{2}\right)^2 - 1 = 0.050625$

   For the remaining 16 years, $j = \left(1 - \frac{0.03}{12}\right)^{-12} - 1 = 0.030493$

   The future value of the four payments of 200 is then:

   $$200(1+i)^9(1+j)^{16} + 200(1+i)^5(1+j)^{16} + 200(1+i)(1+j)^{16} + 200(1+j)^{13}$$
   $$= 504.4080 + 413.9911 + 339.7818 + 295.5404 = 1{,}553.7213$$

   Answer: B

4. First we solve for i; then we evaluate the expression $\delta - d$.

$$v = 8 \cdot i$$

$$\frac{1}{1+i} = 8 \cdot i$$

$$1 = 8 \cdot (i + i^2)$$

$$8 \cdot i^2 + 8 \cdot i - 1 = 0$$

$$i = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 8 \cdot (-1)}}{2 \cdot 8} = \frac{-8 \pm \sqrt{96}}{16} = 0.112372, -1.112372$$

The second root is extraneous (an interest rate less than -100%).
Based on $i = 0.112372$, we calculate $\delta - d$ as follows:

$$\delta = \ln(1+i) = \ln(1.112372) = 0.1064951$$

$$d = \frac{i}{1+i} = \frac{0.112372}{1.112372} = 0.1010205$$

$$\delta - d = 0.005475$$

Answer: B

5. Brian's interest only payments on the loan each year are 40,000(0.04)=1,600. He will be making 20 such payments for a total of 32,000.

Brian's sinking fund payments can be found by setting up a future value of an annuity equation. The effective monthly interest rate is:

$$1.05^{1/12} - 1 = 0.0040741$$

Then:

$$40{,}000 = Xs_{\overline{240}|i} = X\frac{(1+i)^{240} - 1}{i}$$

$$X = 98.5696$$

He will be making 240 such payments for a total of 23,656.7124.
Therefore, Brian's total contribution is 23,656.7124+32,000=55,656.7124. The closest choice is choice C.

Answer: C

6. Let i be the semi-annual effective rate (since payments are made semi-annually).

   In 28 years, the future value will be: $50\ddot{s}_{\overline{56}|} = 50\dfrac{(1+i)^{56}-1}{d}$

   In 14 years, the future value will be: $50\ddot{s}_{\overline{28}|} = 50\dfrac{(1+i)^{28}-1}{d}$

   Setting the 28-year future value equal to four times the 14 year value, we have:

$$50\frac{(1+i)^{56}-1}{d} = 4(50)\frac{(1+i)^{28}-1}{d}$$
$$(1+i)^{56}-1 = 4[(1+i)^{28}-1]$$
$$(1+i)^{56}-4(1+i)^{28}+3=0$$
$$(1+i)^{28} = \frac{4 \pm \sqrt{16-4(1)(3)}}{2} = \frac{4\pm 2}{2} = 2\pm 1$$
$$(1+i)^{28} = 3 \text{ or } (1+i)^{28} = 1$$

   Since i cannot be zero, $i = 3^{1/28}-1 = 0.040016$

   The future value after 28 years is then: $50\dfrac{(1.040016)^{56}-1}{0.040016/1.040016} = 10{,}395.96$

   Ruth contributed 56(50) = 2,800.
   So the interest she earned is 7,595.96.

   Answer: A

7. First calculate the price of the zero coupon bond: $P = 1000 \cdot v_{0.04}^{3} = 888.9963$

   Now use the financial calculator to solve for the yield of a coupon bond with 12 coupons of 30 each, a maturity value of 1,000, and a purchase price of 888.9963:
   Set N=12, PV = -888.9963, PMT = 1,000(0.03) = 30, and FV = 1,000.
   CPT I/Y = 4.196278.

   Remember that this is the semi-annual effective yield. The problem asks for the annual effective yield: $i = (1.041963)^2 - 1 = 0.085686$

   Answer: D

8. Set up an equation of value for the present values:

$$PV = 875 = Xv^{20} + 1{,}000v^{25} + Xv^{30} + 1{,}000v^{35}$$

$$875 - 1{,}000v^{25} - 1{,}000v^{35} = X(v^{20} + v^{30})$$

$$X = \frac{875 - 1{,}000 \cdot 1.045^{-25} - 1{,}000 \cdot 1.045^{-35}}{1.045^{-20} + 1.045^{-30}} = 481.2123$$

Answer: D

9. Think of these as two separate perpetuities-due that with payment periods of 2 years. The "even" one starts at the present time and the "odd" one is deferred one year. First, calculate the two-year effective rate:

$$i = (1.05)^2 - 1 = 0.1025$$

The present value of the "even" perpetuity is

$$PV_{even} = 4\left(\frac{1}{d}\right) = 4\left(\frac{1}{i/(1+i)}\right) = 4\left(\frac{1.1025}{0.1025}\right) = 43.0244$$

The present value of the "odd" perpetuity is similar, but it is deferred by one year.

$$PV_{even} = \frac{7}{1.05}\left(\frac{1}{d}\right) = \frac{7}{1.05}\left(\frac{1+i}{i}\right) = \frac{7}{1.05}\left(\frac{1.1025}{0.1025}\right) = 71.7073$$

The total present value is the sum: 43.0244 + 71.7073 = 114.7317

Answer: D

10. The accumulation function using force of interest is:

$$a(t) = e^{\int_0^t \delta(u)du}$$

The two deposits accumulate to 4,500 at time 8, so we have:

$$300e^{\int_4^8 \frac{6}{u+4}du} + 200e^{\int_t^8 \frac{6}{u+4}du} = 4{,}500 \quad \rightarrow \quad 3e^{6\ln(u+4)\big|_4^8} + 2e^{6\ln(u+4)\big|_t^8} = 45$$

$$e^{6(\ln 12 - \ln(t+4))} = \frac{45 - 3e^{6(\ln 12 - \ln 8)}}{2} \quad \rightarrow \quad \left(\frac{12}{t+4}\right)^6 = \frac{45 - 3\cdot\left(\frac{12}{8}\right)^6}{2} = 5.41406$$

$$\frac{12}{t+4} = 5.41406^{1/6} = 1.32512 \quad \rightarrow \quad t = \frac{12}{1.32512} - 4 = 5.05581$$

Answer: B

11. First calculate amount of the semi-annual sinking fund deposit:
The semi-annual effective rate is $(1+0.03/2)^2-1=3.0225\%$

$SFD \cdot s_{\overline{40}|i=3.0225\%}=100,000$

N = 40, I/Y = 3.0225, PV = 0, and FV = 100,000. CPT PMT = 1,319.49.

Next, determine the amount of interest earned during the 8th year.
$Int_8 = Bal_8 - Bal_7 - 2(SFD)$
To calculate Bal_7:
Use the previous TVM entries, but change N to 14 (end of 7th year).
CPT FV = 22,579.59.
Similarly, for Bal_8, N = 16. CPT FV = 26,644.01.

Int_8 = 26,644.01 – 22,579.59 – 2(1,319.49) = 1,425.45

Interest paid in 8th year (and every other year) = 4(2,250) = 9,000.

Net interest = 9,000 – 1,425.45 = 7,574.55

Answer: C

12. Since the Bond Workbook cannot be used to solve for coupon rate (except by trial and error), we will use the BA II Plus's TVM functions. Since TVM cannot handle values that are not on a payment date (9/1/16 is not a payment date), we need to adjust the purchase price to the value as of a coupon date. We can do this by recognizing that the total price on 9/1/16 is equal to the price on 6/1/16 (the prior coupon date), plus interest for the time from June 1 to September 1. Using 30-day months, this is half of a coupon period (3 months out of 6), so we can calculate the 6/1/16 value by discounting the 9/1/16 price for one-half period:

$$9,552 \cdot \left(1+\frac{0.06}{2}\right)^{-1/2}=9,411.87$$

Having found the price as of 6/1/16, we can solve for the coupon amount, and then calculate the coupon rate:
Set N = 30, I/Y = 3, PV = 9,411.87, and FV = 10,000. CPT PMT = 269.99.

The coupon rate is 2.7% (=269.99/10,000) per 6 months, or 5.4% per year.

Answer: D

13. The formula for the price of a bond is: $P = (Fr)a_{\overline{n}|i} + Fv_i^n$

Plug in what we know for Alex's and Rudy's bonds:

$$P_{Alex} = (1{,}000r)a_{\overline{30}|0.02} + 1{,}000v_{0.02}^{30}$$
$$P_{Rudy} = (800r)a_{\overline{10}|0.05} + 800v_{0.05}^{10}$$

Alex's Price is twice Rudy's price:

$$P_{Alex} = (1{,}000r)a_{\overline{30}|0.02} + 1{,}000v_{0.02}^{30} = 2P_{Rudy} = 2[(800r)a_{\overline{10}|0.05} + 800v_{0.05}^{10}]$$
$$(1{,}000r)a_{\overline{30}|0.02} + 1{,}000v_{0.02}^{30} = (1{,}600r)a_{\overline{10}|0.05} + 1{,}600v_{0.05}^{10}$$
$$r(1{,}000a_{\overline{30}|0.02} - 1{,}600a_{\overline{10}|0.05}) = 1{,}600v_{0.05}^{10} - 1{,}000v_{0.02}^{30}$$
$$r = \frac{1{,}600v_{0.05}^{10} - 1{,}000v_{0.02}^{30}}{1{,}000a_{\overline{30}|0.02} - 1{,}600a_{\overline{10}|0.05}} = \frac{430.1903}{10{,}041.6798} = 0.0428405$$

This is the semi-annual coupon rate, so the annual coupon rate (payable semi-annually) is 2(0.0428405) = 0.085681, or 8.568%

Answer: B

14. Because Jeanette is paying the interest due each period plus repaying 600 of the principal, the principal is being reduced linearly over the 50-year period, and the 30,000 is fully repaid with the 50th payment of 600. Because the principal is declining linearly, we can determine the average loan balance during the 50 years by averaging the balances for the first year and the last year: $(30{,}000 + 600)/2 = 15{,}300$.

The total amount of interest paid is equal to 50 years of interest on this average loan balance: $7\% \times 15{,}300 \times 50 = 53{,}550$.

Answer: B

15. First calculate the quarterly effective interest rate by equating the annuity's value, 2,700, to the present value of its payments.
Set N=10, PV=-2,700, PMT=300, and FV=0. CPT I/Y = 1.96300.

The quarterly effective rate is 1.963%. This is equivalent to a monthly effective interest rate of: $1.01963^{1/3}-1=0.00650097$.

The monthly effective rate of discount is:

$$\frac{0.00650097}{1.00650097}=0.0064590$$

The nominal annual rate of discount convertible monthly is:

$$12\times 0.0064590=0.077508$$

Answer: E

16. There are four payments of 3,500, made at times 0, 1, 2 and 3. We want to find the value of these payments at time 5. If we first find the present value at time 0 we can get the future value at time 5 using the 5-year spot rate.

$$\text{Present value} = 3,500\left(1+\frac{1}{1.04}+\frac{1}{1.045^2}+\frac{1}{1.0525^3}\right)=13,072.38$$

$$\text{Future value} = 13,072.38\left(1.06^5\right)=17,493.79$$

This could also be done using forward rates. We need four forward interest rates:

$i_{(0,5)}$, $i_{(1,5)}$, $i_{(2,5)}$, and $i_{(3,5)}$.

$$(1+i_{(0,5)})^5=\frac{(1+s_5)^5}{1}=1.06^5$$

$$(1+i_{(1,5)})^4=\frac{(1+s_5)^5}{(1+s_1)^1}=\frac{1.06^5}{1.04}$$

$$(1+i_{(2,5)})^3=\frac{(1+s_5)^5}{(1+s_2)^2}=\frac{1.06^5}{1.045^2}$$

$$(1+i_{(3,5)})^2=\frac{(1+s_5)^5}{(1+s_3)^3}=\frac{1.06^5}{1.0525^3}$$

The future value of these payments will then be:

$$\begin{aligned} FV &= 3,500(1+i_{0,5})^5+3,500(1+i_{1,5})^4+3,500(1+i_{2,5})^3+3,500(1+i_{3,5})^2 \\ &= 3,500(1.06^5)+3,500\left(\frac{1.06^5}{1.04}\right)+3,500\left(\frac{1.06^5}{1.045^2}\right)+3,500\left(\frac{1.06^5}{1.0525^3}\right) \\ &= 17,493.79 \end{aligned}$$

Answer: D

17. First calculate the Macaulay duration of the cash flows:

$$D_{\text{mac}} = \frac{2(400v^2)+3(300v^3)+4(200v^4)+6(100v^6)}{400v^2+300v^3+200v^4+100v^6} = 3.0364$$

Then calculate the modified duration:

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{3.0364}{1.045} = 2.9056$$

Answer: D

18. To accumulate 1,000 from t=2 to t=6, we need the values of $a(2)$ and $a(6)$:

$$a(2) = (1+s_2)^2 = 1.036^2 = 1.0733$$

$$a(4) = (1+s_4)^4 = 1.044^4 = 1.1880$$

$$a(5) = a(4)\cdot(1+i_{4,5}) = (1.1880)\cdot(1.053) = 1.2509$$

$$a(6) = a(5)\cdot(1+i_{5,6}) = (1.2509)\cdot(1.055) = 1.3197$$

Accumulated value at t=6 of a payment of 1,000 at t=2:

$$1,000\cdot\frac{a(6)}{a(2)} = 1,000\cdot\frac{1.3197}{1.0733} = 1,229.60$$

Answer: D

19. The effective semi-annual interest rate is: $i = (1.0575)^{1/2} - 1 = 0.0283482$

Use the principal paid in payment 2 to solve for the level payment amount:

$$PRin_2 = 580 = PMT \cdot v^{18-2+1}$$

$$PMT = \frac{580}{v^{17}} = 580(1+i)^{17} = 932.8487$$

Now use this payment amount to solve for the outstanding balance after the 7th payment. Using the prospective method, this will be the present value of the remaining 11 payments:

$$PV = 932.8487a_{\overline{11}|i} = 8,710.8041$$

Answer: C

20. The payments form a geometric annuity:

$$10{,}000 = X \cdot a_{\overline{15}|9\%}^{-13\%} = X \cdot \frac{1-\left((1-0.13)/(1.09)\right)^{15}}{0.09-\left(-0.13\right)} = 4.39094X$$

$$X = \frac{10{,}000}{4.39094} = 2{,}277.42$$

Answer: B

21. Price of a stock using the constant growth model is

$$P = \frac{Div}{i-g}.$$

We have a price of 50, dividend of 5, and interest rate of i =0.12. Solve for g:

$$50 = \frac{5}{0.12-g}$$

g = 0.02

Answer: D

22. In this situation, we are given the change in price (the amount the purchaser overpaid) and asked to estimate the change in interest rate (yield). We start by calculating the percent change in price:

$$\frac{\Delta \text{Price}}{\text{Price}} = \frac{11{,}016}{10{,}800} - 1 = 0.02$$

(The price actually paid was 2% greater than the price that would provide a 6% yield.)

We can apply the modified approximation method to estimate the change in interest rate:

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1+i} = \frac{8.00}{1.06} = 7.54717$$

$$\frac{\Delta \text{Price/Price}}{\Delta i} \approx -D_{\text{mod}} = -7.54717$$

$$\frac{0.02}{\Delta i} \approx -7.54717$$

$$\Delta i \approx \frac{0.02}{-7.54717} = -0.00265$$

Therefore, the yield to maturity for a purchase price of 11,016 is approximately: $6\% - 0.265\% = 5.735\%$

Due to convexity, the actual yield at 11,016 will be slightly different from 5.735%. For a given change in interest rate, convexity causes the price to be slightly higher than we would predict based on duration alone. Our calculation based on duration alone has predicted that a drop of 0.265% in interest rate would cause the price to rise to 11,016. So that drop in interest rate would actually result in a price higher than 11,016. Since the price paid for the bond was 11,106 (and not higher), the yield could not have dropped a full 0.265%. Therefore, the new yield must be slightly more than 5.735%.

Answer: B

23. Because the purchase price is more than 1,000, we know that the bond is a premium bond. Premium callable bonds are priced assuming they will be called at the earliest possible call date. We will write an equation for the price assuming the bond will be called after 6 years.

$$P = 1{,}025 = (1{,}000r)a_{\overline{12}|0.025} + 1{,}000v_{0.025}^{12}$$

$$r = 0.027437$$

The amount of each coupon is then 1,000(0.027437) = 27.44

Answer: D

24. In order to find the fixed interest rate for this swap, we need present value factors for 1, 2, and 3 years:

t	$f^*_{[t-1,t]}$	$a(t)$	P_t
1	0.03200	1.032	0.9690
2	0.04202	1.07536	0.9299
3	0.04603	1.12486	0.8890

$$R = \frac{1-P_3}{P_1+P_2+P_3} = \frac{1-0.8890}{0.9690+0.9299+0.8890} = 0.039816$$

Answer: A

25. First, calculate the amount that Dora will need to repay for the loan in 8 years:

$$FV = 5{,}000\left(1+\frac{0.07}{2}\right)^{16} = 8{,}669.9302$$

Next, determine the future value of the reinvested coupons that Dora is receiving from her bonds. Notice that since the coupon rate is equal to the yield rate, the purchase price of the bonds is the same as the redemption value. She will be investing five coupons of amount 30 every 6 months for 8 years. The future value of the coupons is then:

$$FV = 150 s_{\overline{16}|0.025} = 2907.0337$$

At the end of the 8-years, Dora will have 2,907.0337 from the reinvested coupons and 5,000 from the redemption value of the bonds for a total of 7,907.03. However, she owes 8,669.93 from her original loan. Therefore, what Dora receives at the end of 8 years is $7{,}907.03 - 8{,}669.93 = -762.90$. She has a loss of 762.90.

Answer: C

26. We will calculate the 4-year accumulation factor and then convert it to the equivalent annual effective rate of discount:

$$a(4) = 1.045^2 \cdot e^{\int_2^4 \frac{dt}{1+2t}} = 1.092025 \cdot e^{\left([\ln(1+2t)]/2\right)\big|_2^4}$$

$$= 1.092025 \cdot e^{(\ln 9 - \ln 5)/2} = 1.092025 \cdot \left(\frac{9}{5}\right)^{0.5} = 1.4651$$

$$(1+i)^4 = 1.4651 \qquad i = 1.4651^{0.25} - 1 = 0.100189$$

$$d = i/(1+i) = 0.100189 / 1.100189 = 0.091065$$

Answer: A

27. First, make sure that you have the interest rates translated to the appropriate quarterly rate. Her payments of 75 are earning a quarterly effective rate of $i = \frac{0.06}{4} = 0.015$. Her reinvested interest payments are earning a quarterly effective rate of $j = (1.045)^{1/4} - 1 = 0.011065$.

Let's organize the payments and interest into a table:

Time	0	0.25	0.5	⋯	14.5	14.75	15
Payment	75	75	75	⋯	75	75	0
Total payments	75	150	225	⋯	4,425	4,500	4,500
Interest earned		75(0.015)	150(0.015)	⋯	4,350(0.015)	4,425(0.015)	4,500(0.015)

Sara's total accumulated value is the 75 payments of 60 = 4,500 plus the future value of the reinvested interest payments.

The reinvested interest payments have a future value calculation of

$$\begin{aligned} FV &= 75(0.015)Is_{\overline{60}|j} \\ &= 75(0.015)\frac{\ddot{s}_{\overline{60}|} - 60}{j} \\ &= 75(0.015)\frac{\frac{(1+j)^{60}-1}{d_j} - 60}{j} \\ &= 2,588.7282 \end{aligned}$$

Add the 4,500 from Sara's payments and we have a total accumulated value of 2,588.7282+4500=7,088.73

Answer: E

28. You don't need T to calculate the time weighted yield:

$$1+i=\frac{1{,}100}{1{,}000}\left(\frac{850}{800}\right)\left(\frac{1{,}300}{1{,}250}\right)=1.2155$$

So, i=0.2155, and the time-weighted yield is 21.55%.
Therefore, the dollar-weighted yield is 20.55%
Plug this into the dollar-weighted yield formula to solve for T.

Interest earned is : $1{,}300-1{,}000-(400-300)=200$
Let t be the fraction of a year that remains as of the date T. Then the dollar-weighted formula is:

$$0.2055=\frac{200}{1{,}000-300(1-3/12)+400(t)}$$

$$1{,}000-300(1-3/12)+400(t)=\frac{200}{0.2055}$$

$$400t=\frac{200}{0.2055}+300(3/4)-1000$$

$$t=0.4956$$

The closest date would be July 1st.

Answer: C

29. Assuming a par value of 100 (we could use any amount), the bond's payments are 5 at time 1; 5 at time 3; and 105 at time 5.

The bond's yield to maturity (10%) is the appropriate interest rate to use for calculating the present values for the duration calculation

Macaulay Duration (D_{mac}) = weighted average of the times of the bond's payments, where the weights are based on the present values of the payments.

The weight for the payment at t=3 equals:

$$\frac{\text{PV(pmt. 3)}}{\text{PV(all pmts. combined)}}=\frac{5\cdot v^3}{5\cdot v+5\cdot v^3+105\cdot v^5}$$

$$=\frac{5\cdot 1.10^{-3}}{5\cdot 1.10^{-1}+5\cdot 1.10^{-3}+105\cdot 1.10^{-5}}=0.0511$$

The other weights are 0.0618 and 0.8870 for the payments at t=1 and t=5. To calculate the bond's duration, apply these weights to the times of the payments:

$$D_{\text{mac}}=(0.0618)\cdot 1+(0.0511)\cdot 3+(0.8870)\cdot 5=4.650$$

Answer: C

30. First calculate the amount of the level payments for the original loan. The quarterly effective rate is $i = (1.0675)^{1/4} - 1 = 0.016464$.
Set N = 40, I/Y = 1.6464, PV = 36,000, and FV=0. CPT PMT = -1,235.77.

Then calculate what the outstanding balance would be immediately after the 10th payment if we ignore the extra amounts paid with the 5th and 10th payments.
Set N = 30 (payments remaining) and CPT PV = 29,071.30.

From this balance, subtract the total value of the extra 1,000 Adam added to the 5th and 10th payments:

$$29{,}071.30 - 1{,}000 \cdot 1.016464^5 - 1{,}000 = 26{,}986.22$$

The interest in the 11th payment will be $26{,}986.22 \times 0.01646 = 444.30$.

Answer: C

31. The problem states that the portfolio is "fully immunized." This can be achieved when two asset payments support one liability payment, with the assets and the liability having matching present values and modified durations. The present value and modified duration for the asset portfolio are calculated in the following table (which is the table in the problem with 4 columns added for the present value and modified duration calculations).

Term	Face Amount	Yield	Present Value	Weight	D_{mod}	Wt'd D_{mod}
4	10,000	2.5%	9,059.51	0.7392	3.902	2.885
10	6,000	6.5%	3,196.35	0.2608	9.390	2.449
	Total		12,255.86			5.334

The modified duration for each zero-coupon bond is its term divided by $1+i$, where i is the bond's annual effective yield. The weights are based on the values of the bonds. $D_{\text{mod}}^{Portfolio}$ is 5.334, the total of the weighted D_{mod}'s for the two bonds, 5.334. Since the liability payment at time 6 must also have a duration of 5.334, we can calculate its yield, i:

$$6/(1+i) = 5.334 \qquad i = 6/5.334 - 1 = 12.486\%$$

And since the liability's present value must be 12,255.86, we can calculate that the amount of the liability payment at time 6 is:

$$12{,}255.86 \cdot 1.12486^6 = 24{,}827.51$$

Answer: E

32. The solution to this problem is based on the principle that the amount of the bond premium amortized increases by a factor of 1+i each period, where i is the effective yield per coupon period. From the 4th coupon payment to the 8th, the amount amortized increases by a factor of $(1+i)^4$, so we have:

$$(1+i)^4 = 10.45/7.32 = 1.4276 \qquad i = 1.4276^{1/4} - 1 = 9.3078\%$$

Now we can multiply the amount amortized in the 8th payment by $(1+i)^3$ to find the amount amortized in the 11th payment:

$$10.45(1+i)^3 = 13.6480$$

Answer: A

33. The deposits of X are an annual-payment annuity-due with a future value calculated at 1/1/2030 of $X\ddot{s}_{\overline{13}|0.07}$.

The deposits of 2X can be thought of an annual-payment annuity-immediate with its future value calculated at 7/1/2029. So this future value will need another six-months of interest accumulation to bring it to 1/1/2030:

$$2Xs_{\overline{13}|0.07}1.07^{0.5}$$

The sum of these two terms should equal 15,000, the amount Amy needs to have on 1/1/2030:

$$2Xs_{\overline{13}|0.07}1.07^{0.5} + X\ddot{s}_{\overline{13}|0.07} = 15{,}000$$
$$41.6673X + 21.5505X = 15{,}000$$
$$X = 237.27499$$

Answer: B

34. All 3 statements are true.
Statement I is true, because Treasury bills (unlike notes and bonds) have no coupon payments, so they are zero-coupon bonds.
Statement II is true, because, although the Treasury does not issue the zero-coupon bonds known as "Treasury strips," they are available from investment banks that create zero-coupon bonds from coupon-paying Treasury bonds.
Statement III is true, because both notes and bonds (but not bills) are available with inflation protection (as well as without inflation protection).

Answer: D

35. Because the notional amount of this swap varies from period to period, the calculation of the swap rate will involve not only the present value factors, but also the forward rates for years 2, 3, and 4. These values are shown in the following table, and were calculated using the following formulas, where r_n is the n-year spot rate, and $f^*_{[n-1,n]}$ is the (n-1)-year forward rate (the rate for the period from n-1 to n).

$$P_n = (1+r_n)^{-n} \qquad f^*_{[n-1,n]} = (1+r_n)^n / (1+r_n)^{n-1} - 1$$

n	Spot Rate r_n	PV factor P_n	Fwd. rate $f^*_{[n-1,n]}$
1	5.1%	0.95147	5.100%
2	5.7%	0.89506	6.303%
3	6.1%	0.83725	6.905%
4	6.4%	0.78025	7.305%

The formula for the swap rate is:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

Filling in the values (with notional amounts in millions), we have:

$$R = \frac{Q_2 \cdot f^*_{[1,2]} \cdot P_2 + Q_3 \cdot f^*_{[2,3]} \cdot P_3 + Q_4 \cdot f^*_{[3,4]} \cdot P_4}{Q_2 \cdot P_2 + Q_3 \cdot P_3 + Q_4 \cdot P_4}$$

$$= \frac{3 \cdot 0.06303 \cdot 0.89506 + 2 \cdot 0.06905 \cdot 0.83725 + 1 \cdot 0.07305 \cdot 0.78025}{3 \cdot 0.89506 + 2 \cdot 0.83725 + 1 \cdot 0.78025}$$

$$= 0.066513$$

A spot rate of 6.2% at the end of the 3rd year will require the payer of the fixed rate (6.6513%) to make a net settlement payment at the end of the 4th year of:

$$1{,}000{,}000 \cdot (6.6513\% - 6.2\%) = 4{,}513$$

Answer: B

Practice Exam 9

Exam FM

Questions

1. Chester will receive quarterly payments from a perpetuity, with the first payment occurring three months from today. At a nominal interest rate of 7% convertible semi-annually, the present value of this perpetuity is 18,400. The amount of the first payment is 30. Each subsequent quarterly payment is larger than the previous payment by an amount of X. Calculate X.

   A) 4.50 B) 4.78 C) 5.02 D) 5.11 E) 5.28

2. You are given the following information about an investment account:

Date	Fund Value Before Activity	Deposits	Withdrawals
January 1, 2014	200		
May 1, 2014	X		30
August 1, 2014	210	50	
December 31, 2014	265		

   If the time-weighted rate of return for 2014 was 25%, calculate X.

   A) 195.30 B) 208.56 C) 214.50
   D) 220.35 E) Cannot be determined

3. Which of the following characteristics of a bond tends to result in a higher yield?

   A) greater liquidity

   B) higher seniority in the issuing corporation's capital structure

   C) a higher rating by a bond rating agency

   D) a longer term to maturity

   E) inclusion of an inflation protection feature

4. George buys an 8,000 par value 10-year bond with 7.5% annual coupons to yield an annual effective rate of 6%. The interest portion of the 9th coupon minus the interest portion of the 5th coupon is X. Calculate X.

   A) 23.54 B) 22.20 C) –25.60 D) –23.54 E) –22.20

5. A business takes out an interest-only loan of 10,000 with a loan interest rate of 6.8% per annum, convertible quarterly. The loan requires quarterly interest payments for 10 years. The final payment, at the end of 10 years, will consist of interest for the final quarter, plus 10,000 to repay the loan principal.

   The business also maintains a sinking fund to accumulate the 10,000 principal payment required on the date of the final loan payment. Level deposits are made to the sinking fund at the end of each month (for 10 years). The balance in the sinking fund earns interest at a 6% nominal annual rate, convertible monthly.

   What is the net interest paid by the borrower during the fourth year of this loan?

   A) 511.48 B) 514.98 C) 518.21 D) 523.16 E) 529.02

6. A 1,000-par-value 10-year bond with an 8% annual coupon rate pays semi-annual coupons. The bond is purchased at a price that will generate a 6.00% yield to maturity for the buyer.

   What is the amount of premium amortized at the time of the 6th coupon payment?

   A) 5.52 B) 5.81 C) 6.07 D) 6.23 E) 6.42

7. Brian has an annuity that will make monthly payments of 350 starting three years from today and provide a total of 36 payments. Lisa has an annuity that will make quarterly payments of X starting today and continuing for a total of 24 payments.

   At a 4% annual effective rate, the present values of Lisa's and Brian's annuities are equal. To the nearest 100, what is the future value of (all of) the payments under Lisa's annuity as of a date 6 years from today at a 6% nominal interest rate convertible quarterly?

   A) 10,600 B) 11,500 C) 12,500 D) 13,400 E) 14,300

8. At what force of interest do the following two payment streams have the same present value?

Time of payment:	1	2	3
Payments in Stream 1	-1000	650	450
Payments in Stream 2	100	-300	200

A) 0.060 B) 0.064 C) 0.068 D) 0.072 E) 0.075

9. Tara, who just turned 33, plans to make monthly payments of 250 starting one month from today to a fund earning a nominal interest rate of 4.5% convertible monthly.

Her last payment will be one month before her 65th birthday. Starting on her 65th birthday, Tara plans to make perpetual monthly withdrawals of amount X from the same fund earning the same rate. Calculate X.

A) 798.40 B) 795.42 C) 800.12 D) 799.34 E) 801.40

10. A corporation has a liability that requires a payment of 100,000 in 5 years. It has purchased two zero-coupon bonds that provide full immunization for this liability. The two bonds mature in 3 years and 7 years. All three instruments (the two bonds and the liability) are valued at a 6% annual effective rate.

Assuming that a 6% interest rate will again used to analyze the portfolio 1 year from now, which of the following statements will be true at that time?

I. The present value of the assets equals the present value of the liability.

II. The modified duration of the assets equals the modified duration of the liability.

III. The portfolio is fully immunized.

A) I. only B) I. and II. only C) All
D) None E) Cannot be determined from the information given

11. Marcy takes out a 14-year loan for an amount X. She finances the loan at an annual effective rate of 6.5% and makes level monthly payments. The outstanding balance immediately after her 15th level payment of 960 is 100,710. How much total interest has Marcy paid so far, including her 15th payment?

A) 5,980 B) 6,118 C) 8,212 D) 8,444 E) 21,634

12. Stock B is expected to pay an annual dividend of 4 this year, and to increase its dividend by 4% per year in all future years. If this year's dividend will be paid exactly 6 months from now, what is the value of this stock (to the nearest whole number), based on a 10% annual effective interest rate?

A) 42 B) 50 C) 67 D) 70 E) 100

13. Daniel takes out a monthly-payment home loan for 204,000 at a nominal rate of 6.5% compounded monthly. His first payment is due one month from the loan date, and the term of the loan is 30 years.

Immediately after Daniel's 48th payment, he refinances the remaining balance with a new 30-year loan at a nominal rate of 4.5% convertible monthly. He is also charged 4,000 in closing costs, which are added to the principal at the time of the refinancing.

If Daniel makes all of the scheduled payments under the new loan, how much less will he have paid than if he had kept the original loan?

A) 37,277 B) 48,574 C) 41,277 D) 103,169 E) 79,640

14. Payments of 200, 400, and 600 at times 3, 4, and 5 respectively have a Macaulay duration of 4.3 at an annual effective interest rate of i.

Calculate i.

A) 5.0% B) 5.7% C) 6.1% D) 6.3% E) 6.7%

15. An interest rate swap with a level notional amount has a term of two years and a swap rate of 6.48%. The 1-year spot rate is 6%. Find the 1-year forward rate, $f^*_{[1,2]}$.

A) 6.33% B) 6.49% C) 6.77% D) 6.85% E) 7.00%

16. Ryan buys a 10-year semi-annual-coupon 1,000 par value bond to yield 6% convertible semi-annually. The bond will be redeemed at par. The amount of premium amortized in the 6th coupon is 9.75.

Calculate the original price of Ryan's bond.

A) 875 B) 1,226 C) 1,093 D) 1,024 E) 903

17. The market price of a 1,000 par value bond on February 19th, 2017 is 980. The bond pays coupons of amount *X* on April 15th and October 15th, and the bond will mature on October 15th 2019. The bond is priced to yield 5% convertible semi-annually.

    There are 182 days between coupons dates of October 15th 2016 and April 15th 2017 and 127 days between October 15th 2016 and February 19th, 2017.

    Calculate X.

    A) 26.20 B) 24.40 C) 23.50 D) 22.60 E) 20.90

18. Alicia purchases an annuity that will make level annual payments of 100 starting today. Beginning with the payment 12 years from today, each payment is 5% greater than the previous payment. The last payment will be received 25 years from today.

    This annuity was priced based on an interest rate such that $v^4 = 0.85$. What is the purchase price of this annuity?

    A) 1,822 B) 1,833 C) 1,850 D) 1,858 E) 1,922

19. At time 0, Angelica deposits 3,000 into a fund earning a constant force of interest δ. In six years, Angelica's fund will accumulate to 4,000.

    At time 1, Rick deposits 2,500 into a fund earning interest at an annual effective rate of discount *d*. The accumulated values of Rick's and Angelica's funds are equal at time 4. Calculate *d*.

    A) 13.3% B) 11.7% C) 9.8% D) 8.9% E) 4.8%

20. Brett will make quarterly deposits of *X* to a fund starting today and ending with a last payment 10 years from today. The fund earns a nominal interest rate of 5% compounded quarterly. Brett has determined *X* so that he will earn a total of 1,000 in interest over the 10-year period.

    Calculate X.

    A) 82.42 B) 84.22 C) 76.26 D) 81.49 E) 87.04

21. Given the following definitions for the variables r, i_e, i_u, and c, what is an appropriate expression for the difference between the nominal and real interest rates? (All rates are continuously compounded.)

r	compensation for deferred consumption
i_e	expected rate of inflation
i_u	compensation for unexpected inflation
c	cost of inflation protection

A) $r - c$ B) i_e C) $i_e - c$ D) $i_e + i_u$ E) $i_e + i_u + c$

22. Stuart invests 3,000 at t=0 and an amount X at t=4 into a fund earning a nominal interest rate of 5% convertible monthly.

The accumulated value of the fund at t=3 is one-fifth of its accumulated value at t=6.

Calculate X.

A) 12,105 B) 12,103 C) 11,577 D) 10,409 E) 10,312

23. The following equation defines a yield curve:

$$s_k = 0.08 + 0.003k - 0.002k^2$$

(s_k stands for the annual effective rate of return for a zero coupon bond maturing in k years.)

Using this yield curve, what would be the present value one year from today of a payment of 400 five years from today?

A) 293 B) 325 C) 342 D) 347 E) 387

24. Liz purchased a 5-year bond that pays semi-annual coupons at a 6% annual rate. The bond can be called at its par value of 1,200 on any coupon date, starting at the end of year 4. It was priced to yield at least 4.7% convertible semi-annually. What price did Liz pay for the bond?

A) 1,268.80 B) 1,256.30 C) 1,090.20 D) 1,057.33 E) 1,046.91

25. A 6-year 1,000 par value bond is priced to yield 6% convertible semi-annually. The bond will redeemed at par on its maturity date. Based on its 6% yield rate, the bond's book value immediately after the 5^{th} coupon payment is 1,125. Calculate the book value immediately after the 2^{nd} coupon.

A) 1,909 B) 1,565 C) 1,171 D) 1,203 E) 1,459

26. You are given the following term structure for spot interest rates:

Term (in years)	Spot interest rate
1	7.0%
2	7.2%
3	7.5%
4	X
5	8.0%

Based on these spot rates, the value of a 1,000 par value 5-year bond paying 5.5% annual coupons is 905. Calculate X.

A) 26.6% B) 7.69% C) 7.58% D) 6.93% E) 6.07%

27. Sara starts an investment with an initial deposit of 200 and makes a second deposit of amount X four years later.

Tim invests in the same fund and makes deposits on the same days as Sara, but his initial deposit is 100, and his second deposit (after four years) is 400.

The fund earns a nominal interest rate of 5% convertible monthly for the first three years and 5% convertible semi-annually for the remaining years. At the end of the 8^{th} year, the value of Tim's investment is 100 less than the value of Sara's investment.

Calculate X.

A) 150 B) 196 C) 278 D) 356 E) 360

28. Austin finances 100,000 with a 20-year loan that requires semi-annual payments. The loan rate is 6.4% convertible semi-annually, and the first loan payment is due 6-months after the date of the loan. In order to pay off the loan early, Austin adds 200 to each payment.

How much total interest will Austin pay towards this loan?

A) 71,370 B) 71,480 C) 76,390 D) 78,690 E) 86,680

29. A portfolio consists of the following 3 zero-coupon bonds:

Term	Face Amount
3	1,100
5	1,800
7	2,900

Based on an annual effective interest rate of i, the Macaulay duration of this portfolio is 5.5 years. What is the value of i?

A) 4.5% B) 4.7% C) 4.9% D) 5.1% E) 5.3%

30. A corporation borrows 1,000,000 under a 20-year loan requiring annual interest payments of 80,000, with repayment of the principal at the end of 20 years. In addition to the annual interest payments, the company also makes level annual deposits to a sinking fund in order to accumulate the amount needed to repay the principal of the loan at 20 years.

If the company's total annual payments (the loan interest plus the sinking fund deposit) are equal to the level payment that would be required under a 20-year loan of 1,000,000 at a 9% annual effective rate, what annual effective interest rate is being credited to the balance in the sinking fund?

A) 5.22% B) 6.51% C)7.02% D) 8.51% E) 9.98%

31. An investment pays 1,000 in one year, 2,000 at the end of the second year, and X at the end of the third year. It was purchased to yield an annual effective rate of 9%. If the Macaulay duration is 2.5, calculate X.

A) 4,112 B) 4,531 C) 4,879 D) 5,250 E) 5,744

32. Brian makes a deposit 600 at t=0 and a deposit of 580 at t=2. Lisa makes a deposit of 1,200 at t=1. The accounts earn the same interest rate. At t=5, Brian's and Lisa's accounts have the same balance. What is the balance in each account at t=5?

A) 1,850 B) 2,100 C) 2,350 D) 2,780 E) 2,950

33. Ryan makes a single deposit of 3,000 into a fund that earns interest based on a nominal annual rate of discount of 4% convertible every two months.

    At what rate of simple interest would Ryan's investment accumulate to the same value at the end of two years?

    A) 3.9% B) 4.0% C) 4.1% D) 4.2% E) 4.3%

34. A company has liabilities that require payments of 5,000 in two years and 10,000 in five years. The company purchases two zero-coupon bonds in order to match these liability cash flows exactly. If the 2-year bond has a yield of 4% and the 5-year bond has a yield of 7.5%, what is the internal rate of return for the company's investment in these two bonds?
    (All rates are annual effective rates.)

    A) 5.8% B) 6.1% C) 6.5% D) 6.8% E) 7.2%

35. A two-year bond with a level annual coupon of 4.20 and a maturity value of 100 is priced at 101.291. The one-year spot rate is 0.045.

    Based on this information what would be the swap rate for a two-year interest rate swap with level notional amount?

    A) 3.50% B) 3.52% C) 3.91% D) 4.33% E) 4.77%

Solutions

1. First calculate the quarterly effective rate needed for the present value calculation:

$$\left(1+\frac{0.07}{2}\right)^{1/2}-1=0.0173495$$

The present value of a perpetuity with terms P, $P+Q$, $P+2Q$, etc. is:

$$PV=\frac{P}{i}+\frac{Q}{i^2}=\frac{30}{i}+\frac{X}{i^2}$$

$$18{,}400=\frac{30}{0.0173495}+\frac{X}{0.0173495^2}$$

$$X=0.0173495^2\left(18{,}400-\frac{30}{0.0173495}\right)=5.018$$

Answer: C

2. The time-weighted rate of return formula, based on the given data, is:

$$1+0.25=\left(\frac{X}{200}\right)\left(\frac{210}{X-30}\right)\left(\frac{265}{210+50}\right)$$

$$1.25=\frac{55{,}650X}{52{,}000(X-30)}$$

$$65{,}000X-1{,}950{,}000=55{,}650X$$

$$9{,}350X=1{,}950{,}000 \qquad X=208.5561$$

Answer: B

3. The correct answer is D, a **longer term to maturity**. Just as longer term loans tend to have higher interest rates, the same holds true for bonds.
Choice A, **greater liquidity**, is a characteristic that makes a bond easier to sell, and therefore more attractive to a buyer. This results in a higher price, and thus a **lower** yield.
Choice B, **higher seniority**, and choice C, **higher rating**, also make a bond more attractive and valuable to a buyer, resulting in a higher price and a lower yield.
Choice E, **inflation protection**, is a valuable feature for the bond buyer, because the coupon payments increase with inflation. As with choices A, B, and C, this increases the bond's price and decreases its yield.

Answer: D

4. The principal portion of a coupon is: $F(r-i)v^{n-k+1}$.

   So the principal portion of the 5th coupon is:
   $8{,}000(0.075-0.06)v^{10-5+1}=84.5953$

   Similarly, the principal portion of the 9th coupon is:
   $8{,}000(0.075-0.06)v^{10-9+1}=106.7996$

   The principal portion of the 9th coupon is larger than the principal portion of the 5th coupon by $106.7966-84.5953=22.2043$. Since the principal portion has increased by 22.2043, the interest portion has decreased by 22.2043, so the answer is -22.20.

   Answer: E

5. The amount of the monthly sinking fund deposit is: $=\dfrac{10{,}000}{s_{\overline{120|}\,i=0.005}}=61.02$

   The interest earned in the sinking fund during the 4th year equals the growth in the fund balance during the year minus deposits made during the year:
   $$61.02\cdot(s_{\overline{48|}}-s_{\overline{36|}})-12\cdot 61.02=900.77-732.25=168.52$$
   Interest paid on the interest-only loan during the 4th year (or any year):
   $10{,}000\cdot 0.068=680$ (on a quarterly basis, this is $10{,}000\cdot\dfrac{0.068}{4}\cdot 4=680$)
   The net interest paid by the borrower during the 4th year is:
   $$680-168.52=511.48$$

   Calculator solution:
   N=120, I/Y=0.5, PV=0, and FV=10,000. CPT PMT = 61.02.

   2nd AMORT
   "P1=" 37 ENTER (the number of the first deposit during the 4th year)
   DOWN arrow
   "P2 =" 48 ENTER (the number of the last deposit during the 4th year)
   DOWN arrow x3
   INT =-168.52 (Interest earned on the sinking fund during the 4th year.)

   10,000 x 0.068 = 680 (Interest paid on the loan during any year.)
   680 – 168.52 = 511.48 (Net interest paid during the 4th year.)

   Answer: A

6. Book value of bond after 5th payment $=1,000\cdot(1.03)^{-15}+40\cdot a_{\overline{15}|i=0.03}=1,119.38$

Book value of bond after 6th payment $=1,000\cdot(1.03)^{-14}+40\cdot a_{\overline{14}|i=0.03}=1,112.96$

Amount amortized with the 6th coupon payment $=1,119.38-1,112.96=6.42$

Alternatively:
Premium after 5th payment $=F\cdot(r-i)\cdot a_{\overline{15}|}=1,000\cdot(0.04-0.03)\cdot a_{\overline{15}|}=119.38$
Premium after 6th payment $=F\cdot(r-i)\cdot a_{\overline{14}|}=1,000\cdot(0.04-0.03)\cdot a_{\overline{14}|}=112.96$
Amount of amortization at time of 6th coupon payment $=119.38-112.96=6.42$

Or by formula:
$F\cdot(r-i)\cdot v^{n-t+1}=1,000\cdot(0.04-0.03)\cdot 1.03^{-(20-6+1)}=6.42$

Calculator solution:

Set N=15, I/Y=3, PMT=40, and FV=1,000. CPT PV = 1,119.38 STO 1
N=14. CPT PV = 1,112.96
+/- + RCL 1 = 6.42

Answer: E

7. Let i equal the monthly effective rate for Brian's annuity:

$$i=(1.04)^{1/12}-1=0.00327374$$

Let j equal the quarterly effective rate for Lisa's annuity:

$$j=(1.04)^{1/4}-1=0.009853407$$

First calculate the value today of Brian's payments. Remember that his annuity is deferred three years.

$$PV=\frac{1}{1.04^3}(350\ddot{a}_{\overline{36}|i})=\frac{350}{1.04^3}\cdot\frac{1-1.00327374^{-36}}{0.00327374/1.00327374}=10,584.7484$$

Set this equal to the present value of Lisa's annuity and solve for her payments:

$$10,584.7484=(Pmt)\ddot{a}_{\overline{24}|j}$$

$$Pmt=10,584.7484\div\frac{1-1.009853407^{-24}}{0.009853407/1.009853407}=492.5400$$

The accumulated value of Lisa's payments 6 years from today at a 6% nominal rate convertible quarterly is:

$$(492.54)\ddot{s}_{\overline{24}|0.015}=492.54\cdot\frac{1.015^{24}-1}{0.015/1.015}=14,314.70$$

Answer: E

8. If the present values of the two streams are equal at a force of interest δ, then the present value of the *differences* between the cash flows of the two streams is zero. In other words, the *internal rate of return* of the series of differences (expressed as a force of interest) is δ. So we need to find the IRR of the series (-1,100; 950; 250) and convert it to a force of interest.

   Entering these three cash flows into the calculator's cash flow workbook (as CF_0, C01, and C02) and solving for IRR produces 7.5044%, the annual effective internal rate of return.

   The equivalent force of interest is $\ln(1.075044) = 0.07236$.

   Answer: D

9. First calculate the accumulated value of the payments as of Tara's 65th birthday at a monthly effective rate $i = 4.5\%/12 = 0.375\%$. She does not make a payment on her 65th birthday, so there are 383 $(= 32 \times 12 - 1)$ payments, ending one month before her 65th birthday. The value on that birthday is the accumulated value of a 383-payment annuity-due:

$$250\ddot{s}_{\overline{383}|i} = 250 \cdot \frac{1.00375^{383} - 1}{0.00375 / 1.00375} = 213{,}706.2428$$

   This is the present value of her perpetuity-due.

$$213{,}706.2428 = X \cdot \ddot{a}_{\overline{\infty}|} = \frac{X}{d}$$
$$X = 213{,}706.2428(0.00375 / 1.00375) = 798.4044$$

   Answer: A

10. Statement I. is true. Both the assets and the liability increase in value by a factor of 1.06, so they have equal values after one year.
    Statement II. is true. The Macaulay duration of each instrument decreases by 1 year, and the modified duration decreases by $1/1.06 = 0.94340$ years. So the assets' and the liability's durations both decrease by 0.94340 and remain equal.
    Statement III. is true. When one liability payment is supported by two asset payments (one with a longer and one with a shorter duration than the liability payment) with matching present value and modified duration, the portfolio is fully immunized.

    Answer: C

11. Use the outstanding balance information to solve for the loan amount X using the retrospective method.

The effective monthly rate is: $i = (1.065)^{1/12} - 1 = 0.00526169$

$$100,710 = X(1+i)^{15} - 960 s_{\overline{15}|i}$$
$$X = 106,897.7673$$

The difference between X and the outstanding balance is:

106,897.7673-100,710 = 6,187.7673

This is the amount of principal that was paid with the first 15 payments. However, Marcy's total payments were 15(960) = 14,400. The difference is the amount of interest she has paid:

14,400 – 6,187.7376 = 8,212.26

This can also be done directly on the BA II Plus by first finding the original balance as the PV of 168 payments of 960, and then using the AMORT worksheet with payments P1=1 and P2=15 to find the amount of INT in the first 15 payments.

Answer: C

12. The present value of the projected dividends is:

$$4 \cdot a^{4\%}_{\overline{\infty}|i=10\%} \cdot (1+i)^{0.5} = 4 \cdot \frac{1}{0.10 - 0.04} \cdot (1.10)^{0.5} = 69.92 \approx \underline{70}$$

Answer: D

13. First determine the monthly payment amount for the original loan. The effective monthly interest rate is : $i = 0.065/12 = 0.0054167$.

$$204{,}000 = Pmt \cdot a_{\overline{360}|i} \qquad Pmt = 1{,}289.4188$$

Or N=360, I/Y=6.5/12, PV=204,000, and FV=0. CPT PMT = -1,289.4188.

If Daniel had kept the original loan, he would have paid a total of:

$$1{,}289.4188(360) = 464{,}190.7565$$

The amount of the refinanced loan is the loan balance immediately after the 48th payment plus 4,000 in closing costs.

$$Bal_{48} = 1{,}289.4188 \cdot a_{\overline{312}|i=0.54167\%} = 193{,}921.5279$$

Or N=312. CPT PV = 193,921.5279.
The refinanced loan amount was 193,921.5279 + 4,000 = 197,921.5279.
The interest rate, j, equals $0.045/12 = 0.00375$.
The new monthly payment is calculated as follows:

$$197{,}921.5279 = Pmt \cdot a_{\overline{360}|j} \qquad Pmt = 1{,}002.8393$$

Or N=360, I/Y=0.375, PV=197,921.5279, and FV=0. CPT PMT = 1,002.8393.
Daniel's total payments equal the sum of 48 payments of the original monthly payment amount, plus 360 payments under the refinanced loan:

$$1{,}289.4188(48) + 1{,}002.8393(360) = 422{,}914.2528$$

The difference between the total payments under the original loan and the total payments Daniel will make as a result of refinancing is:

$$464{,}190.76 - 422{,}914.56 = 41{,}276.20$$

Answer: C

14. The formula for the Macaulay duration of these payments is:

$$D_{\text{mac}} = 4.3 = \frac{(3)v^3 200 + (4)v^4 400 + (5)v^5 600}{v^3 200 + v^4 400 + v^5 600}$$

$$4.3(v^3 200 + v^4 400 + v^5 600) = (3)v^3 200 + (4)v^4 400 + (5)v^5 600$$

$$0 = -260v^3 - 120v^4 + 420v^5 = 10v^3(42v^2 - 12v - 26)$$

We can find the value of v by solving the quadratic $42v^2 - 12v - 26 = 0$.
The positive root is $v = 0.9425169$, so i is:

$$i = \frac{1}{v} - 1 = \frac{1}{0.9425169} = 0.060989$$

We could get the same answer on the BAII Plus by using an IRR calculation to find the rate of return for the following cash flows: $CF_3 = -260$, $CF_4 = -120$, and $CF_5 = 420$. The result is: IRR = 6.0989.

Answer: C

15. We are given $r_1 = 0.06$, so we have $P_1 = 1/1.06 = 0.9434$. We next find the 2-year present value factor (or zero-coupon bond price), P_2. The swap rate is the par coupon R. Thus:

$$R = 0.0648 = \frac{1 - P_2}{P_1 + P_2} = \frac{1 - P_2}{0.9434 + P_2}$$
$$0.0611 + 0.0648P_2 = 1 - P_2$$
$$P_2 = 0.8817$$

Then: $$f^*_{[1,2]} = \frac{P_1}{P_2} - 1 = 0.0700$$

Answer: E

16. The amount of premium amortized in the k^{th} payment is: $F(r-i)v^{n-k+1}$.

The amount of premium amortized in the 6^{th} coupon is9.75, so we can solve for the coupon rate, r:

$$9.75 = 1{,}000(r - 0.03)v^{20-6+1} = 1{,}000(r - 0.03)v^{15}$$
$$r = \frac{9.75}{1{,}000 \cdot 1.03^{-15}} + 0.03$$
$$r = 0.04519018$$

The price of the bond is then,
$P = 1{,}000(0.04519018)a_{\overline{20}|0.03} + 1{,}000v^{20} = 1{,}225.9916$
Or N=20, I/Y=3, PMT=45.19018, and FV=1,000. CPT PV = -1,225.9916.

Answer: B

17. The market price of a bond is equal to the price-plus-accrued minus the accrued interest: $MP = P_0(1+i)^t - t(Fr)$, where t is the fraction of the bond period that the seller owned the bond, and P_0 is the price after the previous coupon payment.

The last coupon was on October 15th, 2016. Since the bond matures on October 15th, 2019, there are 6 coupons left after that coupon payment.

$$t = \frac{127}{182} \text{ and } P_0 = (1{,}000r)a_{\overline{6}|0.025} + 1{,}000v_{0.025}^6$$

Since the market price on February 19, 2017, is 980, we have:

$$MP = 980 = [(1{,}000r)a_{\overline{6}|0.025} + 1{,}000v_{0.025}^6](1.025)^{127/182} - \frac{127}{182}(1{,}000r)$$

$$980 - 1{,}000v_{0.025}^6(1.025)^{127/182} = (1{,}000r)a_{\overline{6}|0.025}(1.025)^{127/182} - \frac{127}{182}(1{,}000r)$$

$$r = \frac{980 - 1{,}000v_{0.025}^6(1.025)^{127/182}}{(1{,}000)a_{\overline{6}|0.025}(1.025)^{127/182} - \frac{127}{182}(1{,}000)} = \frac{102.7165}{4{,}906.0537} = 0.02094$$

So the coupon amount is 20.94. The closest answer is E.

Answer: E

18. This problem can be tricky in terms of counting the payments and figuring out when they occur. First, note that there are 26 payments; the first payment is today and the last payment is 25 years later, so there are 25 payments <u>after</u> today, for a total of 26.

Next, consider that these 26 payments consist of a level annuity followed by a geometric annuity with 5% annual increases. The first 5% increase occurs 12 years from today, with the 13th payment, which will be 105. However, we can consider the geometric annuity as beginning with the 12th payment (at time 11), which is the last payment of 100. Then we have 11 level payments of 100, followed by 15 geometrically increasing payments that begin at 100 and have 14 annual 5% increases.

This leads to the following formula for the purchase price (present value) of Alicia's annuity:

$$PV = 100\ddot{a}_{\overline{11}|} + 100v^{11} \cdot \ddot{a}_{\overline{15}|}^{5\%}$$

$$= 100\frac{1-(1+i)^{-11}}{d} + 100(1+i)^{-11} \cdot \frac{1-\left[1.05/(1+i)\right]^{15}}{(i-0.05)/(1+i)}$$

In order to evaluate this expression, we need to know the value of i. The problem states that $v^4 = 0.85$, so we have:

$$v = (0.85)^{1/4} = 0.960185 \text{ and } i = \frac{1}{v} - 1 = 0.041466.$$

Now we can calculate the present value of the annuity's payments at i:

$$100\frac{1-(1.041466)^{-11}}{0.041466/1.041466} + 100(1.041466)^{-11} \cdot \frac{1-\left[1.05/(1.041466)\right]^{15}}{(0.041466-0.05)/(1.041466)}$$

$$= 905.2009 + 1{,}016.4158 = 1{,}921.6167$$

Answer: E

19. First solve for δ:

$$4{,}000 = 3{,}000e^{6\delta} \quad \rightarrow \quad \frac{4}{3} = e^{6\delta}$$

$$\ln\left(\frac{4}{3}\right) = 6\delta \quad \rightarrow \quad \delta = \frac{\ln\left(\frac{4}{3}\right)}{6} = 0.047947$$

Now calculate Angelica's accumulated amount in 4 years:

$$3{,}000e^{4\delta} = 3{,}634.2412$$

For Rick, this is the amount that his investment will accumulate to after 3 years, so we have:

$$2{,}500(1+i)^3 = 3{,}634.2412 \qquad (1+i)^3 = \frac{3{,}634.2412}{2{,}500}$$

$$i = \left(\frac{3{,}634.2412}{2{,}500}\right)^{1/3} - 1 = 0.1328$$

The question asks for an effective rate of discount:

$$d = \frac{i}{1+i} = \frac{0.1328}{1.1328} = 0.11724$$

Answer: B

20. The total interest earned will be the <u>future value</u> of his payments minus the <u>total amount</u> of those payments.

If Brett is making his first quarterly payment today and his last payment 10 years from today, he will make 41 payments. So the total interest earned will be $FV - 41X$. The future value of his payments can be calculated using the future value of an annuity-due plus the 41st payment: $FV = X\ddot{s}_{\overline{40}|i} + X$.

The interest rate, i, is equal to the quarterly effective interest rate:

$$i = \frac{0.05}{4} = 0.0125.$$

Since he wants his total interest to be 1,000, the equation is:

$$1{,}000 = X\ddot{s}_{\overline{40}|i} + X - 41X = X\ddot{s}_{\overline{40}|i} - 40X$$

$$1{,}000 = X(\ddot{s}_{\overline{40}|i} - 40)$$

$$X = \frac{1000}{\ddot{s}_{\overline{40}|i} - 40} = 82.4186$$

(Note that the 41st payment is made at the end of 10 years, so it does not contribute any interest during the 10-year period. So we could have analyzed this as a 40-period annuity-due, ignoring that final payment. Alternatively, we could have analyzed it as a 41-period annuity-immediate with its last payment at the end of 10 years, since $\ddot{s}_{\overline{40}|} + 1 = s_{\overline{41}|}$*.)*

Answer: A

21. We need to write an expression for the nominal rate a lender would charge (for a non-inflation-protected loan), and another expression for the real rate a lender would charge (for an inflation-protected loan). The difference between these two expressions is the difference between the nominal and real interest rates.

Since r is "compensation for deferred consumption," ignoring inflation and the risk of default, and c is what the lender gives up in order to have protection against inflation, the real interest rate (ignoring default) is $r - c$.

In the absence of inflation protection, a lender would increase the interest rate charged by i_e to offset expected inflation, and would also add an amount i_u to compensate for the risk of unexpected inflation. The total rate charged (ignoring the risk of default) would be $r - i_e - i_u$.

The difference between these two rates is $(r + i_e + i_u) - (r - c) = i_e + i_u + c$.

(Note: Both of these expressions ignore the compensation for default risk, which would add the same amount to both rates, and therefore would not affect the difference.)

Answer: E

22. The annual effective interest rate is $i = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 0.05116190$.

The value of the fund in three years is $3{,}000(1+i)^3 = 3{,}484.4167$.

The value of the fund in six years is $3{,}000(1+i)^6 + X(1+i)^2$. We are told this amount is five times the three-year accumulation value, so we have:

$$3{,}000(1+i)^6 + X(1+i)^2 = 5 \times 3{,}484.4167 = 17{,}422.0835$$

$$X = \frac{17{,}422.0835 - 3000(1+i)^6}{(1+i)^2} = 12{,}104.7424$$

Answer: A

23. This can be done quickly by taking the present value at time 0 of a payment of 400 at time 5, and then finding the future value of that amount in 1 year. We need the 1-year and 5-year spot rates.

$$s_1 = .08 + .003(1) - .002(1^2) = .081; \quad s_5 = .08 + .003(5) - .002(5^2) = .045$$

The desired value is given by $\left(\frac{400}{1.045^5}\right)(1.081) = 346.9798$.

The problem can also be done using the language of forward rates. We need to calculate a forward rate from the yield curve that would apply for a 4-year term beginning one year from today. This is $i_{1,5}$.

Using the yield curve we can calculate this forward rate from the spot rates:

$$(1+i_{1,5})^4 = \frac{(1+s_5)^5}{(1+s_1)}$$

The values of s_1 and s_5 are 0.081 and 0.045, as calculated above. Then:

$$(1+i_{1,5})^4 = \frac{(1+0.045)^5}{(1+0.081)} = 1.1528048 \rightarrow i_{1,5} = 1.1528048^{1/4} - 1 = 0.036189$$

The present value one year from today of a payment of 400 five years from today is:

$$\frac{400}{(1+i_{1,5})^4} = \frac{400}{1.1528048} = 346.9798$$

Answer: D

24. This is a premium bond, so it should be priced assuming it will be called at the earliest possible date, which is immediately after the 8th coupon. So the price of the bond is:

$$P = 1,200(0.03)a_{\overline{8}|0.0235} + 1,200v_{0.0235}^{8} = 1,256.2865$$

(Note: Since the problem does not state otherwise, we assume that the bond's maturity value is equal to its par value and that there is no call premium. If there had been a call premium, then we would have calculated the price based on a call and based on no call, and chosen the lower price.)

Answer: B

25. The book value of a bond is the present value of the remaining cash flows, discounted at the yield rate. Immediately after the 5th coupon, there are 7 coupons remaining, so the book value is:

$$1,125 = (1,000r)a_{\overline{7}|0.03} + 1,000v_{0.03}^{7}$$

$$r = \frac{1,125 - 1,000v_{0.03}^{7}}{1,000a_{\overline{7}|0.03}} = 0.05006$$

The bond is paying semi-annual coupons of 50.06 (or, more likely, 50). The book value immediately after the 2nd coupon is the present value of the remaining cash flows as of that date:

$$BV = (50.06)a_{\overline{10}|0.03} + 1,000v_{0.03}^{10}$$
$$= 1,171.1440$$

On the BA II Plus, first find the amount of the coupon:
Set N=7, I/Y=3, PV=-1,125, and FV=1,000. CPT PMT = 50.06.
Then change N to 10 and CPT PV = -1,171.14.

Answer: C

26. The value of the bond is the present value of 5 coupons of 55 each, plus the present value of the redemption value. These present values are based on the spot rates, including X for the 4-year spot rate.

$$905 = \frac{55}{1.07} + \frac{55}{1.072^2} + \frac{55}{1.075^3} + \frac{55}{(1+X)^4} + \frac{1,055}{1.08^5}$$

$$(1+X)^4 = \frac{55}{905 - \left(\frac{55}{1.07} + \frac{55}{1.072^2} + \frac{55}{1.075^3} + \frac{1,055}{1.08^5}\right)} = 1.2658$$

$$(1+X)^4 = 1.2658 \quad \rightarrow \quad (1+X) = 1.06070173 \quad \rightarrow \quad X = 0.06070$$

Answer: E

27. First calculate Tim's accumulated value at time 8. Tim's deposit of 100 is made today at time 0, and his deposit of 400 is made four years later, at time 4. His accumulated value at the end of the 8th year (time 8) is:

$$FV = 100\left(1+\frac{0.05}{12}\right)^{36}\left(1+\frac{0.05}{2}\right)^{10} + 400\left(1+\frac{0.05}{2}\right)^{8}$$
$$= 148.6783 + 487.3612 = 636.0395$$

This amount plus 100 is equal to the future value of Sara's investment:

$$FV = 736.0395 = 200\left(1+\frac{0.05}{12}\right)^{36}\left(1+\frac{0.05}{2}\right)^{10} + X\left(1+\frac{0.05}{2}\right)^{8}$$
$$736.0395 = 297.3565 + X\left(1+\frac{0.05}{2}\right)^{8}$$
$$X = \frac{736.0395 - 297.3565}{\left(1+\frac{0.05}{2}\right)^{8}} = 360.0476$$

Answer: E

28. First calculate what the monthly loan payment would be without the extra 200 that Austin added:

$$100{,}000 = Pmt \cdot a_{\overline{40}|0.032} \quad \rightarrow \quad Pmt = 4{,}467.2096$$

After adding the extra 200, Austin's payments are 4,667.2096. Now, figure out how long it would take to pay off the loan with this payment amount.

$$100{,}000 = 4{,}667.2096 a_{\overline{n}|0.032}$$

Set I/Y=3.2, PV=100,000, PMT=4,667.2096, and FV=0. CPT N = 36.738.

It will take 37 payments to pay off the loan, 36 full payments of 4,667.2096, and one partial payment. To find the total amount paid, we can set N=37 and CPT FV=1,208.4831. This positive value is the amount that Austin would have overpaid if he paid the full 4,667.2096 in the 37th payment. So the total amount he pays is: $37 \times 4{,}667.2095 - 1{,}208.4831 = 171{,}478.2969$. Since 100,000 of this amount is principal, the amount of interest Austin paid is 71,478.30.

Answer: B

29. Based on the information given, we have :

$$D_{\text{mac}} = 5.5 = \frac{3 \cdot 1{,}100 \cdot v^3 + 5 \cdot 1{,}800 \cdot v^5 + 7 \cdot 2{,}900 \cdot v^7}{1{,}100 \cdot v^3 + 1{,}800 \cdot v^5 + 2{,}900 \cdot v^7}$$

This can be simplified as follows:

$$5.5 = \frac{3 \cdot 11 + 5 \cdot 18 \cdot v^2 + 7 \cdot 29 \cdot v^4}{11 + 18 \cdot v^2 + 29 \cdot v^4} = \frac{33 + 90 \cdot v^2 + 203 \cdot v^4}{11 + 18 \cdot v^2 + 29 \cdot v^4}$$

$$5.5 \cdot \left(11 + 18 \cdot v^2 + 29 \cdot v^4\right) = 33 + 90 \cdot v^2 + 203 \cdot v^4$$

$$60.5 + 99 \cdot v^2 + 159.5 \cdot v^4 = 33 + 90 \cdot v^2 + 203 \cdot v^4$$

$$43.5 \cdot v^4 - 9 \cdot v^2 - 27.5 = 0$$

Substituting x for v^2 and applying the quadratic formula, we have:

$$43.5 \cdot x^2 - 9 \cdot x - 27.5 = 0 \qquad x = 0.90525, -0.69835$$

Rejecting the negative root (since v^2 can't be negative), we have:

$$v^2 = 0.090525 \qquad v = 0.95145 \qquad i = \frac{1}{v} - 1 = 0.05103$$

Answer: D

30. First calculate the total annual payment amount, which is equal to the annual payment on a 20-year loan for 1,000,000 at 9%.
Set N=20, I/Y=9, PV=1,000,000, and FV=0. CPT PMT = -109,546.475.

The corporation makes an annual interest payment of 80,000, so the remaining 29,546.475 is the sinking fund deposit. Now we calculate the interest rate credited on the sinking fund.
Set N=20, PV=0, PMT=-29,546.475, and FV=1,000,000.
Then CPT I/Y = 5.2199.

Answer: A

31. The present value of the payments is:

$$P = \frac{1{,}000}{1.09} + \frac{2{,}000}{1.09^2} + \frac{X}{1.09^3}$$

The weights used to calculate the duration are:

$$w_1 = \frac{917.4312}{P} \qquad w_2 = \frac{1683.3600}{P} \qquad w_2 = \frac{X/1.09^3}{P}$$

The Macaulay duration is the weighted average time of the payments:

$$D_{\text{mac}} = 2.5 = \frac{917.4312}{P}(1) + \frac{1{,}683.3600}{P}(2) + \frac{X/1.09^3}{P}(3)$$

$$2.5P = 917.4312(1) + 1{,}683.3600(2) + \frac{X}{1.09^3}(3)$$

$$2.5(917.4312 + 1{,}683.3600 + \frac{X}{1.09^3}) = 917.4312(1) + 1{,}683.3600(2) + \frac{X}{1.09^3}(3)$$

$$6{,}501.978 + \frac{2.5X}{1.09^3} = 4{,}284.1512 + \frac{3X}{1.09^3}$$

$$2{,}217.8268 = \frac{0.5X}{1.09^3} \qquad X = 5{,}744.3000$$

Answer: E

32. The present value of Brian's deposits is: $600 + 580v^2$.
The present value of Lisa's deposit is: $1{,}200v$.

Their deposits earned the same interest rate, and the accumulated values after 5 years are equal. Therefore, the values of their payments would be equal as of any other valuation date, including time 0. So we can set these two present values equal to each other and we have:

$$600 + 580v^2 = 1200v \quad \rightarrow \quad 580v^2 - 1{,}200v + 600 = 0$$

$$v = \frac{1{,}200 \pm \sqrt{1{,}200^2 - 4(600(580))}}{1{,}160} \quad \rightarrow \quad v = 0.8456129,\ 1.223352606$$

If v=0.8456129, then i=0.18257419.
If v=1.223352606, then i=-0.1825741858, which is not reasonable.

The value of Lisa's account five years from today, which is four years after her deposit, is: $1{,}200(1.182574)^4 = 2{,}346.90$.

Answer: C

33. If $d^{(6)} = 0.04$, then the annual effective interest rate is:

$$i = \left(1 - \frac{d^{(6)}}{6}\right)^{-6} - 1 = \left(1 - \frac{0.04}{6}\right)^{-6} - 1 = 0.04095$$

Ryan's accumulated value after 2 years is: $3{,}000(1+i)^2 = 3{,}250.7318$

If j is the simple interest rate that would have accumulated 3,250.7318 in two years, then:

$$3{,}000(1+2j) = 3{,}250.7318 \qquad j = \left(\frac{3{,}250.7318}{3{,}000} - 1\right) \div 2 = 0.04179$$

Answer: D

34. We will calculate the total price paid for the two bonds, and then use that payment and the bonds' redemption payments to calculate the internal rate of return for the investment.

The price for the 2-year bond is: $5{,}000 \cdot 1.04^{-2} = 4{,}622.78$.

The price for the 5-year bond is: $10{,}000 \cdot 1.075^{-5} = 6{,}965.59$.

The total cost of the two bonds is 4,622.78 + 6,965.59 = 11,588.37.

In the Cash Flow worksheet, set CF_0=-11,588.37, C01=0, C02=5,000, C03 and C04=0, and C05=10,000. Then press IRR CPT. The internal rate of return is 6.7811%.

Instead of C03 = C04=0, and C05=10,000, we could have set C03=0, F03=2, and C04=10,000. This is an example of using the "F" values to indicate the number of times that a particular cash flow occurs. (In this case a cash flow of 0 occurred 2 times, at time 3 and time 4.)

(Note: For a reasonableness check on our answer, we can observe that the rate of return had to be between 4% and 7.5% (the yields of the two bonds). Since more money is invested in the 7.5% bond, and it remains invested for a longer period, we expect the overall return for the investment to be closer to 7.5% than to 4%. So 6.78% appears to be reasonable.)

Answer: D

35. The swap rate is the par bond coupon rate. To find this we need to find the unknown two year spot rate x.

The price of the two year bond with 4.2% coupons is101.291 per 100 face, so we have:

$$101.291 = \frac{4.2}{1.045} + \frac{104.2}{(1+x)^2} \rightarrow x = 0.035$$

The prices of the one- and two-year zero-coupon bonds (the present value factors for 1 and 2 years) are:

$$P_1 = \frac{1}{1.045} = 0.9569 \quad P_2 = \frac{1}{1.035^2} = 0.9335\,.$$

The par bond coupon rate is:

$$\frac{1-P_2}{P_1+P_2} = \frac{1-0.9335}{0.9569+0.9335} = 0.0352$$

Answer: B

Practice Exam 10

Exam FM

Questions

1. A homebuyer plans to take out a conventional fixed-rate mortgage for 250,000. A 15-year mortgage is available at an interest rate of 4.8% convertible monthly. A 30-year mortgage is also available, at a rate of 5.7% convertible monthly.

   The monthly payments are lower under the 30-year mortgage, but there are twice as many payments, so the total amount paid is greater than under the 15-year mortgage. To the nearest 1,000, how much more interest would the homebuyer pay under the 30-year mortgage than under the 15-year mortgage?

   A) 165,000 B) 167,000 C) 169,000 D) 170,000 E) 171,000

2. Josie took out a 20-year loan of 100,000 at an interest rate of 8% convertible quarterly. She will pay off the loan with quarterly payments. She will make level quarterly payments of X, starting 3-months after the original loan date. With the 20^{th} payment, she will add an extra 10,000. Josie will pay off the loan exactly with the last full payment of X at the end of 20 years.

   Calculate X.

   A) 2,264 B) 2,347 C) 2,498 D) 2,516 E) 9,167

3. An investor purchases a 1,000 par value bond 8 years before its maturity date for 709.46.

   Assuming that she receives all future semi-annual coupon payments and the redemption payment of 1,000, the bond will provide the investor a 12.00% yield convertible semi-annually.

   What is the bond's annual coupon rate?

   A) 6.00% B) 6.15% C) 6.25%
   D) 6.40% E) a significantly different value

4. The Bates Family bought a house for 265,000. They paid 50,000 down and took out a 30-year mortgage for the balance at 6.5% convertible monthly. After 180 monthly payments (15 years), they sold the house for 305,000. At the time of the sale, they paid closing costs of 2,500 plus 2% of the sale price. After paying off their mortgage, what is the net amount that the Bates paid to own this house for 15 years?

   A) 154,213 B) 151,713 C) 148,113 D) 104,213 E) 90,398

5. A bond with par value X pays semi-annual coupons at a rate of 4% convertible semi-annually. The bond matures at par on February 19, 2011. The bond is purchased on December 31 2009 to yield 2.5% convertible semi-annually. The price-plus-accrued on the purchase date is 2,150. Calculate X.

   A) 2,005 B) 2,054 C) 2,085 D) 2,131 E) 2,147

6. At time 4 Bill deposits X into an account earning an annual effective interest rate of 5%. At time 7 Suzanne deposits $2X$ into an account earning an annual simple interest rate of i. Bill and Suzanne each earn 30 in interest between time 9 and time 9.5. Calculate i.

   A) 3.15% B) 3.57% C) 3.83% D) 3.96% E) 4.25%

7. Given the following information, what is the value of the 3-year forward *rate*, $i_{3,4}$?

   The 3-year spot rate (s_3) is 5.6%.
   The price of a 5-year zero-coupon bond is 75.21 per 100 of maturity value.
   The 4-year-forward, 1-year rate ($i_{4,5}$) is 6.37%.

   All rates are expressed as annual effective rates.

   A) 5.90% B) 6.01% C) 6.08% D) 6.15% E) 6.21%

8. A newly issued 10-year bond with a 10,000 face amount pays semi-annual coupons at a 7% (annual) rate. If the bond is being sold for 10,500, what is its modified duration? Important: the modified duration should be based on <u>the bond's yield convertible semi-annually</u>, not on an annual effective rate.

   A) 7.0 B) 7.2 C) 7.4 D) 7.6 E) 7.9

9. A fund earns interest at a rate of discount of 4% convertible quarterly. An amount X is deposited at time 0 and earns 500 of interest during the period from time 0 to time 4. How much total interest would be earned by that same deposit of X during the period from time 0 to time 15?

   A) 5,238   B) 2,866   C) 2,372   D) 2,349   E) 778

10. An investor has a portfolio consisting of 3,000 worth of a 4-year bond with a modified duration of 3.6; 6,000 worth of a 5-year bond with a modified duration of X; and 7,000 worth of a 6-year bond with a modified duration of 5.8. If the modified duration of the entire portfolio is 5, calculate X.

    A) 5.60   B) 4.92   C) 4.84   D) 4.77   E) 4.52

11. A corporation must make payments of 10,000 at the end of each of the next 5 years (at times 1, 2, 3, 4, and 5). It finds that it can exactly match these payments by purchasing a 5-year annual-coupon bond with an annual effective yield of 8%, and also taking out a 5-year loan of 130,000 at an annual effective interest rate of 9%. The bond's maturity value is equal to its par value. The loan will be fully repaid by a single payment at time 5.

    To the nearest 1,000, what is the company's net outlay at time 0 for these two transactions (the bond purchase and the loan)?

    A) 0   B) 39,000   C) 40,000   D) 42,000   E) 46,000

12. Melanie made annual deposits into a fund for her children's education. At the beginning of the first year, she invested 1,000; at the beginning of the second year, she invested 2,000; at the beginning of each of the third through fifth years, she invested 3,000; at the beginning of the sixth and seventh years, she invested X. At the beginning of the eighth year, the value of her investment was 18,000.

    If the annual effective yield on Melanie's investment was 6.5%, calculate X.

    A) 380   B) 540   C) 880   D) 1,600   E) 3,500

13. Which of the following statements are true of bonds issued by states and municipalities?

    I. Revenue bonds are bonds that are backed by the taxing authority of the state or municipality.

    II. Although they are not considered to be free of default risk, no bond issued by a U.S. state or municipality has ever defaulted.

    III. The coupon rates for bonds issued by U.S. states and municipalities are usually set higher than for other comparable bonds in recognition of their tax status.

    A) I. and II. B) I. and III. C) II. and III. D) All E) None

14. Alexis deposits 500 into an account at the beginning of each month for 15 years. In return, she receives a payment of X at the end of each year forever. The first payment of X occurs 16 years after the first deposit.

    Assuming an annual effective yield of 4%, calculate X.

    A) 4,893 B) 4,909 C) 16,150 D) 50,125 E) 60,500

15. A 3-quarter amortizing interest rate swap has notional amounts of 3 million, 2 million, and 1 million during the 3 quarters of the swap's tenor. The fixed interest rate for the swap is 1.75% (a quarterly effective rate).

    At the end of the first quarter of the swap, the yield curve is as shown in the following table. (The interest rates are quarterly effective rates.)

Term (quarters)	1	2	3	4
Spot rate	1.80%	1.90%	1.95%	1.99%

    To the nearest 100, what is the market value of the payer's position in this swap as of the end of the first quarter?

    A) 3,400 B) 4,300 C) -4,300 D) -3,400 E) A different value

16. A deposit of X is made to a fund at time 1.

    The fund grows at a force of interest $\delta_t = \dfrac{1}{t+1}$.

    If the amount of interest earned from time 1 to time 5 is 6,000, calculate X.

    A) 9,000 B) 6,000 C) 5,500 D) 3,000 E) 2,000

17. A 14-year 100,000 par value bond pays 6% annual coupons. Based on an annual effective yield of Y, the interest portion of the 13th coupon is 1,130. Calculate Y.

A) 0.85% B) 1.03% C) 2.52% D) 2.97% E) 3.38%

18. A perpetuity has monthly payments of 5, 10, 15, 5, 10, 15, etc. that repeat forever in this pattern. The first payment occurs 10 years from today. At an interest rate of 7% convertible semi-annually, what is the present value today of this perpetuity?

A) 1,596 B) 877 C) 872 D) 651 E) 147

19. A lottery drawing jackpot is 40 million. Susan wanted to guarantee a win, so she bought all the possible combinations of numbers, which cost her 5 million. Susan received the jackpot in annual payments of 2 million each, paid at the end of each year for 20 years.

If Susan invests the 2 million at the end of each year into an account that earns an annual effective rate of 5%, after how many years will she have accumulated more money than if she had simply invested her 5 million in an account earning an annual effective rate of 5%?

A) About 3 years B) About 5 years C) About 8 years
D) About 10 years E) Never

20. You are given the following spot rates for a LIBOR yield curve.

Years to maturity	1	2	3	4
Spot rate	2.75%	3.00%	3.40%	4.00%

A four-year annual-payment interest rate swap with a level notional amount of 1,000,000 will enable you to pay a fixed rate and receive 1-year LIBOR. What is the fixed rate?

A) 2.98% B) 3.22% C) 3.46% D) 3.73% E) 3.95%

21. You are given the following information about an investment account:

Date	Value Immediately Before Activity	Deposits	Withdrawals
January 1	48		
February 1	49	$2X$	
August 1	69		X
December 31	58		

The dollar-weighted rate of return for the year is 9.2%. Calculate X, given that $0 < X < 20$.

A) 2.7 B) 4.1 C) 4.9 D) 5.1 E) 5.9

22. Felix deposits 1,500 into an account today. The account earns a rate of i convertible quarterly for the first 2 years and a rate of $3i$ convertible monthly after the first 2 years. If the total amount of interest that Felix earns during the first 7 years is 350, what is the balance in the account at time 5?

A) 1,718 B) 1,700 C) 1,688 D) 1,621 E) 1,595

23. A company has an obligation to pay 100,000 in 6 years. It purchases ten 6-year annual-coupon bonds with an annual coupon rate of 3%, each with a par value of X. The company will reinvest all the coupons into a fund that earns an annual effective rate of 4%. The bonds are redeemed at par.

If the redemption value of the ten bonds plus the accumulated value of the reinvested coupons is used to pay off the company's obligation exactly, what is the value of X?

A) 8,340 B) 8,560 C) 8,990 D) 9,210 E) 9,710

24. A homebuyer takes out a conventional fixed-rate mortgage for 250,000. The interest rate is 4.8% convertible monthly. The amount of interest included in the 30^{th} monthly payment is 910.32.

If the borrower makes all the mortgage payments when they are due, what is the total amount of interest (to the nearest 1,000) that he/she will pay over the term of the mortgage?

A) 122,000 B) 124,000 C) 126,000 D) 128,000 E) 130,000

25. Michael bought a 15-year bond for a price of 1,229. The bond pays semi-annual coupons at an annual coupon rate of 3%. The bond is callable at its par value of 1,200 on any coupon date starting at the end of the 13th year.

Assuming the bond does not default, what is the minimum effective yield that Michael could earn on this bond, expressed as a nominal annual rate convertible semi-annually?

A) 1.39% B) 1.40% C) 2.78% D) 2.80% E) 2.86%

26. Kent took out a 10-year loan at 4.5% convertible semi-annually. The first semi-annual payment, due 6 months after the loan date is 1,030. Each subsequent semi-annual payment increases by 30. What is Kent's loan balance immediately after the 9th payment?

A) 14,216 B) 13,927 C) 12,865 D) 12,655 E) 12,254

27. Tom bought a television for 3,000. He took advantage of the store's 6-month 0% financing offer. However, under the terms of the offer, if the television isn't paid off in 6 months, the customer owes interest from the original date of purchase at a rate of 9% convertible monthly.

Tom did not make any payments for the first 6 months. Starting exactly 6 months from the date of the purchase, Tom makes monthly payments of X to pay off the balance with 12 level payments. Calculate X.

A) 272.34 B) 274.38 C) 262.35 D) 260.40 E) 271.44

28. The following table contains the current spot interest rates. The price implied by this yield curve for a 5-year 1,000 par value bond with 4% annual coupons is 1,027. Calculate the 3-year forward rate.

Term	**Spot Rate**
1 year	0.5%
2 year	1.5%
3 year	X
4 year	3.0%
5 year	3.5%

A) 2.1% B) 2.5% C) 3.2% D) 4.3% E) 4.6%

29. Brett deposited 500 on January 1, 2003, into account A, which earns a nominal rate of i convertible quarterly. On January 1, 2010, he transferred the balance from Account A to account B, which earns a nominal rate of j convertible monthly. On January 1, 2012, the balance in the account is 800.

    If Brett had deposited his money into account B on January 1, 2003 (earning j convertible monthly), the balance in the account on January 1, 2012, would have been 850.

    Calculate i.

    A) 6.0% B) 5.5% C) 5.4% D) 5.2% E) 5.1%

30. Investor A purchased a 15-year bond on its issue date. The bond has a face amount of 1,000, and pays semi-annual coupons at an 8% (annual) coupon rate. Based on the original purchase price, the bond's yield to maturity for Investor A was 6%.

    After 4 years Investor A sold this bond to Investor B. Based on the prices at which Investor A purchased and sold the bond and the coupons he received, his yield on the bond over the 4 years he owned it is 7%.

    Five years later, Investor B sold the bond to Investor C. Based on the prices at which Investor B purchased and sold the bond and the coupons he received, his yield on the bond over the 5 years he owned it is 5%.

    If Investor C holds the bond until its maturity date, what yield will Investor C earn from the bond?
    (All yield rates are nominal rates convertible semi-annually.)

    A) 5.7% B) 5.9% C) 6.1% D) 6.3% E) 6.5%

31. The following two investments have the same net present value at an annual effective rate i.

    (1) Invest 3,000 now and receive 2,000 in one year and 2,600 in two years.
    (2) Invest 1,000 now and receive 500 in one year and 1,000 in two years.

    Calculate i.

    A) not possible B) 10.5% C) 34.5% D) 42.1% E) 74.4%

32. An annuity with 36 monthly payments has a first payment of 600 one month from today. Starting with the second payment, each of the next 15 monthly payments is 4% larger than the previous payment. Then, starting with the 17th payment, each of the next 20 monthly payments is 5% smaller than the previous payment.

At an annual effective interest rate of 6%, what is the accumulated value of this annuity's payments as of the date of the last payment?

A) 24,197 B) 26,312 C) 28,819 D) 28,892 E) 29,115

33. A bond's current price is 1,020 and its yield to maturity at that price is 7.0%. At a price of 1,030, the bond's yield to maturity would be 6.90%. Based on this information, which of the following statements are true?
(The yields in this problem are annual effective rates.)

I. The bond's modified duration is less than 10.

II. At a yield of 7.1%, the bond's price would be less than 1,010.

III. At a price of 1,040, the bond's yield to maturity would be less than 6.8%.

A) I. only B) I. and II. only C) I. and III. only
D) II. and III. E) All

34. A common stock will pay dividends of X one year from now, and at the end of each of the following 13 years. Thereafter, starting with the 15th dividend, the dividends are expected to grow at a fixed rate of 1.5% per year. At a fixed annual effective interest rate of 5%, the price of this stock using the dividend growth model is 123. Calculate X.

A) 5.2 B) 5.1 C) 5.0 D) 4.9 E) 4.6

35. When the Federal Open Market Committee increases its target for the federal funds rate, it tends to cause which of the following:

I. Higher interest rates for mortgages

II. Higher inflation

III. Higher unemployment

A) I. only B) I. and II. only C) I. and III. only
D) II. and III. E) All

Solutions

1. First calculate the monthly payment amount for each mortgage:
30-year: Set N=360, I/Y=0.475, PV=250,000, and FV=0. CPT PMT = -1,451.00.
15-year: Set N=180, I/Y=0.4, PV=250,000, and FV=0. CPT PMT = -1,311.66.

   The total payments are $360 \times 1,451.00 = 522,360$ under the 30-year mortgage, and $180 \times 1,951.04 = 351,186$ under the 15-year mortgage. The difference, $522,360 - 351,186 = 171,174$, equals the amount of additional interest the homebuyer would pay under the 30-year mortgage.

   Answer: E

2. First calculate the present value of the extra payment of 10,000 and subtract that from the loan amount. At a quarterly effective rate of i = 0.02, the present value of 10,000 paid at the end of 20 quarters, is:

$$\frac{10,000}{(1+i)^{20}} = 6,729.7133$$

   Subtract this present value from the original loan amount.

$$100,000 - 6,729.7133 = 93,270.2867$$

   Finally, calculate the payment amount, X, needed to pay off a loan of 93.270.2867 at a quarterly effective rate of 2%, with 80 quarterly payments. Set N=80, I/Y=2, PV=93,270.2867, and FV=0. CPT PMT = 2,346.7462.

   Answer: B

3. The equation of value for this bond is:

$$\text{Price} = 1,000 \cdot v^n + 1,000 \cdot r \cdot a_{\overline{16}|6\%}$$

$$709.46 = 1,000 \cdot (1.06)^{-16} + 1,000 \cdot r \cdot \frac{1 - 1.06^{-16}}{0.06}$$

$$r = \left(709.46 - 1,000 \cdot (1.06)^{-16}\right) / \left(1,000 \cdot \frac{1 - 1.06^{-16}}{0.06}\right) = 0.03125$$

   The coupon rate <u>per period</u> is 3.125%, so the coupon rate is 6.25%.

   Answer: C

4. After the down payment of 50,000, the Bates family has a mortgage of 215,000. Using the financial calculator to find the monthly payment, N=360, I/Y=6.5/12, PV=215,000, FV=0; CPT PMT = 1,358.95. At the time of the sale (after 180 payments) they still have an outstanding balance of 156,002.15. (To find this using the financial calculator by the retrospective method, leave everything in the calculator as is, except change N to 180; then CPT FV.)

   Their costs at the time of the sale are 2,500 plus 305,000(0.02)=6,100, for a total cost of 8,600. The sale price is 305,000. They owe 156,002.15 on their mortgage and 8,600 in closing costs, so they receive a net amount of 140,397.85 from the sale.

   The Bates family paid an initial down payment of 50,000 and 180 payments of 1,358.95, so their total payments were:
$$50,000 + 180 \cdot 1,358.95 = 294,611$$
   They received a net amount of 140,397.85 at the time of sale, so their net outlay for the house over the 15 years was $294,611 - 140,398 = 154,213$.

   Answer: A

5. This problem involves calculating the price of a bond between coupon dates. The coupons will be paid on February 19 and August 19 of each year. The purchase date is December 31, 2009, so the most recent coupon was paid on August 19, 2009, and the next coupon is due on February 19, 2010.

   First, calculate t = number of days from last coupon date to purchase date / number of days in the bond period. Using "exact days" on the financial calculator, t = 134/184. The price-plus-accrued is $P_0(1+i)^t$, where P_0 is the price on the previous coupon date, and i is the purchaser's yield per coupon period, 1.25%. As of the previous coupon date, there are 3 coupons remaining.
$$P_0 = X(0.02)a_{\overline{3}|0.0125} + Xv_{0.0125}^3$$
   Price-plus-accrued = $2,150 = \left(X(0.02)a_{\overline{3}|0.0125} + Xv_{0.0125}^3\right)(1+i)^{134/184}$
$$X = \frac{2,150}{\left((0.02)a_{\overline{3}|0.0125} + v_{0.0125}^3\right)(1.0125)^{134/184}} = 2,084.88$$

   To solve this problem using less algebra, start by calculating the price-plus-accrued for a similar bond with a face amount of 100. (This can be done using the method applied above, or by entering values into the Bond worksheet.) This value is 103.12. Since 2,150 was paid for the bond in the problem, its par value (X) is $2,150 \div 103.12 = 20.8488$ times as large. This means that X is $20.8488 \times 100 = 2,084.88$.

   Answer: C

6. Since the problem involves interest earned during a 6-month period, translate Bill's 5% annual effective rate to a 6-month effective interest rate: $j = 1.05^{0.5} - 1 = 0.024695077$. Bill earns 30 from time 9 to time 9.5, so his balance at time 9 was: $\dfrac{30}{0.024695077} = 1,214.82$.

   Bill made a deposit of X at time 4, which is 10 half-years before time 9, so:

   $$X(1.024695077)^{10} = 1,214.82 \qquad X = \frac{1,214.82}{(1.024695077)^{10}} = 951.84$$

   Suzanne also earns 30 from time 9 to 9.5, and her deposit of 2X was made at time 7. Since she earns simple interest, her interest-earning principal remains constant at 2X, so her interest earned during *any* 6-month period is constant at 30. Therefore, we have:

   $$2(951.8409)(0.5i) = 30$$
   $$0.5i = \frac{30}{2(951.8409)} = 0.015759$$
   $$i = 2 \cdot (0.015759) = 0.031518$$

   Answer: A

7. We will find the value of $i_{3,4}$ by working with values of the accumulation function, $a(t)$.

   $$a(3) = (1+s_3)^3 = 1.056^3 = 1.17758$$
   $$a(5) = \frac{100}{75.21} = 1.3296$$
   $$a(4) = \frac{a(5)}{1+i_{4,5}} = \frac{1.3296}{1.0637} = 1.24999$$
   $$1 + i_{3,4} = \frac{a(4)}{a(3)} = \frac{1.24999}{1.17558} = 1.06148$$
   $$i_{3,4} = 6.15\%$$

   Answer: D

8. In order to calculate the bond's duration, we need to know its yield. Using the BA II Plus, set:
N = 20, PV = 10,500, PMT = 350, and FV = 10,000. CPT I/Y = 3.159.

The yield is 3.159% per coupon period, so $i^{(2)} = 6.318\%$

Next calculate the Macaulay Duration:

$$D_{\text{mac}} = \frac{Cpn \cdot (Ia)_{\overline{n}|} + n \cdot Face \cdot v^n}{Cpn \cdot a_{\overline{n}|} + Face \cdot v^n} = \frac{350 \cdot (Ia)_{\overline{20}|3.159\%} + 20 \cdot 10,000 \cdot (1.03159)^{-20}}{350 \cdot a_{\overline{20}|3.159\%} + 10,000 \cdot (1.03159)^{-20}}$$

$$D_{\text{mac}} = \frac{155,979.23}{10,500} = 14.855$$

(The denominator is, of course, the price of the bond, 10,500.)

All of the above calculations have been done in terms of coupon periods (half-years), so the Macaulay Duration is 14.855 half-years, or 7.428 years. To find the Modified Duration, we need to divide by 1 plus the effective interest rate per conversion period. Since the interest rate is convertible semi-annually, the effective rate per conversion period is $i^{(2)}/2 = 3.159\%$. So the Modified Duration is $7.428/1.03159 = 7.200$.

(Note: Modified duration with respect to nominal interest rates is discussed on page M7-18. The relevant formulas are **(7.36)** *and* **(7.37)***.)*

Answer: B

9. Use the rate of discount (4% convertible quarterly) to find i and solve for X:

$$1 + i = \left(1 - \frac{0.04}{4}\right)^{-4} = 1.04102 \qquad i = 0.04102$$

$$X\left((1+i)^4 - 1\right) = 500 \qquad X = \frac{500}{\left((1.04102)^4 - 1\right)} = 2,866.05$$

The amount of interest earned in 15 years on a deposit of X is:

$$X\left((1+i)^{15} - 1\right) = 2,866.0461\left((1.04102)^{15} - 1\right) = 2,372.02$$

Answer: C

10. The modified duration of a portfolio is $D_{\text{mod}} = w_1 D^1_{\text{mod}} + w_2 D^2_{\text{mod}} + w_3 D^3_{\text{mod}}$,

where the weights are $w_k = \dfrac{Y_k}{Y_1 + Y_2 + Y_3}$

(Y_k is the value of each investment)

$$w_1 = \frac{3{,}000}{3{,}000 + 6{,}000 + 7{,}000} = 0.1875 \qquad w_2 = \frac{6{,}000}{3{,}000 + 6{,}000 + 7{,}000} = 0.375$$

$$w_1 = \frac{7{,}000}{3{,}000 + 6{,}000 + 7{,}000} = 0.4375$$

Solving for *X*, we have:

$$5 = 0.1875(3.6) + 0.375(X) + 0.4375(5.8) \qquad X = 4.7667$$

Answer: D

11. In order to match the five payments of 10,000, the bond's annual coupon must be 10,000. At time 5, the bond will pay 10,000 plus its redemption value. In order to have an exact matching to the required payment of 10,000 at time 5, the redemption value of the bond must be offset by repayment of the loan. The cost of repaying the loan in 5 years will be:

$$130{,}000 \cdot 1.09^5 = 200{,}021.11$$

Therefore, the face amount of the bond is 200,021.11. Its coupons are 10,000, so the price of the bond (based on its 8% yield) can be found with the following calculator settings:
N=5, I/Y=8, PMT=10,000, and FV=200,021. CPT PV = -176,058.11.

The corporation pays 176,058.11 for the bond and takes out a loan for 130,000, so the net outlay at time 0 is 46,058.11.

Answer: E

12. Set up a future value equation (as of time 7, which is the beginning of the eighth year) and solve for *X*:

$$18{,}000 = 1{,}000(1.065)^7 + 2{,}000(1.065)^6 + 3{,}000(1.065)^5 + 3{,}000(1.065)^4 + 3{,}000(1.065)^3 + X(1.065)^2 + X(1.065)$$

$$18{,}000 - \left[16{,}065.7791\right] = X\left[(1.065)^2 + (1.065)\right]$$

$$X = \frac{18{,}000 - 16{,}065.7791}{(1.065)^2 + (1.065)} = 879.5011$$

Answer: C

13. None of the statements are true.

Statement I. is false because revenue bonds are supported by the revenue from a particular project, such as a toll road, not by the issuing authority's power to collect taxes.

Statement II. is false because there have been defaults on bonds issued by municipalities, including New York City, Detroit, and Orange County (CA).

Statement III. is false. Coupons on municipal bonds are typically set lower than on other bonds, because the coupons are not subject to federal tax, and therefore purchasers accept a lower yield on municipal bonds. A lower coupon rate keeps the coupon rate closer to the yield rate, so that the bond will trade near par.

Answer: E

14. First find the future value at time 15 of the deposits that Alexis will make. The future value of the annuity-due is $500\ddot{s}_{\overline{180}|i}$, where $i = (1.04)^{1/12} - 1$:

$$500\ddot{s}_{\overline{180}|i} = 122,728.99$$

This is the present value of the perpetuity.

The first payment of X will occur one year after the valuation date we used for the annuity (which was time 15). So we can view 122,728.99 as the present value of a perpetuity-immediate at an annual effective rate of 4%.

$$122,728.99 = X\frac{1}{0.04} \qquad X = 4,909.16$$

Answer: B

15. The market value of the payer's position in this swap (which could be either positive or negative) is the present value of the amounts the payer will receive (based on current interest rates at time 0.25), minus the present value of the amounts the payer must pay. The amounts the payer must pay were determined on the inception date, and are based on the fixed swap rate of 1.75%. But the present value of these payments at any time depends on the interest rates in effect at that time. In this case, we will be doing calculations based on the rates as of the end of the first quarter of the swap.

To calculate the present value of the payer's payments, we will need the present value factors for payments 1 and 2 quarters from now:

$$P_{0.25} = 1.018^{-1} = 0.98232 \qquad P_{0.50} = 1.019^{-2} = 0.96306$$

The present value (in millions) of the payer's two remaining payments, based on the fixed rate of 1.75%, is:

$$0.0175 \cdot (2 \cdot 0.98232 + 1 \cdot 0.96306) = 0.051235$$

The present value of the payer's payments is 51,235.

The present value of payments by the receiver (the variable rate payer) is based on the forward rates for the two remaining settlement periods, so we need to calculate those forward rates:

$$f^*_{[0,0.25]} = s_{0.25} = 0.018000 \qquad f^*_{[0.25,0.050]} = \frac{(1+s_{0.50})^2}{1+s_{0.25}} - 1 = \frac{1.0190^2}{1.0180} - 1 = 0.020001$$

The present value (in millions) of the receiver's payments is:

$$0.0180 \cdot 2 \cdot 0.98232 + 0.020001 \cdot 1 \cdot 0.96306 = 0.054626$$

This present value is 54,626.

The market value of the payer's position is the difference between these two present values (amount to be received minus amount to be paid):

$$54{,}626 - 51{,}235 = 3{,}391$$

Answer: A

16. The amount of interest earned from time 1 to time 5 is the difference between the ending amount at time 5 and the starting amount at time 1.

At time 1 the value of the fund is X. At time 5 the value of the fund is:

$$Xe^{\int_1^5 \delta_t dt} = Xe^{\int_1^5 \frac{1}{t+1} dt} = Xe^{\ln(t+1)|_1^5} = Xe^{\ln 6 - \ln 2} = 3X$$

The amount of interest earned is the value at time 5 minus the value at time 1:

$$3X - X = 6{,}000 \qquad X = 3{,}000$$

Answer: D

17. The interest portion of a bond's coupon is the coupon minus the principal portion. The principal portion of a bond's coupon is $\text{Pr}_k = F(r-i)v^{n-k+1}$.

The coupons are each 6,000, and the interest portion of the 13th coupon is 1,130. Therefore, the principal portion of the 13th coupon is 4,870.

$$\text{Pr}_k = 4{,}870 = 100{,}000(0.06-i)v^{14-13+1}$$
$$4{,}870 = \frac{100{,}000(0.06-i)}{(1+i)^2}$$
$$4{,}870(1+i)^2 = 6{,}000 - 100{,}000i$$
$$4{,}870 + 9{,}740i + 4{,}870i^2 = 6{,}000 - 100{,}000i$$
$$4{,}870i^2 + 109{,}740i - 1{,}130 = 0$$

Using the quadratic formula, $i = 0.0102924$ (discarding the second solution, where $i <$ -100%). So the yield, Y, is 1.029%.

Answer: B

18. To simplify the payment pattern, we will first determine the present value of 3 monthly payments (of 5, 10, and 15), and then treat the perpetuity in this problem as a series of *quarterly* payments, with each quarterly payment having the value we calculated for those 3 payments.

The monthly annuity-due of 5, 10, 15 can be evaluated as either $5(I\ddot{a})_{\overline{3}|i}$ or $5\left(1+2v+3v^2\right)$, where i is the monthly effective rate:

$$i = \left(1+\frac{0.07}{2}\right)^{1/6} - 1 = 0.005750039$$

Using either of the above expressions for the 3-payment increasing annuity, the present value is 29.7718.

As of time 10, the payments have a value equal to a perpetuity-due with quarterly payments of 29.7718. This value can be calculated by finding the *quarterly* effective rate and computing the value of this perpetuity-due:

$$i = \left(1+\frac{0.07}{2}\right)^{1/2} - 1 = 0.017349497$$
$$29.7718 \cdot \ddot{a}_{\overline{\infty}|} = \frac{29.7718}{d} = \frac{29.7718}{0.017349497 / 1.017349497} = 1{,}745.78$$

The present value of the perpetuity-due at time 0 is:

$$1{,}745.78 \cdot \left(1+\frac{0.07}{2}\right)^{-20} = 877.37$$

Answer: B

19. Another way of stating the question is:
Susan had 5 million at time 0. How many years of end-of-year payments of 2 million would be required in order to match that 5 million present value?
When stated that way, we see that the equation of value is:

$$5 = 2 \cdot a_{\overline{n}|0.05} = 2 \cdot \frac{1 - 1.05^{-n}}{0.05}$$

Solving this equation either by algebra (using logarithms) or by the financial calculator, we find that n=2.737.

It is important to understand that the above formula for $a_{\overline{n}|}$ is valid only for *integer* values of n. So Susan's accumulation from her lottery-winning annuity does *not* match the accumulation from 5 million at time 2.737. (In fact, she couldn't possibly have caught up with a 5 million investment after only 2.737 years, because at that point she has received only 4 million in winnings.) But this calculated value of n tells us that Susan's accumulated winnings are *less* than the accumulated value of 5 million at time 2, and are *more* than the accumulation of 5 million at time 3.

(Note: This problem is not realistic, in that a lottery that posed the potential for a guaranteed profit would have many people pursuing the winning strategy, which would either bankrupt the lottery operator or eliminate the possibility of a guaranteed profit.)

Answer: A

20. The swap rate is the par coupon rate:

$$R = \frac{1 - \dfrac{1}{1.04^4}}{\dfrac{1}{1.0275} + \dfrac{1}{1.03^2} + \dfrac{1}{1.034^3} + \dfrac{1}{1.04^4}} = 0.0395$$

Answer: E

21. Use the equation for dollar-weighted rate of return to solve for X.

$$I = 58 - 48 - [2X - X] = 10 - X$$

$$0.092 = \frac{10 - X}{48 + 2X(1 - 1/12) - X(1 - 7/12)}$$

$$4.416 + 0.16867X - 0.03833X = 10 - X$$

$$1.13034X = 5.584$$

$$X = 4.9401$$

Answer: C

22. The total amount of interest earned during the first 7 years is 350. The total amount in the account at time 7 is $1,500+350=1,850$.

The balance in the account at time 7 is the future value of the deposit of 1,500. The deposit of 1,500 accumulates for the first 2 years at a quarterly effective rate of $\frac{i}{4}$, and for the following 5 years at a monthly effective rate of $\frac{3i}{12}=\frac{i}{4}$.

The future value at time 7 of the deposit of 1,500 is:

$$1,500\left(1+\frac{i}{4}\right)^{8}\left(1+\frac{3i}{12}\right)^{60}=1,500\left(1+\frac{i}{4}\right)^{68}=1,850$$

Solving for i:

$$i=\left[\left(\frac{1,850}{1,500}\right)^{\frac{1}{68}}-1\right]\cdot 4=0.012355545$$

The balance in the account at time 5 is:

$$1,500\left(1+\frac{i}{4}\right)^{8}\left(1+\frac{3i}{12}\right)^{36}=1,718.01$$

Answer: A

23. Each bond will pay $0.03X$ in coupons annually for 6 years. If there are ten bonds, then the company is investing $0.03X\cdot 10=0.3X$ annually into a fund that earns an annual effective rate of 4%. The future value of these reinvested coupons at time 6, plus the ten redemption values of X must total 100,000.

$$0.3Xs_{\overline{6}|0.04}+10X=100,000$$

$$X=\frac{100,000}{0.3s_{\overline{6}|0.04}+10}=8,340.36$$

Answer: A

24. The interest in the 30th monthly payment (910.32) equals the monthly effective interest rate (0.4%) applied to the outstanding balance after the 29th monthly payment: $Int_{30} = Bal_{29} \cdot 0.004$.

Therefore, $Bal_{29} = 910.32 / 0.004 = 227,580$.

Now we know the balance at t=0 (which is the loan amount, 250,000), the balance at t=29, and the interest rate $i = 0.004$. We can write an equation of value (based on the retrospective formula for a loan balance) and solve for the monthly payment:

$$Bal_n = Bal_0 \cdot (1+i)^n - Pmt \cdot s_{\overline{n}|i}$$

$$227,580 = 250,000 \cdot (1.004)^{29} - Pmt \cdot \frac{1.004^{29} - 1}{0.004}$$

Solving by algebra or using the financial calculator, the amount of the monthly payment is 1,730.67.
(N=29, I/Y=0.4, PV=250,000, and FV=-227,580. CPT PMT = -1,730.67.)

Next we can find the term of the mortgage (in months) by solving for n in the equation of value as of time 0:

$$250,000 = 1,730.67 \cdot \frac{1 - 1.004^{-n}}{0.004}$$

Again using algebra or the BA II Plus, we find that n is 216. So the mortgage has 216 monthly payments. (It is an18-year mortgage.)
(FV=0. CPT N = 216.)

The problem asks for the total amount of interest that the borrower will pay. This is simply the total amount of all 216 payments, less the amount of the loan: $216 \cdot 1,730.67 - 250,000 = 123,825$.

Answer: B

25. This is a premium callable bond, since the price (1,229) is greater than the par value (1,200). Therefore, it should be priced assuming it will be called (redeemed) at the earliest possible call date, the end of the 13th year: n=26.

The coupon rate is r = 0.015, so each coupon is a payment of 1,200(0.015)=18.

Using the financial calculator, set N=26, PV = -1,229, PMT = 18, and FV=1,200. Then CPT I/Y = 1.38863%, the semi-annual effective rate. So the nominal annual rate of interest convertible semi-annually would be twice that value, or 2.7773%.

Answer: C

26. Using the prospective method, the outstanding balance after the 9th payment is the present value of the remaining payments.

There are 20 total payments, so we want the present value of the last 11 payments. The first payment is 1,030, the second payment is 1,060, and the nth payment is 1,030+(n-1)30. So the 10th payment is 1,300 and the 20th payment is 1,600.

We are finding the value immediately after the 9th payment, so this is an annuity-immediate. We want to calculate the present value of payments of 1,300; 1,330; 1,360; …, 1,600:

$$PV = Pa_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

$$PV = 1{,}300a_{\overline{11}|} + 30\left(\frac{a_{\overline{11}|} - 11v^{11}}{0.0225}\right)$$

$$PV = 13{,}926.87$$

Answer: B

27. Tom didn't pay off the balance in 6 months, so he owes 6 months of interest at 9% convertible monthly. At 6 months, Tom owes:

$$3{,}000\left(1+\frac{0.09}{12}\right)^6 = 3{,}137.5567$$

Starting immediately, Tom will make 12 payments of X to pay off this balance. So the first payment will occur now (6 months after the purchase), and 3,137.5567 is the present value of an annuity-due with 12 payments of X:

$$3{,}137.5567 = X\ddot{a}_{\overline{12}|0.09/12} \qquad X = 272.34$$

Or set N=12, I/Y=0.75, PV=3,137.5567, and FV=0. CPT PMT = -272.34.

Answer: A

28. Use the price of the bond to solve for the unknown rate X.

$$1{,}027 = \frac{40}{1.005} + \frac{40}{1.015^2} + \frac{40}{(1+X)^3} + \frac{40}{1.03^4} + \frac{1{,}040}{1.035^5}$$

$$1{,}027 - \left(\frac{40}{1.005} + \frac{40}{1.015^2} + \frac{40}{1.03^4} + \frac{1{,}040}{1.035^5}\right) = \frac{40}{(1+X)^3}$$

$$(1+X)^3 = \frac{40}{1{,}027 - \left(\frac{40}{1.005} + \frac{40}{1.015^2} + \frac{40}{1.03^4} + \frac{1{,}040}{1.035^5}\right)}$$

$$X = 0.02466$$

X is the 3-year spot rate. Solving for the three-year forward rate, we have:

$$1 + i_{3,4} = \frac{(1+s_4)^4}{(1+s_3)^3} = \frac{(1.03)^4}{(1.02466)^3} = 1.0461876 \qquad i_{3,4} = 0.0461876$$

Answer: E

29. First solve for j by using the fact that if Brett's money had been in account B for the entire term, the balance would have been 850:

$$500\left(1 + \frac{j}{12}\right)^{108} = 850 \qquad j = 12 * \left[\left(\frac{850}{500}\right)^{1/108} - 1\right] = 0.059104$$

Then solve for i, using the information given about account A:

$$500\left(1 + \frac{i}{4}\right)^{28}\left(1 + \frac{j}{12}\right)^{24} = 800$$

$$i = 4 * \left[\left(\frac{800}{500\left(1 + \frac{j}{12}\right)^{24}}\right)^{1/28} - 1\right] = 0.050616$$

Answer: E

Note: Instead of solving for the nominal rate j, we could have solved for the equivalent annual effective rate, and the equations would have been slightly less messy. We don't really need to know j itself, but only the accumulation rate for account B.

30. In order to find Investor C's yield to maturity, we need to know the price C paid for the bond at time 9.

We begin by finding Investor A's purchase price. A bought the bond at a price to yield 6% to maturity. We can find the purchase price as follows:
N=30, I/Y=3, PMT=40, and FV=1,000. CPT PV = -1,196.00

A sold the bond to B after 4 years (8 coupon periods) at a price such that his yield was 7%. Solving for the sale price (B's purchase price), we have:
N=8, I/Y=3.5, PV=-1,196.00, and PMT=40. CPT FV = 1,212.84.

We also know that B sold the bond 9 years after it was issued (5 years after B purchased it), and that B earned a yield of 5% over those 5 years. We find B's sale price (C's purchase price) as follows:
N=10, I/Y=2.5, PV = -1,212.84 (recall this value from the FV register (press RCL FV), change its sign, and enter it into PV), and PMT=40.
CPT FV=1,104.40.

Now we know C's purchase price, so we can find C's yield to maturity:
N=12, PV=-1,104.40, PMT=40, and FV = 1,000. CPT I/Y = 2.954.
C's yield (convertible semi-annually) is $2 \times 2.954\% = 5.908\%$.

Answer: B

31. The two net present value equations are:

$$NPV_1 = -3000 + 2000v + 2600v^2$$
$$NPV_2 = -1000 + 500v + 1000v^2$$

Set the two equations equal to each other and solve for v.

$$-3000 + 2000v + 2600v^2 = -1000 + 500v + 1000v^2$$
$$0 = -1600v^2 - 1500v + 2000$$

Using the quadratic formula, we find that $v = \pm 0.743573$. We discard the negative value (v cannot be negative) and solve for i:

$$i = \frac{1}{v} - 1 = 0.3448582$$

Answer: C

Note that the NPVs are negative at 34.48582% (which is possible).
Note also that this problem can be solved using the BA II Plus's CF worksheet, as follows: Calculate the differences between the cash flows for the two investments, enter these differences into the CF worksheet, and solve for IRR. Important: This method works for any number of cash flows, whereas the quadratic formula method (used above) cannot handle more than 3 cash flows, and they must be equally spaced.

32. One way to solve for the future value is first to find the present value and then multiply it by a factor of $(1+i)^{36}$, where i is the monthly effective interest rate.

The present value equation for the 36 payments is:

$$PV = 600v + 600(1.04)v^2 + 600(1.04^2)v^3 + \ldots + 600(1.04^{15})v^{16} +$$
$$600(1.04^{15})(0.95)v^{17} + 600(1.04^{15})(0.95^2)v^{18} + \ldots + 600(1.04^{15})(0.95^{20})v^{36}$$

To evaluate this formula, express it as the sum of two geometric series:

$$PV = 600v[1 + (1.04)v + (1.04^2)v^2 + \ldots(1.04^{15})v^{15}] +$$
$$600(1.04^{15})(0.95)v^{17}[1 + (0.95\ \ )v\ \ + \ldots + (0.95^{19})v^{19}]$$

$$PV = 600v\left[\frac{1-(1.04v)^{16}}{1-1.04v}\right] + 600(1.04^{15})(0.95)v^{17}\left[\frac{1-(0.95v)^{20}}{1-0.95v}\right]$$

The monthly effective interest rate is $i = (1.06)^{1/12} - 1 = 0.00486755$
$PV = 12{,}517.9130 + 11{,}679.4754 = 24{,}197.39$

This is the present value. The future value at the time of the 36th payment is:

$$24{,}197.39(1+i)^{36} = 28{,}819.48$$

This problem can also be solved using the geometric annuity formula:
$$600 \cdot s_{\overline{15|}}^{4\%} \cdot (1+i)^{21} + 600 \cdot 1.04^{15} \cdot s_{\overline{21|}}^{-5\%}$$
$$= 600 \cdot \left(\frac{1.00486755^{15} - 1.04^{15}}{0.00486755 - 0.04} \cdot 1.00486755^{21} + 1.04^{15\cdot} \cdot \frac{1.00486755^{21} - 0.95^{21}}{0.00486755 - (-0.05)} \right)$$
$$= 28{,}819.48$$
In this case, we have divided the 36 payments into groups of 15 and 21, instead of groups of 16 and 20 (as was done with the first solution), thus treating the 16th payment as the first payment of the decreasing geometric annuity, rather than the last payment of the increasing geometric annuity.

Answer: C

33. Only Statement I. is true.

Statement I: We can approximate the modified duration by comparing the percentage change in price, $(1{,}030 - 1{,}020) \div 1{,}020 = 0.009804$, to the change in yield that caused the price change, $0.069 - 0.070 = -0.001$. The negative of the ratio of these two values is an estimate of the modified duration:

$$D_{\text{mod}} \approx -\frac{\Delta\text{Price/Price}}{\Delta\text{yield}} = -\frac{0.009804}{-0.001} = 9.804$$

Our estimate for the modified duration is a bit less than 10, suggesting that Statement I. is true. But could the actual modified duration be greater than 10? If it were 10 or higher, then we would estimate a price higher than 1,030 when the yield falls to 6.9%. Since the first-order modified approximation (using duration, but not convexity) always underestimates the price, a modified duration greater than 10 is not possible. (An estimate higher than 1,030 is obviously not an underestimate of the actual price at 6.9%, which we know to be 1,030.)

Statement II: A 10-basis-point (0.10%) drop in yield caused the price to increase by 10. Would a 10-basis-point increase in yield cause the price to drop by more than 10? No. Because of the convexity of the price-yield curve (see the graph on page M7-16), successive increases in the yield cause smaller and smaller decreases in the price, so if a change from 6.9% to 7.0% decreases the price by 10 (from 1,030 to 1,020), then a change from 7.0% to 7.1% would cause a decrease of less than 10, resulting in a price greater than 1,010.

Statement III: Again referring to the graph on page M7-16, we can see that the convexity (the curvature, or positive second derivative) of the price-yield curve has the following effect: Each successive increase in price (by an amount of 10 in this case, from 1,030 to 1,040) corresponds to a smaller decrease in yield than the previous increase (from 1,020 to 1,030). So we know that a price of 1,040 will cause the yield to decrease by less than 10bp from the yield at 1,030. Therefore, it will not be less than 6.8%, and the statement is not true.

Answer: A

34. The first component of the stock price is the present value of the first 14 dividends, which are a level annuity of 14 payments of X. After a deferral period of 14 years, the second component fits the dividend growth model for stock valuation: $P = \dfrac{Div}{i-g}$, where Div is the first dividend <u>after</u> the period of 14 level dividends. Div in the formula is $X(1.015)$.

    Set the total present value of these two components to the price of 123.

$$123 = Xa_{\overline{14}|0.05} + v^{14}\frac{X(1.015)}{0.05-0.015}$$
$$X = 123 \div \left(a_{\overline{14}|0.05} + v^{14}\frac{(1.015)}{0.05-0.015} \right) = 5.0111$$

    Answer: C

35. Items I. and III. (mortgage rates and unemployment) tend to increase as a result of an increase in the target for the federal funds rate. Item II. (inflation) does not.

    A higher federal funds rate makes it more expensive for a bank to borrow reserve funds in order to meet its daily reserve requirement. This reduces the bank's ability and willingness to make loans to its customers, including mortgage loans, so **rates for mortgages increase**. It also raises the cost of business loans and reduces their availability. This discourages businesses from expanding, which reduces employment levels, **causing unemployment to rise**.

    The effects described above discourage consumer spending, which tends to **reduce the rate of inflation**.

    Answer: C

Practice Exam 11

Exam FM

Questions

1. A 20-year bond with 5.5% semi-annual coupons is callable with a 5% call premium at any coupon date on or after its 10^{th} anniversary. Ben purchases the bond on its 2^{nd} anniversary at a price that assures a yield of at least 5% convertible semi-annually (if the bond does not default).

   If the bond is called on its 15^{th} anniversary, what is Ben's actual yield?

   A) 5.04% B) 5.07% C) 5.12% D) 5.15% E) 5.19%

2. Money is withdrawn continuously and at a constant rate from a fund that earns an annual effective interest rate of $i = 4\%$. If the total amount withdrawn from this fund over 10 years is 3,000 and this completely depletes the fund, what was the fund's initial balance?

   A) 2,433 B) 2,482 C) 2,614 D) 3,602 E) 3,673

3. An investor purchases a 5-year bond with a par value and maturity value both equal to X and 6% annual coupons. The purchase price is 1,120. Coupons are reinvested in a fund that earns a 7% annual effective rate. If the annual effective yield on the overall investment at the end of 5 years is 5.5%, calculate X.

   A) 1,223 B) 1,158 C) 1,097 D) 1,088 E) 1,053

4. A homebuyer borrows 200,000 under a 20-year variable-rate conventional mortgage with monthly payments. The interest rate for this mortgage loan during its first year is 3.6%.

   If the required monthly payment increases by 270.60 at the beginning of the mortgage's second year, what is the new interest rate for the second year? (Each interest rate is expressed as a nominal annual rate, convertible monthly.)

   A) 6.00% B) 6.18% C) 6.24% D) 6.36% E) 6.60%

5. Jessica deposits 8,000 into a fund A that earns a nominal rate of 4% convertible quarterly. At the end of each quarter for 5 years, Jessica withdraws 400 from fund A and deposits it into fund B, which earns a nominal interest rate of 5.5% convertible quarterly.

   After 5 years, what is the total accumulated value in funds A and B?

   A) 9,136 B) 9,879 C) 10,090 D) 10,100 E) 10,477

6. Consider the following two bonds:
   - A zero-coupon bond that will pay X at the end of 10 years and has an annual effective yield of 5%.
   - A bond with a par value of 2,400 that pays 7% annual coupons and has a redemption value of X at the end of 10 years. This bond is callable (at its redemption value, X) on any coupon date, starting at the end of the 6^{th} year. It is priced to yield an annual effective rate of at least 6%.

   The price of the coupon bond is 60% greater than the price of the zero-coupon bond. Calculate X.

   A) 2,400 B) 2,865 C) 2,917 D) 2,926 E) 2,979

7. The following table shows the history of account balances, deposits, and withdrawals for an account:

Date	Value Before Dep/Wdl	Deposit / Withdrawal
January 1, 2016	0	1,000
April 1, 2016	1,050	200
August 1, 2016	X	-525
January 1, 2017	800	

   If the account's internal rate of return for this one-year period is equal to its time-weighted return for the same period, what is the value of X?
   (Both the IRR and the time-weighted return are annual effective rates.)

   A) 1,226 B) 1,290 C) 1,375 D) 1,460 E) 1,573

8. At t=1 a deposit of 100 is made in an account that grows at a varying force of interest $\delta_t = \frac{1}{2t+5}$. There are no other deposits or withdrawals.

At t=X, the account balance has increased to 140. What is the value of X?

A) 3.96 B) 4.16 C) 4.36 D) 4.56 E) 4.76

9. A company must make liability payments of 20,000 and 40,000 at the end of years 1 and 2, respectively. The only investments available to the company are the following annual-coupon bonds, both of which are redeemable at par. The par value of Bond B is 1,000; the par value of Bond A is not given.

Bond	Maturity	Annual Coupon rate	Annual Effective Yield	Par Value
A	1 year	5%	4%	?
B	2 years	6%	4.5%	1,000

What is the total cost of purchasing bonds to match the liability cash flows?

A) 55,850 B) 55,890 C) 56,110 D) 57,450 E) 58,990

10. A 10-year annual-coupon bond has a current price of 98 and a yield to maturity of 6% (an annual effective rate). The derivative of this bond's price with respect to its yield to maturity is -725.

What is the bond's Macaulay Duration (in years)?

A) 7.0 B) 7.2 C) 7.4 D) 7.6 E) 7.8

11. Janet deposits 300 into an account at time 0 and 600 nine years later. The account earns a constant force of interest δ for the first 4 years. After 4 years, the account earns a nominal rate of 3% convertible semi-annually. If the account balance at time 15 is 1,250, what is the value of δ?

A) 15.4% B) 14.4% C) 10.2% D) 6.2% E) 4.7%

12. Tino and Angela each have 1,500 available every month for housing and investing. Each of them takes out a mortgage of 200,000. Tino takes out a 30-year monthly-payment mortgage at an interest rate of 6.5% convertible monthly. Angela takes out a 20-year monthly-payment mortgage at a rate of 6.1% convertible monthly. Their first payments are due at the end of the first month.

    Any money that each one has left over (from the 1,500-a-month budget) after making the mortgage payment is deposited into an account that earns an annual effective rate of 5%.

    By how much does Angela's balance exceed Tino's at the end of 30 years? (Assume Angela continues depositing money into her account after her mortgage is paid off.)

    A) 60,160   B) 61,785   C) 74,250
    D) 75,970   E) Tino's balance exceeds Angela's

13. You have a liability that requires a payment of 10,000 eighteen months (1.5 years) from now. The liability is valued at a 6% annual effective rate. You decide to purchase bonds to create a bond portfolio that has the same present value and modified duration as the liability.

    The following zero-coupon bonds are available for purchase:
    - a 1-year zero-coupon bond with a 4% annual effective yield
    - a 2-year zero-coupon bond with a 7% annual effective yield

    What is the total face amount of the bonds you will buy?

    A) 9,975   B) 9,990   C) 10,000   D) 10,010   E) 10,025

14. A stock is expected to pay a dividend of 5 in one year. Each subsequent annual dividend is expected to be 2% larger than the preceding one. At a valuation interest rate of 4%, calculate the Macaulay duration of the stock.

    A) 45   B) 50   C) 52   D) 55   E) 65

15. A perpetuity has annual payments of 6 at time 2, 7 at time 3, 8 at time 4, etc. At an annual effective interest rate of 5.5%, calculate the present value of this perpetuity as of time 0.

A) 395 B) 417 C) 434 D) 440 E) 457

16. At the beginning of the year, Howard gives his money manager 100,000 to invest. In 6 months, Howard's account is worth 105,000 and Howard immediately gives the manager an additional 95,000. At the end of the year, the account is worth 220,000.

What is the value of this account's time-weighted rate of return minus its dollar-weighted rate of return?

A) 2.8% B) 1.4% C) 0% D) –1.4% E) -2.8%

17. Michael takes out a 10-year loan of 50,000 with a 15% nominal interest rate convertible quarterly. Michael initially makes level payments of Y at the end of each quarter, planning to use the entire term of 10 years to pay off the loan. However, Michael adds X to the amount of the 20^{th} payment (just to that one payment). As a result, the loan is exactly paid off with the 30^{th} payment (in an amount of Y).

Calculate X.

A) 2,430 B) 5,350 C) 10,110 D) 13,830 E) 19,980

18. A U.S. Treasury bill matures in 260 days for its face amount of 1,000. Its rate is quoted as 5.85%. What is this T-bill's current price?

A) 944.73 B) 950.01 C) 954.00 D) 957.75 E) 960.31

19. The following is a yield curve table for spot rates.

Year	Spot Rate
1	6.0%
2	6.5%
3	X
4	Y
5	8.0%

The 3-year forward rate is 9.5%, and the 4-year forward rate is 10%. Calculate the coupon rate for a par coupon bond with a maturity of 4 years.

A) 7.2% B) 7.3% C) 7.4% D) 7.5% E) 7.6%

20. Carolyn buys a perpetuity that will pay 100 at the end of each month for the first 10 years, 50 at the end of each month for the following 15 years, and 10 at the end of each month after 25 years. Using a nominal interest rate of 3% convertible monthly, calculate the cost of this perpetuity.

A) 21,596 B) 19,212 C) 17,613 D) 16,482 E) 15,456

21. An annuity-due has 40 initial quarterly payments of X, followed by a perpetuity of quarterly payments of 20 starting 10 years from the present. If the present value of these payments at a nominal rate of 6% convertible monthly is 950, calculate X.

A) 6.9 B) 7.4 C) 13.1 D) 22.1 E) 30.3

22. A lender requires an annual effective return of 6% per annum. It sets the contractual rate on its loans based on its estimate of the probability that the borrower will default.

Assuming that this lender sets its loan rates at the minimum level that is expected to produce its required rate of return, what is the difference between the annual effective rates charged on two 3-year loans of 10,000, if the estimated probabilities of default for the two borrowers are 0.9% and 3.6%?

Assume that the loans are to be repaid with interest at the end of 3 years, and that there is no recovery in the event of default.

A) 0.9% B) 1.0% C) 1.1% D) 1.3% E) 1.4%

23. A 6-year 1,500 par bond with 5% semi-annual coupons is sold to yield a nominal interest rate convertible semi-annually of x. If the discount for the bond is 200, calculate x.

A) 7.8% B) 7.9% C) 8.0% D) 8.1% E) 8.2%

24. Account A and Account B both earn compound interest at a rate x ($x > 0$). However, Account A earns interest at an annual effective rate of x, while Account B earns interest at a nominal rate of x, convertible semi-annually.

An amount of 1,000 is deposited into each of these accounts at t=0. No other deposits or withdrawals occur. At the end of 10 years, the balance in one of the accounts is 5% larger than the balance in the other account.
What is the value of x?

A) 7% B) 9% C) 11% D) 13% E) 15%

25. Given the following current prices for zero-coupon bonds, what would be the level swap rate for a 3-year accreting interest rate swap with notional amounts of 2 million, 3 million and 4 million during its three settlement periods?

Term (years)	1	2	3	4
Zero-coupon bond Price per 1,000	957.85	910.49	854.04	802.64

A) 5.24% B) 5.42% C) 5.73% D) 5.95% E) 6.04%

26. Joe and Scott contribute to separate funds for 20 years. Both funds earn an annual effective interest rate of i. Joe contributes X annually, with the first payment at time 3. Scott contributes X annually with the first payment at time 1.

Which of the following represents the difference between Joe's and Scott's accumulated values immediately after their contributions at time 10?

A) $X\left(a_{\overline{10}|i} - a_{\overline{8}|i}\right)$

B) $X\left(a_{\overline{9}|i} - a_{\overline{7}|i}\right)$

C) $X(1+i)^7\left(a_{\overline{9}|i}(1+i)^2 - a_{\overline{7}|i}\right)$

D) $X(1+i)^8\left(a_{\overline{10}|i}(1+i)^2 - a_{\overline{8}|i}\right)$

E) $X(1+i)^{20}\left(a_{\overline{20}|i}(1+i)^2 - a_{\overline{20}|i}\right)$

27. Chris deposits X into an account at the beginning of each year for 8 years. These payments earn an annual effective interest rate of 5%, but the interest earned each year in this account is immediately reinvested in a second account at an annual effective interest rate of 3.5%. If Chris has a total of 9,500 at the end of 8 years, calculate X.

A) 954 B) 986 C) 997 D) 1,011 E) 1,050

28. A 30-year semi-annual-payment loan for 125,000 is offered at an annual effective rate of 6%. The borrower arranges to make increasing payments, where the first payment (6-months after the loan date) is X and each subsequent payment is 100 more than the previous payment. Find the outstanding balance immediately after the end of the fourth year.

A) 110,287 B) 116,086 C) 120,115 D) 133,914 E) not solvable

29. A 20-year bond with a face amount (and maturity value) of 1,000 will mature on February 1, 2022. The bond pays semi-annual coupons at a 4% (annual) rate. The bond was purchased on September 20, 2011, at a price such that its yield to maturity is 5.6% (a nominal rate, convertible semi-annually).

The bondholder adjusts the bond's book value at each coupon date to maintain a constant rate of return.

By what amount was the bond's book value adjusted on August 1, 2015?

A) -5.59 B) -5.29 C) +5.29 D) +5.43 E) +5.59

30. The following function generates the t-year forward rates for a particular yield curve:

$$i_{t,t+1} = -0.00015t^2 + 0.005t + 0.044 \qquad 0 \le t \le 10$$

Calculate the 3-year spot rate using forward rates based on this function.

A) 4.9% B) 5.2% C) 5.8% D) 6.8% E) 15.3%

31. Janet has a 14-year 6% semi-annual coupon bond purchased to yield 4.5% convertible semi-annually. The amount of premium amortized in the 4^{th} coupon is 14. If the bond is redeemable at par, calculate the book value of the bond immediately after the 6^{th} coupon.

A) 3,256 B) 3,676 C) 3,759 D) 4,413 E) 4,711

32. Which of the following indicates the correct relationship among the annuity functions?

A) $a_{\overline{n}|i} < \bar{a}_{\overline{n}|i} < \ddot{a}_{\overline{n}|i}$
B) $a_{\overline{n}|i} < \ddot{a}_{\overline{n}|i} < \bar{a}_{\overline{n}|i}$
C) $\bar{a}_{\overline{n}|i} < a_{\overline{n}|i} < \ddot{a}_{\overline{n}|i}$
D) $\ddot{a}_{\overline{n}|i} < \bar{a}_{\overline{n}|i} < a_{\overline{n}|i}$
E) $\ddot{a}_{\overline{n}|i} < a_{\overline{n}|i} < \bar{a}_{\overline{n}|i}$

33. Gavin deposits 1,000 into an account that accumulates based on a nominal rate of discount of 5% convertible semi-annually for the first three years, and a nominal rate of discount d convertible monthly for the next four years. If the account balance is 1,600 after seven years, calculate d.

A) 7.3% B) 7.5% C) 7.9% D) 8.1% E) 8.4%

34. The yield curve for spot rates is given by:

$$r_n = .0025n^2 + .0025n + .025$$

Find the fixed rate for a 3-year interest rate swap with level notional amount.

A) 4.8% B) 4.9% C) 5.0% D) 5.2% E) 5.4%

35. A loan of 46,000 is made for a term of 15 years at a 3.5% annual effective interest rate. The borrower pays the lender quarterly payments of interest only until the end of the 15th year when the 46,000 must also be repaid.

In order to save for the principal repayment, the borrower makes a deposit at the end of each year into a sinking fund earning a 2.6% annual effective interest rate. Suppose that the borrower and the lender agree to terminate the loan at the time of the 12th interest payment, and the borrower makes that interest payment and also repays the entire principal.

After applying the balance in the sinking fund immediately before the 12th loan interest payment is due (and before the 3rd sinking fund deposit is made), what additional amount must the borrower pay in order to pay off the entire loan including interest due.

A) 11,066 B) 15,791 C) 40,841 D) 41,104 E) 41,238

Solutions

1. Because the bond has a call premium, we can't be certain whether it should be priced based on a call at the first call date, or based on continuing to its maturity date. We will calculate both prices and choose the lower one (since paying the higher price based on one of the two scenarios would result in realizing a less-than-5% yield if the other scenario occurs). We don't know the bond's par value, but that is not needed to solve for the actual yield. We will assume a par value of 100.

   First calculate the bond's price based on a 5% yield to first call:
   Set N=16, I/Y=2.5, PMT=2.75, and FV=105. CPT PV = -106.63.
   Remember that the bond was purchased on its second anniversary, so there are 16 coupon periods until the first call date. The maturity value (FV) of 105 includes the 5% call premium.

   Then calculate the bond's price based on a 5% yield to maturity:
   Set N=36, I/Y=2.5, PMT=2.75, and FV=100. CPT PV = -105.89.

   Even though this is a premium bond (because the price is greater than the par value), the price is lower if we assume it will not be called. That is because the premium is only 5.89, and by the time of the 10^{th} anniversary, enough of the premium will have been amortized that the book value will be less than the call price of 105. Therefore, a call would increase the buyer's yield from the bond, so it is not wise to assume that a call will occur.

   Now we know that the purchase price is 105.89, and we need to calculate the yield if the bond is called on its 15^{th} anniversary.
   Set N=26, PV = -105.89, PMT=2.75, and FV=105. CPT I/Y = 2.574.

   The effective yield per coupon period is 2.574%, so the yield convertible semi-annually is $2 \times 2.574\% = 5.148\%$

   Answer: D

2. This is the present value of a continuous annuity that pays 300 per year for 10 years:

$$300\bar{a}_{\overline{10}|i=4\%} = 300\left(\frac{1-v^{10}}{\delta}\right) = 300\left(\frac{1-1.04^{-10}}{\ln(1.04)}\right) = 2{,}481.616$$

   Answer: B

3. The yield on the overall investment is 5.5%, and the amount invested is the price of the bond, 1,120. Therefore, at the end of 5 years there will be a return of:

$$1,120(1.055)^5 = 1,463.7952$$

This amount is the sum of the bond's face value, X, plus the future value of the reinvested coupons:

$$X + X(0.06)s_{\overline{5}|0.07} = 1,463.7952$$
$$X = \frac{1,463.7952}{\left(1+(0.06)s_{\overline{5}|0.07}\right)} = 1,088.2877$$

Answer: D

4. First calculate the monthly payment for the first year. The equation of value is: $200,000 = Pmt \cdot 12 \cdot a^{(12)}_{\overline{240}|0.3\%}$. We can use the BA II Plus to solve for *Pmt*:
Set N=240, I/Y=0.3, PV=200,000, and FV=0. CPT PMT = -1,170.22.

Next find the outstanding balance at the end of 1 year. Because there are 228 payments remaining, set N=228 and CPT PV = 193,043.28.

The payment increases by 270.60, so the new payment at the beginning of the second year is 1,170.22 + 270.60 = 1,440.82.
Set PMT=-1,440.82 and CPT I/Y = 0.51500.

The new interest rate (convertible monthly) is $12 \times 0.515\% = 6.18\%$

Answer: B

5. Fund B's balance is the future value of an annuity-immediate with quarterly payments of 400 for 5 years at a quarterly effective rate of 0.055/4 = 0.01375.

$$FV_B = 400s_{\overline{20}|0.01375} = 9,136.4801$$

Fund A's balance is the accumulated value of the 8,000 initial balance, minus the future value of an annuity-immediate with quarterly payments of 400 for 5 years, both calculated at a quarterly effective interest rate of 0.04/4 = 0.01:

$$FV_A = 8,000(1.01)^{20} - 400 \cdot s_{\overline{20}|1\%} = 953.9147$$

The total amount in fund A and fund B at the end of 5 years is:
9,136.4801 + 953.9147 = 10,090.3948

Answer: C

6. The price of the zero-coupon bond is: $P_Z = X \cdot 1.05^{-10} = 0.6139X$.

We don't know the coupon bond's redemption value (X), so we can't be sure whether it is a premium bond. We will have to solve the problem two ways: assuming the bond will be called at time 6, and assuming it will not be called (and will mature at time 10). Whichever assumption produces a lower price will lead to the correct value for X.

If we assume that the bond will be called, we have:

$$P_C = 2{,}400 \cdot (0.07)a_{\overline{6}|0.06} + X \cdot v_{0.06}^{6}$$

The price of the coupon bond is 60% larger than the price of the zero coupon bond, so we have:

$$2{,}400 \cdot (0.07)a_{\overline{6}|0.06} + X \cdot v_{0.06}^{6} = 1.6\left(X \cdot v_{0.05}^{10}\right)$$

$$168a_{\overline{6}|0.06} = X\left[1.6 \cdot v_{0.05}^{10} - v_{0.06}^{6}\right]$$

$$X = \frac{168a_{\overline{6}|0.06}}{1.6 \cdot 1.05^{-10} - 1.06^{-6}} = \frac{826.1105}{0.2773} = 2{,}979.11$$

This is the correct value of X only if the coupon bond is a premium bond and will be called at time 6. We need to check whether the bond's price based on a redemption value of 2,979.1147 exceeds 2,979.1147. Using the BA II Plus: Set N=6, I/Y=6, PMT=168, and FV=2,979.11. CPT PV = 2,926.27.

The price (2,926.27) is <u>less</u> than the redemption value (2,979.11), so this is <u>not</u> a premium bond, and we should <u>not</u> assume that it will be called. We will now repeat the above calculations assuming that the bond is <u>not</u> called.

The price of the bond, assuming it is not called, is:

$$P_C = 2{,}400 \cdot (0.07)a_{\overline{10}|0.06} + X \cdot v_{0.06}^{10}$$

The price of the coupon bond is 60% larger than the price of the zero-coupon bond, so we have:

$$2{,}400 \cdot (0.07)a_{\overline{10}|0.06} + X \cdot v_{0.06}^{10} = 1.6\left(X \cdot v_{0.05}^{10}\right)$$

$$168a_{\overline{10}|0.06} = X\left[1.6 \cdot v_{0.05}^{10} - v_{0.06}^{10}\right]$$

$$X = \frac{168a_{\overline{10}|0.06}}{1.6 \cdot 1.05^{-10} - 1.06^{-10}} = \frac{1{,}236.4946}{0.423866} = 2{,}917.18$$

This is the correct value of X if the coupon bond will <u>not</u> be called at time 6. Using the BA II Plus to find the bond's price:
Set N=10, I/Y=6, PMT=168, and FV=2,917.18. CPT PV = 2,865.43.
The price is <u>less</u> than our calculated value of $X = 2{,}917.18$. So it is <u>not</u> a premium bond, and the redemption value is 2,917.18.

Answer: C

7. Find the <u>monthly</u> IRR for these cash flows using the Cash Flow worksheet:

CF_0 = 1,000
C01 = 0 F01 = 2 (no CF's for 2/1 and 3/1)
C02 = 200 F02 = 1
C03 = 0 F03 = 3 (no CF's for 5/1, 6/1, and 7/1)
C04 = -525 F04 = 1
C05 = 0 F05 = 4 (no CF's for 9/1 through 12/1)
C06 = -800 F06 = 1
Then press: IRR CPT IRR = 1.049184% (monthly effective rate)
Annual effective IRR $= (1+0.01049184)^{12} - 1 = 0.1334275$

$$1+i^{TW} = \frac{1,050}{1,000}\cdot\frac{X}{1,250}\cdot\frac{800}{X-525} = 1.1334275$$
$$1.05\cdot X\cdot 800 = 1,250\cdot(X-525)\cdot 1.1334275$$
$$840X - 1,250\cdot(1.1334275)\cdot X = 1,250\cdot(-525)\cdot 1.1334275 = -743,811.80$$
$$X = -743,811.80/(840 - 1,250\cdot 1.1334275) = 1,289.58$$

Answer: B

8. The accumulated value in this account at time t is $100\cdot e^{\int_{u=1}^{t}\frac{du}{2u+5}}$. We will evaluate the integral and then find the value of t that produces a balance of 140 in the account.

$$\int_{u=1}^{t}\frac{du}{2u+5} = \left.\frac{\ln(2u+5)}{2}\right|_{u=1}^{t} = \frac{1}{2}\cdot\left[(\ln 2t+5) - \ln 7\right] = \frac{1}{2}\cdot\ln\frac{2t+5}{7}$$

Substitute this value for the integral in the formula for the accumulated value and set the value at time t equal to 140:

$$Bal_t = 100\cdot e^{\frac{1}{2}\cdot\ln\frac{2t+5}{7}} = 100\cdot\left(\frac{2t+5}{7}\right)^{1/2} = 140 \quad\rightarrow\quad \left(\frac{2t+5}{7}\right)^{1/2} = \frac{140}{100} = 1.4$$
$$2t+5 = 7\cdot(1.4)^2 = 13.72 \quad\rightarrow\quad t = 4.36$$

The balance will equal 140 at time 4.36 years, so X = 4.36.

Answer: C

9. This type of problem should be worked backwards in time. First calculate how many Bond B's are needed. Each Bond B will pay 1,060 at time 2 (the redemption value plus the coupon). Since the liability payment at time 2 is 40,000, the company needs 40,000/1,060 = 37.7358 units of Bond B. This amount of Bond B will provide 37.7358(60) = 2,264.15 in coupons at time 1, so the remaining liability payment at time 1 is:

   20,000 – 2,264.15 = 17,735.85.

   The company needs to buy enough of Bond A to provide 17,735.85 at time 1. Since all of Bond A's payments occur at time 1, the amount invested in Bond A is simply the present value of 17,735.85. (We don't need to know its par value.)

$$PV_A = \frac{17,735.85}{1.04} = 17,053.70$$

   To calculate the cost of the Bond B's using the financial calculator, we have:
   FV=1,000, I/Y = 4.5, N=2, PMT = 60. CPT PV = -1,028.0900.
   Price of Bond B is 1,028.0900 per 1,000 of face amount.
   The total cost of the Bond B's is:

$$PV_B = 37.7358 \cdot (1,028.09) = 38,795.80$$

   The total cost is 17,053.70 + 38,795.80 = 55,849.50

   Answer: A

   Note: We did not need to know the par value of <u>either</u> bond in order to solve this problem. Regardless of the face amount of each Bond B, the total face amount of Bond B that is needed will be 37,735.80, and the cost will be 38,795.80.

10. Because we know the derivative of the bond's price with respect to a change in the interest rate, we can write an expression for its modified duration. Then we can use the bond's modified duration (and its yield of 6%) to calculate its Macaulay duration:

$$D_{\text{mod}} = \frac{-dP/di}{P} = \frac{-(-725)}{98} = 7.398$$
$$D_{\text{mac}} = D_{\text{mod}} \cdot (1+i) = 7.398 \cdot (1.06) = 7.842$$

   Answer: E

11. The accumulated value at time 15 is:

$$300e^{4\delta}\left(1+\frac{0.03}{2}\right)^{22}+600\left(1+\frac{0.03}{2}\right)^{12}=1,250$$

$$e^{4\delta}=\frac{1,250-600(1.015)^{12}}{300(1.015)^{22}}=1.27953068$$

$$\delta=\ln(1.27953068)/4=0.061623$$

Answer: D

12. Using a financial calculator, we can calculate Tino and Angela's mortgage payments.

For Tino, N=360, I/Y=6.5/12, PV=200,000, FV=0, CPT PMT = 1,264.14.
His payment is 1,264.14.

For Angela, N=240, I/Y=6.1/12, PV=200,000, FV=0, CPT PMT = 1,444.42.
Her payment is 1,444.42.

Subtracting from 1,500, Tino invests 235.86 at the end of each month for 30 years. Angela invests 55.58 at the end of each month for 20 years, and then 1,500 per month for the following 10 years.

They are both investing at an annual effective rate of 5%. The monthly effective rate is $i=1.05^{1/12}-1=0.004074$.

The future value of Tino's annuity is: $235.86s_{\overline{360}|0.4074\%}=192,317.78$

The future value of Angela's annuity is:

$$55.58s_{\overline{240}|i}(1.004074)^{120}+1,500s_{\overline{120}|0.4074\%}$$
$$=36,739.0884+231,544.7420=268,281.10$$

Thus, Angela is ahead of Tino by 268,281.10 – 192,317.78 = 75,963.31.

Answer: D

Note: The intermediate results in the above solution (235.86 and 55.58) are shown to the nearest cent. However, the calculations are done using the full precision of the calculator. The calculated value is not exactly equal to any of the answer choices, and when this happens, we select the closest choice.

13. We begin by determining the present value and modified duration of the liability and each of the bonds.

Liability: $PV_L = 10,000 \cdot 1.06^{-1.5} = 9,163.07$ $\qquad D_{\text{mod}}^{L} = 1.5 / 1.06 = 1.4151$

Bond 1: $PV_1 = B_1 \cdot 1.04^{-1} = 0.96154 \cdot B_1$ $\qquad D_{\text{mod}}^{1} = 1 / 1.04 = 0.96154$

Bond 2: $PV_2 = B_2 \cdot 1.07^{-2} = 0.87344 \cdot B_2$ $\qquad D_{\text{mod}}^{2} = 2 / 1.07 = 1.86916$

(B_1 and B_2 are the face amounts of the two bonds.)

To match present values and modified durations, we have:

$$0.96154 \cdot B_1 + 0.87344 \cdot B_2 = 9,163.07$$

$$\frac{0.96154 \cdot (0.96154 \cdot B_1) + 1.86916 \cdot (0.87344 \cdot B_2)}{9,163.07} = 1.4151$$

Solving for B_1 and B_2 produces: B_1 = 4,767.47 and B_2 = 5,242.46.
The total face amount of the bonds is $4,767.42 + 5,242.46 = 10,009.88$.

Answer: D

14. We can find the modified duration of this stock from the formula:

$$D_{\text{mod}} = \frac{-\left(\frac{dP}{di}\right)}{P}$$

The present value of the dividends is: $P = \frac{Div}{(i-g)}$. So: $\frac{dP}{di} = \frac{-Div}{(i-g)^2}$

At an interest rate of 4% and a dividend growth rate of 2%:

$$D_{\text{mod}} = \frac{-\left(\frac{dP}{di}\right)}{P} = \frac{-\left(\frac{-Div}{(i-g)^2}\right)}{\frac{Div}{(i-g)}} = \frac{1}{i-g} = \frac{1}{0.04-0.02} = 50$$

From this, we can find the Macaulay duration:

$$D_{\text{mod}} = D_{\text{mac}} / (1+i) \qquad D_{\text{mac}} = (1+i) \cdot D_{\text{mod}} = 1.04 \cdot 50 = 52$$

Answer: C

15. This is a perpetuity of the form P, $P+Q$, $P+2Q$, etc. The present value equation is $\frac{P}{i}+\frac{Q}{i^2}$.

This formula calculates the present value one period before the first payment of P. We have a P of 6 and a Q of 1. We want the present value at time 0, which is 2 time units before the first payment of P, so we need to discount our formula by one year.

$$PV = v\left(\frac{6}{0.055}+\frac{1}{0.055^2}\right) = 416.7483$$

Answer: B

16. The time-weighted rate of return is:

$$j = \left(\frac{105}{100}\right)\left(\frac{220}{200}\right) - 1 = 1.155 - 1 = 0.155$$

The dollar-weighted rate of return is:

$$i = \frac{220-100-95}{100\cdot(1)+95\cdot(1-0.5)} = 0.16949$$

The time-weighted return minus the dollar-weighted return is:
0.155 – 0.16949 = - 0.01449

Answer: D

17. First, calculate Y using a financial calculator. N=40, I/Y = 15/4=3.75, PV=50,000, FV=0, CPT PMT. The payment is 2,432.9728.

With the 20th payment, Michael is adding X so that he can pay off the loan with 10 more payments of 2,432.9728. This means that the loan balance immediately after the 20th payment should be the present value of 10 payments of 2,432.9728:

$$PV = 2,432.9728\cdot a_{\overline{10}|0.0375} = 19,981.4881$$

However, with the original loan terms, the loan balance immediately after the 20th payment (by the prospective method) would be the present value of the last 20 payments of 2,432.9728:

$$PV = 2,432.9728\cdot a_{\overline{20}|0.0375} = 33,809.0871$$

Michael's payment of X is the difference between these two present values:

33,809.0871 – 19,981.4881 = 13,827.5990

Answer: D

18. U.S. Treasury bill rates are quoted based on the following formula:

$$\text{Quoted Rate} = \frac{360}{\text{Days to Maturity}} \times \frac{\text{Amount of Interest}}{\text{Maturity Value}}$$

For this T-bill, we can fill in all of the values except the amount of interest:

$$0.0585 = \frac{360}{260} \times \frac{\text{Amount of Interest}}{1{,}000}$$

$$\text{Amount of Interest} = \frac{260}{360} \times 58.50 = 42.25$$

Since the amount of interest is 42.25, the current price of the T-bill is:

$$1{,}000 - 42.25 = 957.75$$

Answer: D

19. We can calculate the 4-year par coupon bond's coupon rate if we have the spot rates for 1, 2, 3, and 4 years. We are missing the 3- and 4-year spot rates, but the information in the problem allows us to calculate their values.

Using the 3-year and 4-year forward rates, $i_{3,4}$ and $i_{4,5}$, and the 5-year spot rate, s_5, all of which are given, we can calculate the 3- and 4-year spot rates:

$$(1+s_4)^4 \cdot (1+i_{4,5}) = (1+s_5)^5$$

$$s_4 = \left[(1+s_5)^5 / (1+i_{4,5})\right]^{1/4} - 1 = \left[(1.08)^5 / (1.10)\right]^{1/4} - 1 = 0.075057$$

$$(1+s_3)^3 \cdot (1+i_{3,4}) = (1+s_4)^4$$

$$s_3 = \left[(1+s_4)^4 / (1+i_{3,4})\right]^{1/3} - 1 = \left[(1.075057)^4 / (1.095)\right]^{1/3} - 1 = 0.068490$$

A par coupon bond is a coupon bond that is selling at par. Let c be the annual coupon rate that causes a 4-year bond to sell at par. Then, using a par value of 100, we have:

$$100c \cdot \left[(1+s_1)^{-1} + (1+s_2)^{-2} + (1+s_3)^{-3} + (1+s_4)^{-4}\right] + 100 \cdot (1+s_3)^{-3} = 100$$

$$c \cdot \left[1.06^{-1} + 1.065^{-2} + 1.068490^{-3} + 1.075057^{-4}\right] + 1 \cdot 1.075057^{-4} = 1$$

$$c = \frac{1 - 1.075057^{-4}}{\left[1.06^{-1} + 1.065^{-2} + 1.068490^{-3} + 1.075057^{-4}\right]} = 0.074071$$

The 4-year par coupon bond has a coupon rate (and a yield) of 7.4071%.

Answer: C

20. To figure out the present value of this perpetuity, split it into an annuity of 10 years followed by a deferred annuity of 15 years, then followed by a deferred perpetuity.

$$100a_{\overline{120}|0.0025} + v^{120}50a_{\overline{180}|0.0025} + v^{300}10\left(\frac{1}{0.0025}\right)$$
$$= 10{,}356.18 + 5{,}365.73 + 1{,}891.24 = 17{,}613.15$$

Another way to analyze this perpetuity (without using deferred annuities) is the following, which produces the same answer:

$$10a_{\overline{\infty}|0.0025} + 40a_{\overline{300}|0.0025} + 50a_{\overline{120}|0.0025}$$
$$= 4{,}000 + 8{,}435.06 + 5{,}178.09 = 17{,}613.15$$

Answer: C

21. This is an annuity-due followed by a deferred perpetuity-due:

$$PV = 950 = X\ddot{a}_{\overline{40}|i} + v^{40}20\ddot{a}_{\overline{\infty}|i}$$

We are given the nominal interest rate convertible monthly, but we need the quarterly effective rate, i.

$$i = \left(1 + \frac{0.06}{12}\right)^3 - 1 = 0.015075$$

Then the present value equation becomes:

$$950 = X \cdot \frac{1 - 1.015075^{-40}}{0.015075 / 1.015075} + 1.015075^{-40} \cdot 20 \cdot \left(\frac{1}{0.015075 / 1.015075}\right)$$
$$950 = X \cdot (30.3252) + 740.1843$$
$$X = 6.9188$$

Answer: A

22. The expected value of the repayment this lender requires for each loan is:

$$10{,}000 \cdot 1.06^3 = 11{,}910.16$$

Because of the borrowers' estimated default rates, the probabilities of receiving the repayment are $(1-0.009)=0.991$ for one borrower, and $(1-0.036)=0.964$ for the other. The lender will set the repayment amounts for the two borrowers at:

$$\frac{11{,}910.16}{0.991} = 12{,}018.32 \quad \text{and} \quad \frac{11{,}910.16}{0.964} = 12{,}354.94$$

Converting these repayment amounts to annual effective rates for the loans, we have:

$$\left(\frac{12{,}018.32}{10{,}000}\right)^{1/3} - 1 = 0.063199 \quad \text{and} \quad \left(\frac{11{,}910.16}{10{,}000}\right)^{1/3} - 1 = 0.073034$$

The difference between these two rates is 0.009835

Answer: B

23. The discount is 200, so the price of the bond is 1,500-200 = 1300. We can solve for the nominal rate x using the financial calculator.

N=12, PV = -1,300, PMT = 37.50, FV = 1,500, CPT I/Y = 3.91358

This is the semi-annual effective rate. Multiply by 2 to get x, the nominal rate convertible semi-annually:

$$x = (3.91358\%)\,(2) = 7.82716\%$$

Answer: A

24. Account B will have a larger balance because its interest is compounded more frequently. The amounts deposited into each account (1,000) don't matter, as long as each account received the same deposit.

To find x, solve the following equation:

$$\left(1+\frac{x}{2}\right)^{20} = 1.05 \cdot (1+x)^{10}$$

$$\left(1+\frac{x}{2}\right)^{2} = 1.05^{0.1} \cdot (1+x) = 1.0048909 \cdot (1+x)$$

$$1 + x + \frac{x^2}{4} = 1.0048909 + 1.0048909x$$

$$x^2 - 4 \cdot (0.0048909x) - 4 \cdot (0.0048909) = 0$$

$$x^2 - 0.019564x - 0.019564 = 0$$

$$x = \frac{0.019564 \pm \left[(-0.019564)^2 - 4 \cdot 1 \cdot (-0.019564)\right]^{\frac{1}{2}}}{2}$$

$$= \frac{0.019564 \pm 0.280424}{2} = 0.14999, -0.1304$$

Answer: E

25. The general formula for the swap rate is:

$$R = \frac{\sum_{k=1}^{n}\left(Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k}\right)}{\sum_{k=1}^{n}\left(Q_{t_k} \cdot P_{t_k}\right)}$$

To apply this formula, we need to calculate the 0-, 1-, and 2-year forward rates:

$$f^*_{[0,1]} = \frac{1}{P_1} - 1 = \frac{1{,}000}{957.85} - 1 = 0.044005$$

$$f^*_{[1,2]} = \frac{P_1}{P_2} - 1 = \frac{957.85}{910.49} - 1 = 0.052016$$

$$f^*_{[2,3]} = \frac{P_2}{P_3} - 1 = \frac{910.49}{854.04} - 1 = 0.066098$$

Inserting the forward rates, notional amounts (in millions), and present value factors into the swap rate formula, we have:

$$\begin{aligned} R &= \frac{1 \cdot f^*_{[0,1]} \cdot P_1 + 2 \cdot f^*_{[1,2]} \cdot P_2 + 3 \cdot f^*_{[2,3]} \cdot P_3}{1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3} \\ &= \frac{1 \cdot 0.044005 \cdot 0.95785 + 2 \cdot 0.052016 \cdot 0.91049 + 3 \cdot 0.066098 \cdot 0.85404}{1 \cdot 0.95785 + 2 \cdot 0.91049 + 3 \cdot 0.85404} \\ &= 0.057334 \end{aligned}$$

Answer: C

26. Joe's future value is $Xs_{\overline{8}|i}$.

Scott's future value is $Xs_{\overline{10}|i}$.

The difference between the accumulated values is:

$Xs_{\overline{10}|i} - Xs_{\overline{8}|i} = X\left(s_{\overline{10}|i} - s_{\overline{8}|i}\right)$

However, the solutions are written using present value annuity functions, so we need to convert the above expression to one using present values and accumulation factors. Since $s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n$, we can rewrite the solution as:

$$X\left(a_{\overline{10}|i}(1+i)^{10} - a_{\overline{8}|i}(1+i)^8\right) = X(1+i)^8\left(a_{\overline{10}|i}(1+i)^2 - a_{\overline{8}|i}\right)$$

Answer: D

27. The total in the first account is just the 8 payments of X.

The deposits to the second account occur at the end of each year, and are the interest payments of $X(0.05)$, $2X(0.05)$, …$8X(0.05)$. We want the future value of these deposits plus the 8 payments of X to equal 9,500.

$$8X + X(0.05)(Is)_{\overline{8}|0.035} = 9{,}500$$
$$X\left(8 + (0.05)(Is)_{\overline{8}|0.035}\right) = 9{,}500$$
$$X = \frac{9{,}500}{8 + (0.05)(Is)_{\overline{8}|0.035}} = \frac{9{,}500}{8 + 1.95499} = 954.2949$$

Answer: A

28. First solve for X. We want to calculate the present value of payments of X, X+100, X+200, X+300, … X+5,900 at a 6-month effective interest rate of $i = \sqrt{1.06} - 1 = 0.0295630141$. Using the PQ formula, we have:

$$PV = Pa_{\overline{n}|i} + Q\left(\frac{a_{\overline{n}|i} - nv^n}{i}\right)$$
$$125{,}000 = Xa_{\overline{60}|i} + 100\left(\frac{a_{\overline{60}|i} - 60v^{60}}{i}\right)$$
$$125{,}000 = X(27.9366) + 59{,}161.7089$$
$$X = 2{,}356.7038$$

We can find the outstanding balance at time 4 by the prospective method (the present value of the remaining payments). Immediately after the fourth year, there are 52 payments remaining starting with the 9th payment. The 9th payment is X+8(100). We want to calculate the present value of payments of X+8(100), X+9(100), …., X+59(100). Again using the PQ formula with the value of X calculated above, we have:

$$PV = 3{,}156.7038a_{\overline{52}|i} + 100\left(\frac{a_{\overline{52}|i} - 52v^{52}}{i}\right)$$
$$PV = 83{,}307.7650 + 50{,}605.8808 = 133{,}913.6458$$

Note: This outstanding balance is greater than the original loan amount. That is because in the first few years of the loan, the payments are less than the interest due, resulting in negative amortization.

Answer: D

29. It is important to recognize that the purchase date does not matter. Only the yield and the bond's characteristics (face amount, coupon rate, maturity date) are needed for amortization calculations.

One way to solve this problem is by using the formula for the amount of premium amortized in a given period:

$$F(r-i)v^{n-k+1} = 1{,}000(0.020-0.028)1.028^{-(40-27+1)} = -5.43$$

There was -5.43 of premium amortized on August 1, 2015, which means that 5.43 of discount was amortized on that date. So the book value of the bond was increased by 5.43.

Another approach is to calculate the interest earned during the period and subtract the coupon paid. Start by calculating the bond's book value as of the beginning of the coupon period in question, February 1, 2015. On that date there are 14 coupon periods remaining:
Set N=14, I/Y=2.8, PMT=20, and FV=1,000. CPT PV = -908.39.
The interest earned from 2/1/2015 to 8/1/2015 was: $908.39 \cdot 0.028 = 25.43$.

The coupon payment is 20. This provides the bondholder with part of the 25.43 of interest that was earned. The remaining 5.43 of interest is capitalized by increasing the book value of the bond.

Alternatively, after calculating the bond's value with 14 periods remaining, as shown above, we could set N=13 and CPT PV = -913.82. The bond's value has increased from 908.39 to 913.82, an increase of 5.43.

Answer: D

30. The 3-year spot rate can be calculated from the relation:

$$(1+s_3)^3 = (1+i_{0,1})(1+i_{1,2})(1+i_{2,3})$$

Use the given function for forward rates to calculate each $i_{t,t+1}$:

$$i_{0,1} = -0.00015(0)^2 + 0.005(0) + 0.044 = 0.044$$
$$i_{1,2} = -0.00015(1)^2 + 0.005(1) + 0.044 = 0.04885$$
$$i_{2,3} = -0.00015(2)^2 + 0.005(2) + 0.044 = 0.0534$$

Now solve for the 3-year spot rate:

$$(1+s_3)^3 = (1+0.044)(1+0.04885)(1+0.0534) = 1.15347$$
$$s_3 = 1.15347^{1/3} - 1 = 0.048743$$

Answer: A

31. The amount of premium amortized in period k is:

$$F(r-i)v^{n-k+1}$$

Plug in what you know and solve for F:

$$14 = F(0.03-0.0225)v^{28-4+1}$$

$$F = \frac{14}{(0.03-0.0225)v^{28-4+1}} = 3,255.7398$$

The book value immediately after the 6th coupon payment is the present value of future payments with 22 coupon periods remaining.

$$BV = 3,255.7398(0.03)a_{\overline{22}|0.0225} + 3,255.7398v_{0.0225}^{22} = 3,675.8121$$

Answer: B

32. This problem is easy if we just observe that $\ddot{a}_{\overline{n}|i}$ is the value of payments made at the beginning of each period, which is obviously more valuable than $a_{\overline{n}|i}$, which represents the value of payments made at the end of each period. The continuous annuity, $\bar{a}_{\overline{n}|i}$, is intermediate in value, because its payments occur throughout the period, which means they occur after the beginning of the period and before the end.

We can also analyze the annuity formulas in more detail. The present value of an annuity-due is always larger than the present value of the corresponding annuity-immediate, since $\ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i}$. Therefore, we can eliminate choices D and E. The question is, how does the present value of a continuous annuity relate to the other two?

The formula for the present value of a continuous annuity is $\bar{a}_{\overline{n}|i} = \dfrac{1-v^n}{\delta}$, which is the same as an annuity-due and annuity-immediate, except for the denominator. The annuity-due equation has d in the denominator, and the annuity-immediate has i in the denominator, and we know that $d < i$. Now the question is, how do i, d, and δ relate?

You may know that δ lies between d and i. If not, we can calculate the values of d *and* δ for a particular value of i. We know that $\delta = \ln(1+i)$. Using 5% as an example, we have i = 0.05, d = 4.7619, and $\delta = 4.8790$. We can see that $i > \delta > d$. Since these are the constants in the denominator, this means that $a_{\overline{n}|i} < \bar{a}_{\overline{n}|i} < \ddot{a}_{\overline{n}|i}$.

Answer: A

33. The future value equation for Gavin's deposit will be (note the negative exponent, which converts a present value factor to an accumulation factor):

$$1{,}000\left(1-\frac{0.05}{2}\right)^{-6}\left(1-\frac{d}{12}\right)^{-48}=1{,}600$$

$$d=-12\cdot\left[\left(\frac{1.6}{\left(1-\frac{0.05}{2}\right)^{-6}}\right)^{-1/48}-1\right]=0.079261$$

Answer: C

34. The yield curve given by the polynomial is:

n	1	2	3
r_n	3.0%	4.0%	5.5%

The swap rate is the par bond coupon rate, which is given by:

$$R=\frac{1-\frac{1}{1.055^3}}{\frac{1}{1.03}+\frac{1}{1.04^2}+\frac{1}{1.055^3}}=0.054$$

Answer: E

35. The payoff payment at time 12 consists of an interest payment and the 46,000 principal repayment. However, the balance in the sinking fund is used to pay part of this amount. We need to find the balance in the sinking fund immediately before the end of the 3rd year and subtract that from the amount due under the loan. The net amount of the payoff payment is:

(46,000 + Interest payment) – balance in the sinking fund

The interest payment equals the quarterly interest rate times 46,000. The quarterly interest rate is $i = 1.035^{1/4} - 1 = 0.008637446$. The interest payment is 46,000(0.008637446)=397.32, and the total amount due is 46,397.32.

The balance in the sinking fund immediately before the end of the 3rd year includes 2 previous annual payments plus accrued interest.

To calculate the annual sinking fund payment, X, using the BA II Plus:
Set N=15, I/Y=2.6, PV=0, and FV=46,000. CPT PMT = 2,546.64.

The balance in the sinking fund one year after the second deposit is:

$$2,546.64(1.026^2 + 1.026) = 5,293.64$$

This is the balance in the sinking fund at the end of the 3rd year (before a third deposit is made). So the net payoff payment is:

$$46,397.32 - 5,293.64 = 41,103.68$$

Answer: D

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